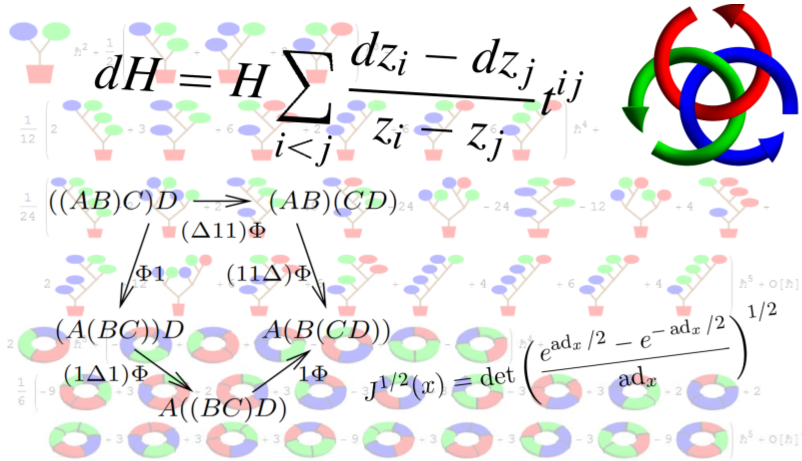
© | << < ? > >> | Dror Bar-Natan: Talks:
CRM 50 Home: Algebra:




# Expansions, Lie Algebras, and Invariants

### Workshop at CRM, Montreal, July 1-31, 2019

**Organizers: Anton Alekseev, Dror Bar-Natan, and Roland van der Veen**

**Jump to:** Bar-Natan, Alekseev, Naef, VanDerVeen, Schneps, Massuyeau, Sikora, Ito, Thurston, Kuehn, Moussard, Costantino, Robertson, Severa, Kuno, Suzuki, Dancso, Ens, Le, Bar-Natan (OU), Massuyeau (2), Kawazumi, Bar-Natan (GDO), Murakami, Pym, Furusho, Enriquez.

**Agenda.** Our workshop will bring together a number of experts working on "expansions" and a number of experts working on "invariants" in the hope that the two groups will learn from each other and influence each other. "Expansions" are solutions of a certain type of intricate equations within graded spaces often associated with free Lie algebras; they include Drinfel'd associators, solutions of the Kashiwara-Vergne equations, solutions of various deformation quantization problems, and more. By "invariants" we refer to quantum-algebra-inspired invariants of various objects within low-dimensional topology; these are often associated with various semi-simple Lie algebras. The two subjects were born together in the early days of quantum group theory, but have to a large extent evolved separately. We believe there is much to gain by bringing the two together again.

**Rough Plan.** In the first week (July 2-5) the organizers will give some introductory talks, explaining why we are here. The second week (July 8-12) will be "conference"-style, albeit with relatively few and long talks. The remainder of the time (July 15-31) will be in workshop style.

Group Photo on July 10 (thanks, Louis Pelletier!): Schneps, Costantino, Naef, Moussard, Severa, Suzuki, Ito, Kuno, Frohlich, Le, Thurston, Scherich, Kuehn, van der Veen, Dancso, Faes, Robertson, Massuyeau, Bar-Natan, Ens, Alekseev.

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| |Canada Day|Intros| | | | | |Conference| | | | | | |Workshop| | | | | | | | | | | | | | | | | | | |
|Alekseev|■|■|■|■|■|■|■|■|■|■|■|■| | | | |■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Bar-Natan|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Bryden|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | |
|Costantino| |■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | |
|Dancso|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Enriquez| | | | | | | | | | | | | | | | | | | |■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|
|Ens| | | | | | | |■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | | |
|Faes|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | | |
|Frohlich|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | | |
|Furusho| | | | | | | | | | | | | | | | | | | | |■|■|■|■|■|■|■|■|■|■|■|■|■|■|
|Ito|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Kawazumi| | | | | | | | | | | | | | | | |■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Kuehn|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Kuno|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Le|■|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | |
|Massuyeau|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Moussard|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
|Murakami J| | | | | | | | | | | | | | | | | | | |■|■|■|■|■|■|■|■|■|■|■| | | | |
|Naef|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |
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|Scherich| |■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | |
|Schneps|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | |
|Severa| | | | | |■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | |
|Sikora| |■|■|■|■|■|■|■|■| | | | | | | | | | | | | | | | | | | | | | | | | |
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|van der Veen|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■|■| | | |

Approximate Attendance Schedule

**Participants.** Anton Alekseev, Dror Bar-Natan, John Bryden, Francois Costantino, Zsuzsanna Dancso, Benjamin Enriquez, Travis Ens, Quentin Faes, Jesse Frohlich, Hidekazu Furusho, Tetsuya Ito, Nariya Kawazumi, Ulf Kuehn, Yusuke Kuno, Thang Le, Gwenael Massuyeau, Delphine Moussard, Jun Murakami, Florian Naef, Marcy Robertson, Nancy Scherich, Leila Schneps, Pavol Severa, Adam Sikora, Sakie Suzuki, Dylan Thurston, and Roland van der Veen.

Map of the Area

**Location.** At the CRM (see http://www.crm.umontreal.ca/coord/coord_an.shtml), mostly in room 5340.

## Lectures and Videos

Tuesday July 2.

**Dror Bar-Natan**, *Expansions, Lie algebras and Invariants*.

> This will be an introduction/survey of several of the main topics of this workshop. Much of these lectures could have been given twenty years ago and nothing will be newer than as of five years ago.
>
> Video 1, Video 2, Video 3.

Wednesday July 3.

**Anton Alekseev**, *Goldman brackets, Turaev cobrackets and formality*.

> We will recall the notions of Goldman brackets and Turaev cobrackets on the vector space spanned by homotopy classes of free loops on an oriented 2-manifold. We will then set up the formality question for the Goldman-Turaev Lie bialgebra. This talk is an introduction to the talks by Yusuke Kuno and Florian Naef.
>
> Video.

**Florian Naef**, *Formality of string topology in dimensions* $\geq 2$ *and Kashiwara-Vergne*.

> String topology studies operations on the (circle-equivariant) homology of the free loop space of a manifold. In particular, there is a bracket and cobracket operation, defined in the surface case by Goldman and Turaev, respectively, and in the general case by Chas-Sullivan and Goresky-Hingston, respectively. In the surface case these structures are connected to the Kashiwara-Vergne problem, namely solutions to the Kashiwara-Vergne problem gives "nice" algebraic descriptions of these operations. We will generalize this to general closed manifolds, that is, we give an algebraic description that depends on evaluating certain Feynman diagrams up to loop order 1.
>
> Video.

Thursday July 4.

**Roland van der Veen**, *Universal invariants and quasi-triangular Hopf algebras* and *Polynomial time knot invariants*.

> The first half is a survey on quasi-triangular Hopf algebras and the corresponding universal quantum knot invariants. In fact, the universal invariants provide a convenient graphical calculus to discuss the algebra. The Drinfeld double construction and relations to Poisson Lie groups and their quantization will also be mentioned. In the second half we present some techniques for efficient computation in Hopf algebras of PBW type. As an application we discuss how this leads to a wealth of strong knot invariants that are still computable in polynomial time. Our main example is related to the $n$-loop approximations to the universal $sl_2$ invariant. Joint work with Dror Bar-Natan.
>
> Video 1, Video 2.

Friday July 5.

**Leila Schneps**, *The mould theory approach to multiple zetas* and *The mould theory approach to Kashiwara-Vergne* and *The mould theory approach to elliptic multizetas and Kashiwara-Vergne*.

Monday July 8.

**Gwenael Massuyeau**, *Generalized Dehn twists on surfaces and homology cylinders*.

> (Joint work with Yusuke Kuno.) Given an oriented surface $S$ and a simple closed curve $C$ in $S$, the "Dehn twist" along $C$ is the homeomorphism of $S$ defined by "twisting" $S$ around $C$ by a full twist. If the curve $C$ is not simple, this transformation of $S$ does not make sense anymore, but one can consider two possible generalizations: one possibility is to use the homotopy intersection form of $S$ to "simulate" the action of a Dehn twist on the (Malcev completion of) the fundamental group of $S$; another possibility is to view $C$ as a curve on the top boundary of the cylinder $S \times [0, 1]$, to push it arbitrarily into the interior so as to obtain, by surgery along the resulting knot, a homology cylinder. In this talk, we will relate two those possible generalizations of a Dehn twist and we will give explicit formulas using a symplectic expansion of the fundamental group of $S$.
>
> Video.

**Adam Sikora**, *Quantization and Toric Degenerations of Character Varieties of Surfaces*.

> We explore the properties of character varieties of surfaces and their knot theory inspired quantizations, called skein algebras, by applying the theory of pseudo- Anosov diffeomorphisms and of measured foliations. In particular, we prove that every sufficiently generic measured foliation of a surface defines a toric degeneration of the character variety and a quantum toric degeneration of the skein algebra.
>
> Video.

**Tetsuya Ito**, *Garside theory and braid group representations*.

> A Garside structure is a combinatorial structure of groups that gives rise to a normal form solving the word and conjugacy problem. In this talk I discuss relations of braid group representations (of course, that is closely related to associator and knot invariants, the main theme of this Workshop) and Garside structures. Schematically speaking, I explain that we are able to see the normal form of braids `directly' from the image of braid representations. This seems to suggest rich combinatorial structures on braid group representations.
>
> Video.

Tuesday July 9.

**Dylan Thurston**, *Sutured Manifolds and Hopf algebras*.

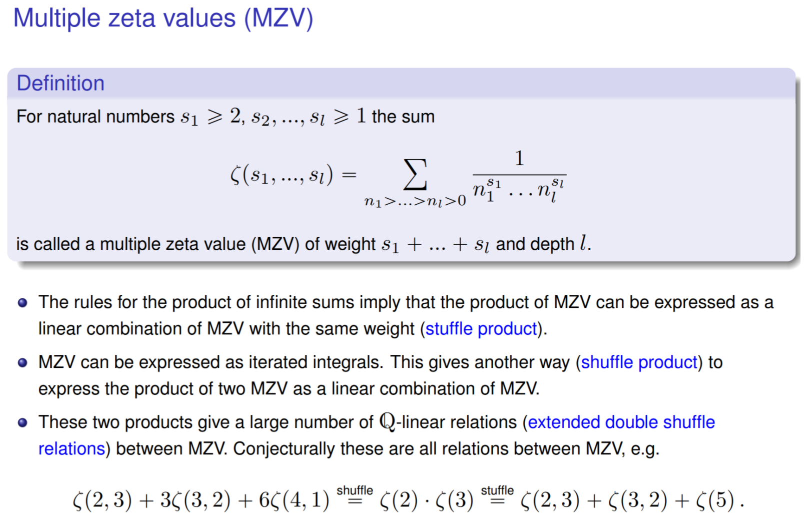
Kuperberg showed how Hopf algebras can give invariants of (framed) 3-manifolds. We look at possible extensions to open expressions, with free variables; these appear to be related to marked sutured 3-manifolds. In particular, there is a geometric interpretation of the Drinfel'd double construction. Some of this was done by Rohit Thomas, and portions are joint work with Dror Bar-Natan.

Video.

**Ulf Kuehn**, *Lie-algebras associated to multiple q-zeta values*.

The Lie Algebras associated with multiple zeta values are conjecturally related to Lie algebras of the Grothendieck Teichmueller groups and the Kashiwara Vergne problem. In this talk I present Lie Algebras that are associated with multiple q-zeta values and indicate how they complement the multiple zeta picture.

Slides, Video.



Kuehn's Slides

**Delphine Moussard**, *Finite type invariants of knots in homology 3- spheres*.

For null-homologous knots in rational homology 3- spheres, there are two equivariant invariants obtained by universal constructions a la Kontsevich, one due to Kricker and defined as a lift of the Kontsevich integral, and the other constructed by Lescop by means of integrals in configuration spaces. In order to explicit their universality properties and to compare them, we study a theory of finite type invariants of null-homologous knots in rational homology 3-spheres. We give a partial combinatorial description of the space of finite type invariants, graded by the degree. This description is complete for knots with a trivial Alexander polynomial, providing explicit universality properties for the Kricker lift and the Lescop equivariant invariant and proving the equivalence of these two invariants for such knots.

Video.

Wednesday July 10.

**François Costantino**, *Stated skein algebras of surfaces*.

(Joint work with Thang Le) After providing the definition of stated skein algebras and surfaces and discussing their relations with standard skein algebras, I will state a result detailing their algebraic behavior under topological operations. This, together with the identification of the algebra of the bigon with $O_q(sl_2)$, will allow us to discuss an interesting functor into the category of $U_q(sl_2)$ bimodules and their tensor products. If time permits I will detail how this fits into a framework of non symmetric operads in the sense of Markl.

Video.

**Marcy Robertson**, *Weak modular operads as a model for the Teichmueller tower*.

The Grothendieck-Teichmueller group is an explicitly defined (profinite) group introduced by Drinfeld which is closely related to (and contains) the absolute Galois group. The idea was based on Grothendieck's suggestion that one should study the absolute Galois group by relating it to its action on the Teichmueller tower of fundamental groupiods of the moduli stacks of genus g curves with n marked points. In this talk, we give an reimagining of the Teichmueller tower in terms of a profinite completion of an algebraic object called a Segal modular operad." Using this reinterpretation, we show that the homotopy automorphisms of this model for the Teichmuller tower is isomorphic to a subgroup of the (profinite) Grothendieck-Teichmüller group introduced by Hatcher, Lochak and Schneps. We then show a non-trivial action of the absolute Galois group on our tower -- giving evidence that this is a good model for the Teichmueller tower. This talk will be aimed a general audience and will not assume any previous knowledge of the Grothendieck-Teichmueller group or operads.

Video.

Thursday July 11.

**Pavol Severa**, *Quantization of Poisson Hopf algebras*.

I will describe a method for quantization of Poisson Hopf algebras (in arbitrary $\mathbb{Q}$-linear symmetric monoidal categories) which is compatible with tensor products. The main idea comes from nerves of groups: they are symmetric simplicial sets. Nerves of Hopf algebras then turn out to be braided rather than symmetric and nerves of Poisson Hopf algebras to be infinitesimally braided. The problem is thus solved via the standard machinery of Drinfeld associators. A little bit can be said also if groups are replaced with groupoids and Hopf algebras with Hopf algebroids. Joint work with Jan Pulmann.

Video.

**Yusuke Kuno**, *Symplectic/special expansions for surfaces*.

The 1-formality of a free group of finite rank is captured by the notion of group-like expansions. In this talk, we consider the situation where the free group under consideration is given as the fundamental group of a compact oriented surface with boundary. Then it is natural to ask group-like expansions to respect the topology of the surface, and the notion of special/symplectic expansions arises. I will present several examples of such expansions and results related to the Goldman bracket and the Tuarev cobracket.

Video.

**Sakie Suzuki**, *Factorizations of the universal $R$ matrix and the universal quantum invariant for framed 3-manifolds*.

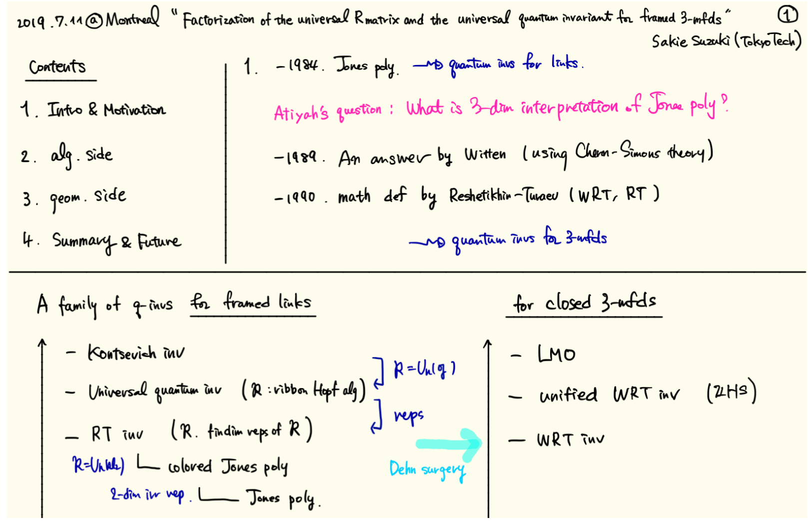
Take the Drinfeld double $D(B)$ of the Borel subalgebra $B$ of the quantized enveloping algebra of $sl_2$. We consider two embeddings of $D(B)$ as an algebra, into a double of Heisenberg double and into a quantum torus algebra. With both embeddings, each image of the universal $R$ matrix has a factorization into a product of four elements each satisfying a pentagon relation. This setting leads us to the Kashaev invariant of links and to quantum Teichmüller theory. In this talk I will explain these situations and show our trials to unify these studies in a view point of the universal $S$ tensor and framed 3-manifolds. This talk includes a joint work with Y. Terashima.

Slides, Video.

Friday July 12.

**Zsuzsanna Dancso**, *Topological approaches to Kashiwara-Vergne*.

I will give an overview of a problem in topology equivalent to the KV problem, concerning finite type invariants of a class of 4-dimensional tangles (joint with Dror Bar-Natan). I will point out some key differences from the Alekseev-Kawazumi-Kuno-Naef topological approach, described in previous talks, and mention a - so far not entirely successful - attempt at re-phrasing AKKN in more knot-theoretic terms. The true goal would be to clarify the mysterious, but likely existing, topological connection between these two very different-looking constructions.



Suzuki's Slides

Video.

**Travis Ens**, *Braidors: A Simplified Theory of Drinfel'd's Associators?*

I will present some incomplete work on a potentially equivalent theory to Drinfel'd's theory of associators, but which is simpler in some respects and in addition is more natural from a topological perspective. The fundamental idea is to replace braids in a disk by braids in an annulus. I will present some results which have been proven and several unproven conjectures which are backed by by computational evidence.

Video.

**Thang Le**, *On Witten's finiteness conjecture*.

Witten conjectured that over the field of rational functions in the Kauffman variable $A$, the skein module of any closed oriented 3-manifold is finite-dimensional. We will show that the conjecture holds true if the 3-manifold is a Dehn filling of a compact manifold with toroidal boundary satisfying certain $q$-holonomic condition. In particular, the conjecture holds for Dehn fillings of any 2-bridge knots or links, torus knots, and $(-2, 3, 2n+1)$-pretzel knots.

Video.

Tuesday July 16.

**Dror Bar-Natan**, *Over-then-Under-tangles and the Drinfel'd Double*.

If you attend, at the end of the talk it should be clear to you why the Drinfel'd Double construction is natural and appealing from a knot-theory perspective, yet in what sense I still don't understand it.

Video.

Wednesday July 17.

**Gwenael Massuyeau**, *Turaev's loop operations on surfaces and their formal descriptions*.

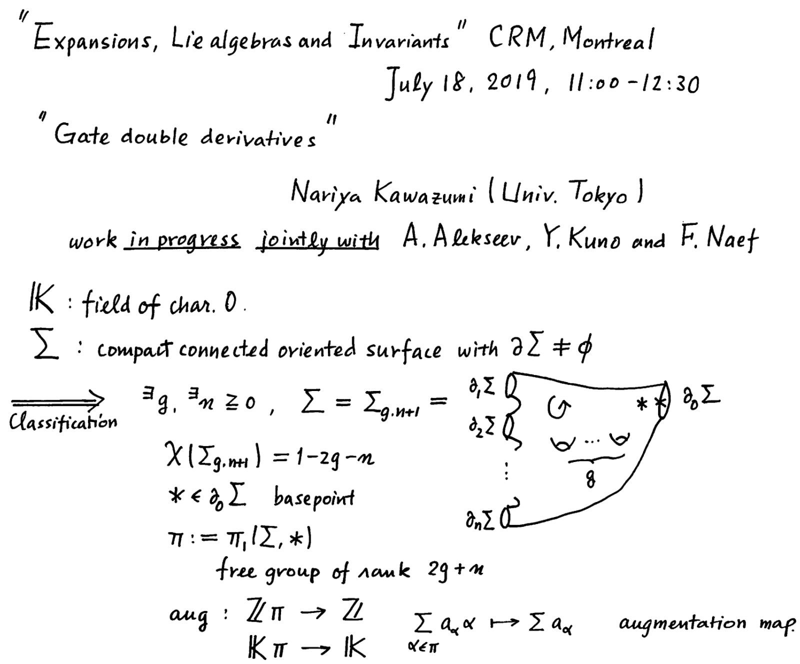
Turaev introduced in 1978 two operations on the fundamental group of a surface with boundary. The first operation measures the intersection of two loops on a surface, while the second operation measures the self-intersection of a single loop. In this "informal" talk, we will survey these loop operations before addressing the problem of their "formal" descriptions. We will see that a "formality" isomorphism for the self-intersection operation of a punctured disk arises from any Drinfeld associator. The proof is based on some 3-dimensional formulas for Turaev's loop operations.

Thursday July 18.

**Nariya Kawazumi**, *Gate double derivatives*.

Recently Turaev introduced the notion of a gate derivative on the group ring of the fundamental group of an oriented surface. Its double version gives a topological interpretation of a double divergence which connects the homotopy intersection form and the Turaev cobracket. We will explain the defintion of a gate double derivative and some of its properties. This is a joint work in progress with Anton Alekseev, Yusuke Kuno and Florian Naef.

Video, slides.



Kawazumi's Slides

Friday July 19.

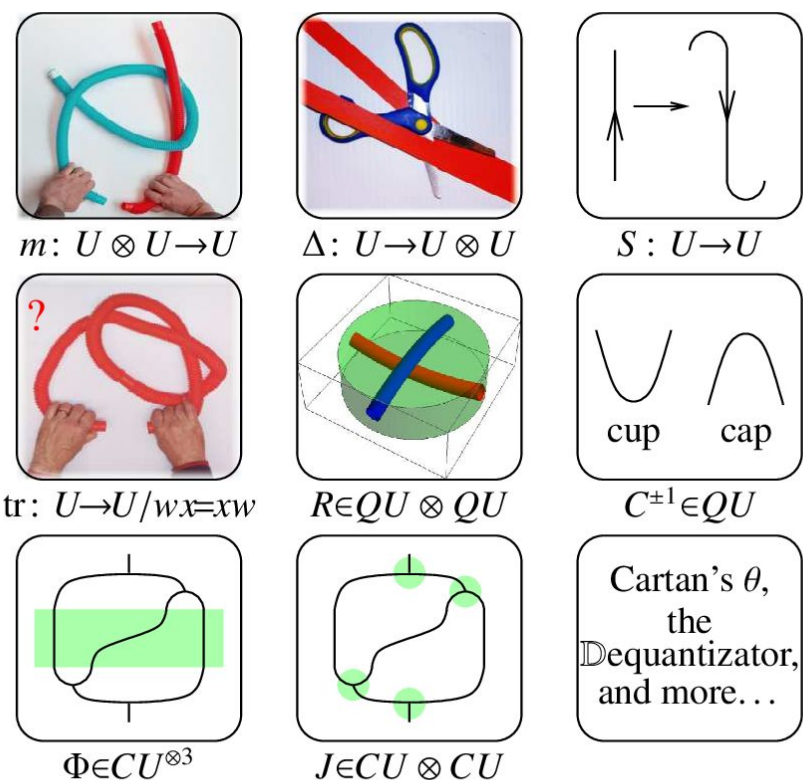
**Dror Bar-Natan**, *Everything around $sl_{2+}^{\epsilon}$ is **DoPeGDO**. So what?*.

I'll explain what "everything around" means: classical and quantum $m$, $\Delta$, $S$, $tr$, $R$, $C$, and $\theta$, as well as $P$, $\Phi$, $J$, $\mathbb{D}$, and more, and all of their compositions. What **DoPeGDO** means: the category of Docile Perturbed Gaussian Differential Operators. And what $sl_{2+}^{\epsilon}$ means: a solvable approximation of the semi-simple Lie algebra $sl_2$.

Knot theorists should rejoice because all this leads to very powerful and well-behaved poly-time-computable knot invariants. Quantum algebraists should rejoice because it's a realistic playground for testing complicated equations and theories.

This is joint work with Roland van der Veen and continues work by Rozansky and Overbay.

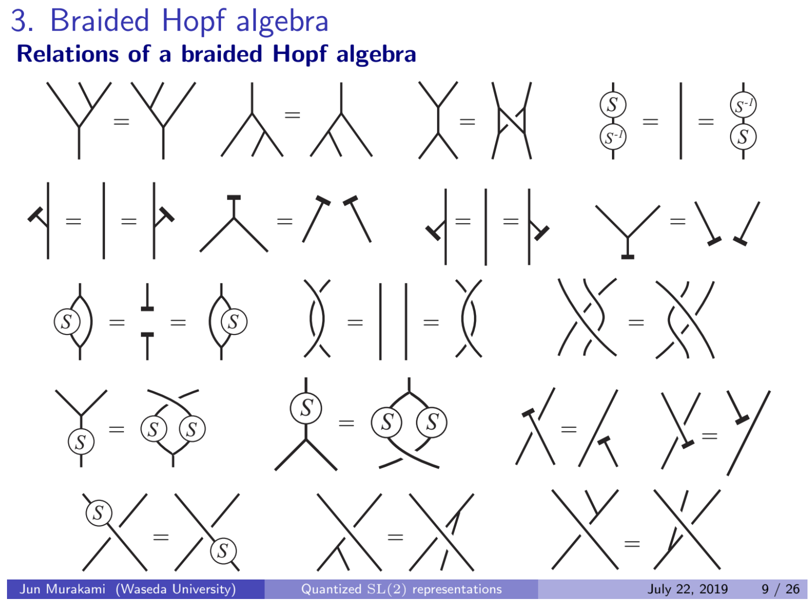
Video 1, Video 2, handout: PDF, HTML.



Bar-Natan's Handout

Monday July 22.

**Jun Murakami**, *Quantized $SL(2)$ representations of knot groups*



Murakami's Slides

For a braided Hopf algebra $A$ with braided commutativity, we introduce the space of $A$ representations of a knot $K$ as a generalization of the $G$ representation space of $K$ defined for a group $G$. By rebuilding the $G$ representation space from the view point of Hopf algebras, it is extended to any braided Hopf algebra with braided commutativity. Applying this theory to $\mathrm{BSL}(2)$ which is the braided quantum $\mathrm{SL}(2)$ introduced by S. Majid, we get the space of $\mathrm{BSL}(2)$ representations. It is a non-commutative algebraic scheme which provides quantized $\mathrm{SL}(2)$ representations of $K$.

Video, slides.

Tuesday July 23.

**Brent Pym**, *Multiple zeta values in deformation quantization*

(From arXiv:1812.11649, joint with Peter Banks and Erik Panzer) Kontsevich's 1997 formula for the deformation quantization of Poisson brackets is a Feynman expansion involving volume integrals over moduli spaces of marked disks. We develop a systematic theory of integration on these moduli spaces via suitable algebras of polylogarithms, and use it to prove that Kontsevich's integrals can be expressed as integer-linear combinations of multiple zeta values. Our proof gives a concrete algorithm for calculating the integrals, which we have used to produce the first software package for the symbolic calculation of Kontsevich's formula.

Video.

Thursday July 25.

**Hidekazu Furusho**, *The associator relation and the confluence relations*

I will talk about the confluence relation, a relation among multiple zeta values, introduced by Hirose and Sato. I will explain that it is equivalent to Drinfeld's associator relation.

Video.

Friday July 26.

**Benjamin Enriquez**, *On double shuffle relations for MZVs*

By a work of Furusho (2011), associator relations between multiple zeta values imply double shuffle relations (this result was also announded in a unfinished preprint by Deligne and Terasoma in 2005). The talk presents a new proof of this result. It turns out to be a consequence of the construction of a coproduct over a module over the underlying algebra of the harmonic coproduct, of "Betti" counterparts of both the harmonic coproduct and its module version, and of the statement that an arbitrary associator relates both coproducts with their Betti counterparts. In their turn, these statements rely on a geometric interpretation in terms of moduli spaces of the harmonic coproduct (following Deligne and Terasoma) and of its module version as well as of their Betti counterparts, and on the property of associators of relating braid groups with their infinitesimal counterparts. (Joint w. H. Furusho.)

[Video](), [slides]().

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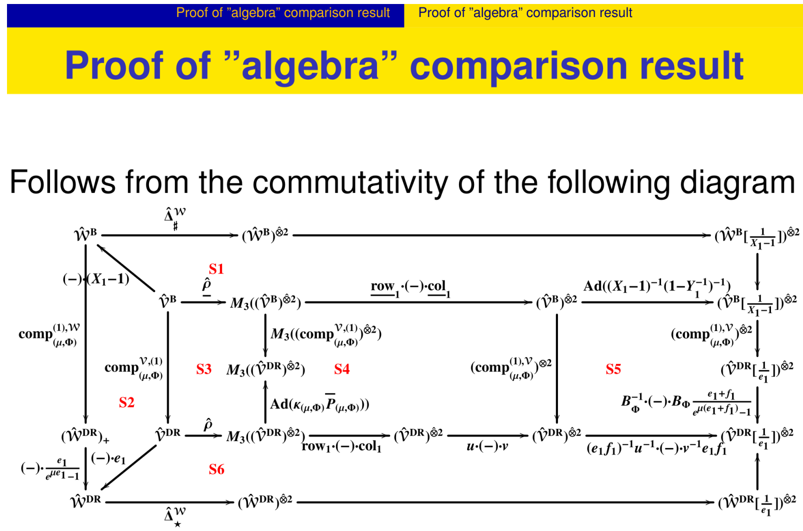
**Housekeeping.** [http://drorbn.net/m19](http://drorbn.net/m19), [pensieve]().




Enriquez' Slides