

Pensieve header: Solving for associators in TAUT/SAUT.

```

SetDirectory["C:/drorbn/AcademicPensieve/Projects/Braidors"];
<< ../WKO4/FreeLie.m;
$SeriesShowDegree = 3;

FreeLie` implements / extends
{*, +, **, $SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw,
CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ħ, ↦, ↠}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to
support it within reason until July 15, 2022. This is version 150814.

AngleBracket /:
  λ1_AngleBracket ** λ2_AngleBracket /; Support[λ1] == Support[λ2] := BCHtb[λ1, λ2];
AngleBracket /: λ1_AngleBracket ** λ2_AngleBracket := NonCommutativeMultiply[
  λ1 ∪ ((# → LS[0]) & /@ <Complement[Support@λ2, Support@λ1]>),
  λ2 ∪ ((# → LS[0]) & /@ <Complement[Support@λ1, Support@λ2]>)
];
to[us_List → vs_List][ser_LieSeries | ser_CWSeries | ser_AngleBracket] :=
  ser // LieMorphism[Thread[(LW/@us) → (LW/@vs)]];
to[u_, v_] := to[{u} → {v}];
to[us_List → vs_List][ξ_Es] := to[us → vs] /@ ξ;
ho[xs_List → ys_List][λ_AngleBracket] :=
  Union[λ \ xs, <Thread[ys → Table[λx, {x, xs}]]>];
ho[x_, y_] := ho[{x} → {y}];
do[as_List → bs_List][ξ_] := ξ // to[as → bs] // ho[as → bs];
do[a_, b_][ξ_] := ξ // to[a, b] // ho[a, b];
λ_AngleBracket // dΔ[a_, b_, c_] :=
  (λ \ a) ∪ <b → λa, c → λa> // LieMorphism[LW@a → LW@b + LW@c];
ξ_AngleBracket // σ[s___List] := Module[{ξ1},
  ξ1 = ξ // do[Range[Length@{s}] → First /@ {s}];
  Do[
    ξ1 = ξ1 // dΔ[{s}[[i, 1]], {s}[[i, 1]], {s}[[i, j]],
    {i, Length@{s}}, {j, 2, Length@{s}[[i]]}
  ];
  ξ1
];
AngleBracket /: (ξ_AngleBracket)s-σ := ξ // s;

```

**DKS**[**b**[**t**[1, 3], **t**[2, 3]]/24] // **αMap**[1, 2, 3]

$$\left\langle 1 \rightarrow \text{LS}\left[0, \frac{\overline{23}}{24}, 0, \dots\right], 2 \rightarrow \text{LS}\left[0, -\frac{\overline{13}}{24}, 0, \dots\right], 3 \rightarrow \text{LS}\left[0, \frac{\overline{12}}{24}, 0, \dots\right] \right\rangle$$

**α** = **LS**[{1, 2, 3}, **αs**]; **β** = **LS**[{1, 2, 3}, **βs**]; **γ** = **LS**[{1, 2, 3}, **γs**];  
**αs**[1] = **αs**[2] = **αs**[3] = **βs**[1] = **βs**[2] = **βs**[3] = **γs**[1] = **γs**[2] = **γs**[3] = 0;  
**αs**[2, 3] = 1/24; **βs**[1, 3] = -1/24;  
**αs**[1, 3] = **αs**[1, 2] = **βs**[1, 2] = 0;  
**Φ**<sub>0</sub> = <1 → **α**, 2 → **β**, 3 → **γ**>;  
**Φ**<sub>0</sub>@{2}

$$\left\langle 1 \rightarrow \text{LS}\left[0, \frac{\overline{23}}{24}, \dots\right], 2 \rightarrow \text{LS}\left[0, -\frac{\overline{13}}{24} + \overline{23}\beta_s[2, 3], \dots\right], \right. \\ \left. 3 \rightarrow \text{LS}\left[0, \overline{12}\gamma_s[1, 2] + \overline{13}\gamma_s[1, 3] + \overline{23}\gamma_s[2, 3], \dots\right] \right\rangle$$

**SeriesSolve**[{**α**, **β**, **γ**}, **ħ**<sup>-1</sup>(**b**[**LW**@1, **α**] + **b**[**LW**@2, **β**] + **b**[**LW**@3, **γ**] ≡ **LS**[0]) &&  
**Φ**<sub>0</sub><sup>σ</sup>[3,2,1] ≡ -**Φ**<sub>0</sub> && **Φ**<sub>0</sub> \*\* **Φ**<sub>0</sub><sup>σ</sup>[1,23,4] \*\* **Φ**<sub>0</sub><sup>σ</sup>[2,3,4] ≡ **Φ**<sub>0</sub><sup>σ</sup>[12,3,4] \*\* **Φ**<sub>0</sub><sup>σ</sup>[1,2,34] ];

**Φ**<sub>0</sub>@

{2}

$$\left\langle 1 \rightarrow \text{LS}\left[0, \frac{\overline{23}}{24}, \dots\right], 2 \rightarrow \text{LS}\left[0, -\frac{\overline{13}}{24}, \dots\right], 3 \rightarrow \text{LS}\left[0, \frac{\overline{12}}{24}, \dots\right] \right\rangle$$

**Φ**<sub>0</sub>@{4}

$$\left\langle 1 \rightarrow \text{LS}\left[0, \frac{\overline{23}}{24}, 0, -\frac{\overline{1123}}{1440} + \frac{\overline{71223}}{5760} + \frac{\overline{1233}}{5760} - \right. \right. \\ \left. \frac{\overline{72223}}{5760} + \frac{\overline{72233}}{5760} + \frac{1}{480} \frac{\overline{1213}}{1213} + \frac{\overline{1323}}{2880} + \frac{\overline{71232}}{2880} - \frac{\overline{2333}}{1440}, \dots\right], \\ 2 \rightarrow \text{LS}\left[0, -\frac{\overline{13}}{24}, 0, \frac{\overline{1113}}{1440} + \frac{\overline{71223}}{1920} - \frac{\overline{71132}}{5760} - \frac{\overline{1133}}{5760} + \frac{\overline{71213}}{5760} + \right. \\ \left. \frac{\overline{71323}}{2880} + \frac{\overline{71232}}{1920} + \frac{\overline{71322}}{5760} + \frac{\overline{71332}}{5760} + \frac{\overline{1333}}{1440}, \dots\right], \\ 3 \rightarrow \text{LS}\left[0, \frac{\overline{12}}{24}, 0, -\frac{\overline{1112}}{1440} + \frac{\overline{1123}}{5760} + \frac{\overline{71223}}{5760} + \frac{\overline{71122}}{5760} + \frac{\overline{1132}}{5760} - \right. \\ \left. \frac{\overline{1233}}{1440} + \frac{\overline{1213}}{5760} + \frac{\overline{1323}}{1440} - \frac{\overline{71222}}{5760} - \frac{\overline{71322}}{5760} - \frac{\overline{1332}}{1440}, \dots\right] \right\rangle$$

**R** = **DKS**[**t**[1, 2]/2] // **αMap**[1, 2]

$$\left\langle 1 \rightarrow \text{LS}\left[\frac{\overline{2}}{2}, 0, 0, \dots\right], 2 \rightarrow \text{LS}\left[\frac{\overline{1}}{2}, 0, 0, \dots\right] \right\rangle$$

$$\left( R^{\sigma[12,3]} \equiv \bar{\mathfrak{e}}_0 ** R^{\sigma[2,3]} ** (-\bar{\mathfrak{e}}_0)^{\sigma[1,3,2]} ** R^{\sigma[1,3]} ** \bar{\mathfrak{e}}_0^{\sigma[3,1,2]} \wedge \right. \\ \left. (-R)^{\sigma[12,3]} \equiv \bar{\mathfrak{e}}_0 ** (-R)^{\sigma[2,3]} ** (-\bar{\mathfrak{e}}_0)^{\sigma[1,3,2]} ** (-R)^{\sigma[1,3]} ** \bar{\mathfrak{e}}_0^{\sigma[3,1,2]} \right) @ \{6\}$$

BS[7 True, ...]

**Braidors & weak associators:  $B = \Phi^{012} R_u^{12} \Phi^{-021}$**   
 $B^{012} B^{02,1,3} B^{023} = B^{01,2,3} B^{013} B^{03,1,2}.$

$R = \text{DKS}[t[1, 2] / 2];$

$B_0 = \bar{\mathfrak{e}}_0^{\sigma[1,2,3]} ** R^{\sigma[2,3]} ** (-\bar{\mathfrak{e}}_0)^{\sigma[1,3,2]}$

$\text{DKS}\left[\frac{\overline{t_{23}}}{2}, \frac{1}{12} \overline{t_{13} t_{23}}, 0, -\frac{\overline{t_{13} t_{23} t_{23} t_{23}}}{5760} + \frac{1}{720} \overline{t_{13} t_{13} t_{23} t_{23}} - \frac{1}{720} \overline{t_{13} t_{13} t_{13} t_{23}}, \dots\right]$

$\text{lhs} = B_0^{\sigma[1,2,3]} ** B_0^{\sigma[13,2,4]} ** B_0^{\sigma[1,3,4]}$

$\text{DKS}\left[\frac{\overline{t_{23}}}{2} + \frac{\overline{t_{24}}}{2} + \frac{\overline{t_{34}}}{2}, \frac{1}{12} \overline{t_{13} t_{23}} + \frac{1}{12} \overline{t_{14} t_{24}} + \frac{1}{12} \overline{t_{14} t_{34}} + \frac{1}{24} \overline{t_{24} t_{34}}, 0, \right. \\ -\frac{\overline{t_{13} t_{23} t_{23} t_{23}}}{5760} - \frac{\overline{t_{14} t_{24} t_{24} t_{24}}}{5760} + \frac{\overline{t_{14} t_{34} t_{24} t_{24}}}{2880} + \frac{\overline{t_{14} t_{34} t_{34} t_{24}}}{2880} - \frac{\overline{t_{14} t_{34} t_{34} t_{34}}}{5760} \\ - \frac{7 \overline{t_{24} t_{34} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{24} t_{34} t_{24}}}{1440} + \frac{1}{480} \overline{t_{14} t_{24} t_{14} t_{34}} + \frac{1}{360} \overline{t_{14} t_{34} t_{24} t_{34}} + \\ \frac{1}{720} \overline{t_{13} t_{13} t_{23} t_{23}} - \frac{1}{720} \overline{t_{13} t_{13} t_{13} t_{23}} + \frac{1}{720} \overline{t_{14} t_{14} t_{24} t_{24}} + \frac{1}{360} \overline{t_{14} t_{14} t_{34} t_{24}} + \\ \frac{1}{720} \overline{t_{14} t_{14} t_{34} t_{34}} + \frac{1}{360} \overline{t_{14} t_{24} t_{34} t_{34}} - \frac{1}{720} \overline{t_{14} t_{14} t_{14} t_{24}} - \frac{1}{720} \overline{t_{14} t_{14} t_{14} t_{34}} + \\ \left. \frac{\overline{t_{14} t_{14} t_{24} t_{34}}}{1440} - \frac{1}{576} \overline{t_{14} t_{24} t_{24} t_{34}} + \frac{7 \overline{t_{24} t_{24} t_{34} t_{34}}}{5760} - \frac{\overline{t_{24} t_{24} t_{24} t_{34}}}{1440}, \dots\right]$

$\text{rhs} = B_0^{\sigma[12,3,4]} ** B_0^{\sigma[1,2,4]} ** B_0^{\sigma[14,2,3]}$

$\text{DKS}\left[\frac{\overline{t_{23}}}{2} + \frac{\overline{t_{24}}}{2} + \frac{\overline{t_{34}}}{2}, \frac{1}{12} \overline{t_{13} t_{23}} + \frac{1}{12} \overline{t_{14} t_{24}} + \frac{1}{12} \overline{t_{14} t_{34}} + \frac{1}{24} \overline{t_{24} t_{34}}, 0, \right. \\ -\frac{\overline{t_{13} t_{23} t_{23} t_{23}}}{5760} - \frac{\overline{t_{14} t_{24} t_{24} t_{24}}}{5760} + \frac{\overline{t_{14} t_{34} t_{24} t_{24}}}{2880} + \frac{\overline{t_{14} t_{34} t_{34} t_{24}}}{2880} - \frac{\overline{t_{14} t_{34} t_{34} t_{34}}}{5760} \\ - \frac{7 \overline{t_{24} t_{34} t_{34} t_{34}}}{5760} + \frac{\overline{t_{14} t_{24} t_{34} t_{24}}}{1440} + \frac{1}{480} \overline{t_{14} t_{24} t_{14} t_{34}} + \frac{1}{360} \overline{t_{14} t_{34} t_{24} t_{34}} + \\ \frac{1}{720} \overline{t_{13} t_{13} t_{23} t_{23}} - \frac{1}{720} \overline{t_{13} t_{13} t_{13} t_{23}} + \frac{1}{720} \overline{t_{14} t_{14} t_{24} t_{24}} + \frac{1}{360} \overline{t_{14} t_{14} t_{34} t_{24}} + \\ \frac{1}{720} \overline{t_{14} t_{14} t_{34} t_{34}} + \frac{1}{360} \overline{t_{14} t_{24} t_{34} t_{34}} - \frac{1}{720} \overline{t_{14} t_{14} t_{14} t_{24}} - \frac{1}{720} \overline{t_{14} t_{14} t_{14} t_{34}} + \\ \left. \frac{\overline{t_{14} t_{14} t_{24} t_{34}}}{1440} - \frac{1}{576} \overline{t_{14} t_{24} t_{24} t_{34}} + \frac{7 \overline{t_{24} t_{24} t_{34} t_{34}}}{5760} - \frac{\overline{t_{24} t_{24} t_{24} t_{34}}}{1440}, \dots\right]$

**(lhs == rhs) @ {6}**

BS[7 True, ...]

**Bs[2, 1] = Bs[3, 1] = 0; Bs[3, 2] = 1/2; B1 = DKS[3, Bs];**

**SeriesSolve[B1, B1<sup>σ[1,2,3]</sup> \*\* B1<sup>σ[13,2,4]</sup> \*\* B1<sup>σ[1,3,4]</sup> == B1<sup>σ[12,3,4]</sup> \*\* B1<sup>σ[1,2,4]</sup> \*\* B1<sup>σ[14,2,3]</sup>];**

**B1 @ {6}**

SeriesSolve::ArbitrarilySetting: In degree 3 arbitrarily setting {Bs[3, 1, 2, 2] → 0}.

SeriesSolve::ArbitrarilySetting: In degree 5 arbitrarily setting {Bs[3, 1, 1, 1, 2, 2] → 0}.

$$\begin{aligned}
 & \text{DKS} \left[ \frac{t_{23}}{2}, \frac{1}{12} t_{13} t_{23}, 0, -\frac{t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{1}{720} t_{13} t_{13} t_{23} t_{23} - \frac{1}{720} t_{13} t_{13} t_{13} t_{23}, \right. \\
 & -\frac{t_{13} t_{23} t_{23} t_{23} t_{23}}{7680} + \frac{t_{13} t_{23} t_{13} t_{23} t_{23}}{8640} + \frac{t_{13} t_{13} t_{23} t_{23} t_{23}}{3840}, \\
 & \frac{t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{645120} - \frac{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{145152} - \frac{71 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{2903040} - \\
 & \frac{23 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{20160} + \frac{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{22680} + \\
 & \left. \frac{13 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{161280} - \frac{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{15120} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{30240}, \dots \right]
 \end{aligned}$$

**B0 @ {6}**

$$\begin{aligned}
 & \text{DKS} \left[ \frac{t_{23}}{2}, \frac{1}{12} t_{13} t_{23}, 0, -\frac{t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{1}{720} t_{13} t_{13} t_{23} t_{23} - \frac{1}{720} t_{13} t_{13} t_{13} t_{23}, \right. \\
 & -\frac{t_{13} t_{23} t_{23} t_{23} t_{23}}{7680} + \frac{t_{13} t_{23} t_{13} t_{23} t_{23}}{8640} + \frac{t_{13} t_{13} t_{23} t_{23} t_{23}}{3840}, \\
 & \frac{t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{645120} - \frac{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{145152} - \frac{71 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{2903040} - \\
 & \frac{23 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{20160} + \frac{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{22680} + \\
 & \left. \frac{13 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{161280} - \frac{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{15120} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{30240}, \dots \right]
 \end{aligned}$$

**(B0 == B1) @ {7}**

SeriesSolve::ArbitrarilySetting: In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting: In degree 7 arbitrarily setting {Bs[3, 1, 1, 1, 1, 1, 2, 2] → 0}.

BS[8 True, ...]

