

Pensieve header: Checking that  $B(\phi)$  indeed solves the braidor equations and solving for a fresh braidor.

```
SetDirectory["C:/drorbn/AcademicPensieve/Projects/Braidors"];
<< ../WKO4/FreeLie.m;
$SeriesShowDegree = 4;
Φs[2, 1] = Φs[3, 1] = Φs[3, 2] = 0; Φs[3, 1, 2] = 1/24; Φ₀ = DKS[3, Φs];
SeriesSolve[Φ₀, Φ₀σ[3,2,1] ≡ -Φ₀ && Φ₀ ** Φ₀σ[1,23,4] ** Φ₀σ[2,3,4] ≡ Φ₀σ[12,3,4] ** Φ₀σ[1,2,34]];
Φ₀@{6}
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FreeLie` implements / extends

- {\*, +, \*\*, \$SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ħ, ↦, ↪}

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 150814.

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Φs[3, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Φs[3, 1, 1, 1, 2] → 0}.

$$DKS\left[0, \frac{1}{24} \overline{\overline{\overline{t_{13} t_{23}}}}, 0, -\frac{7}{5760} \overline{\overline{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}}} + \frac{7}{5760} \overline{\overline{\overline{\overline{t_{13} t_{13} t_{23} t_{23}}}}} - \frac{t_{13} t_{13} t_{13} t_{23}}{1440}, \right.$$

$$0, \frac{31}{967680} \overline{\overline{\overline{\overline{\overline{\overline{t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}}}}}} - \frac{157}{1935360} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}}}} - \frac{31}{387072} \overline{\overline{\overline{\overline{\overline{t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}}}} -$$

$$\frac{31}{483840} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}}}} + \frac{11}{290304} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}}}} + \frac{31}{725760} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}}}} +$$

$$\frac{83}{967680} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}}}} - \frac{13}{241920} \overline{\overline{\overline{\overline{\overline{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}}}} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{60480}, \dots]$$

$\Phi_0^{\sigma[1,2,3]}$

$$DKS\left[0, \frac{1}{24} \overline{\overline{\overline{t_{13} t_{23}}}}, 0, -\frac{7}{5760} \overline{\overline{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}}} + \frac{7}{5760} \overline{\overline{\overline{\overline{t_{13} t_{13} t_{23} t_{23}}}}} - \frac{t_{13} t_{13} t_{13} t_{23}}{1440}, \dots\right]$$

**Braidors & weak associators:  $B = \Phi^{012} R_u^{12} \Phi^{-021}$**   
 $B^{012} B^{02,1,3} B^{023} = B^{01,2,3} B^{013} B^{03,1,2}$ .

$$R = \text{DKS}[t[1, 2] / 2];$$

$$B_0 = \mathfrak{B}_0^{\sigma[1,2,3]} ** R^{\sigma[2,3]} ** (-\mathfrak{B}_0)^{\sigma[1,3,2]}$$

$$\text{DKS}\left[\frac{t_{23}}{2}, \frac{1}{12} \overline{t_{13} t_{23}}, 0, -\frac{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}}{5760} + \frac{1}{720} \overline{t_{13} \overline{t_{13} t_{23} t_{23}}} - \frac{1}{720} \overline{t_{13} t_{13} \overline{t_{13} t_{23}}}, \dots\right]$$

$$\text{lhs} = B_0^{\sigma[1,2,3]} ** B_0^{\sigma[13,2,4]} ** B_0^{\sigma[1,3,4]}$$

$$\begin{aligned} \text{DKS}\left[\frac{t_{23}}{2} + \frac{t_{24}}{2} + \frac{t_{34}}{2}, \frac{1}{12} \overline{t_{13} t_{23}} + \frac{1}{12} \overline{t_{14} t_{24}} + \frac{1}{12} \overline{t_{14} t_{34}} + \frac{1}{24} \overline{t_{24} t_{34}}, 0, \right. \\ \left. -\frac{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}}{5760} - \frac{\overline{\overline{t_{14} t_{24} t_{24} t_{24}}}}{5760} + \frac{\overline{\overline{t_{14} t_{34} t_{24} t_{24}}}}{2880} + \frac{\overline{\overline{t_{14} t_{34} t_{34} t_{24}}}}{2880} - \frac{\overline{\overline{t_{14} t_{34} t_{34} t_{34}}}}{5760} - \right. \\ \left. \frac{7 \overline{\overline{t_{24} t_{34} t_{34} t_{34}}}}{5760} + \frac{\overline{\overline{t_{14} t_{24} t_{34} t_{24}}}}{1440} + \frac{1}{480} \overline{\overline{t_{14} t_{24} t_{14} t_{34}}} + \frac{1}{360} \overline{\overline{t_{14} t_{34} t_{24} t_{34}}} + \right. \\ \left. \frac{1}{720} \overline{t_{13} \overline{t_{13} t_{23} t_{23}}} - \frac{1}{720} \overline{t_{13} t_{13} \overline{t_{13} t_{23}}} + \frac{1}{720} \overline{t_{14} \overline{t_{14} t_{24} t_{24}}} + \frac{1}{360} \overline{t_{14} \overline{t_{14} t_{34} t_{24}}} + \right. \\ \left. \frac{1}{720} \overline{t_{14} \overline{t_{14} t_{34} t_{34}}} + \frac{1}{360} \overline{t_{14} \overline{t_{24} t_{34} t_{34}}} - \frac{1}{720} \overline{t_{14} t_{14} \overline{t_{14} t_{24}}} - \frac{1}{720} \overline{t_{14} t_{14} \overline{t_{14} t_{34}}} + \right. \\ \left. \frac{\overline{t_{14} t_{14} \overline{t_{24} t_{34}}}}{1440} - \frac{1}{576} \overline{\overline{t_{14} t_{24} t_{24} t_{34}}} + \frac{7 \overline{\overline{t_{24} t_{24} t_{34} t_{34}}}}{5760} - \frac{\overline{\overline{t_{24} t_{24} t_{24} t_{34}}}}{1440}, \dots\right] \end{aligned}$$

$$\text{rhs} = B_0^{\sigma[12,3,4]} ** B_0^{\sigma[1,2,4]} ** B_0^{\sigma[14,2,3]}$$

$$\begin{aligned} \text{DKS}\left[\frac{t_{23}}{2} + \frac{t_{24}}{2} + \frac{t_{34}}{2}, \frac{1}{12} \overline{t_{13} t_{23}} + \frac{1}{12} \overline{t_{14} t_{24}} + \frac{1}{12} \overline{t_{14} t_{34}} + \frac{1}{24} \overline{t_{24} t_{34}}, 0, \right. \\ \left. -\frac{\overline{\overline{t_{13} t_{23} t_{23} t_{23}}}}{5760} - \frac{\overline{\overline{t_{14} t_{24} t_{24} t_{24}}}}{5760} + \frac{\overline{\overline{t_{14} t_{34} t_{24} t_{24}}}}{2880} + \frac{\overline{\overline{t_{14} t_{34} t_{34} t_{24}}}}{2880} - \frac{\overline{\overline{t_{14} t_{34} t_{34} t_{34}}}}{5760} - \right. \\ \left. \frac{7 \overline{\overline{t_{24} t_{34} t_{34} t_{34}}}}{5760} + \frac{\overline{\overline{t_{14} t_{24} t_{34} t_{24}}}}{1440} + \frac{1}{480} \overline{\overline{t_{14} t_{24} t_{14} t_{34}}} + \frac{1}{360} \overline{\overline{t_{14} t_{34} t_{24} t_{34}}} + \right. \\ \left. \frac{1}{720} \overline{t_{13} \overline{t_{13} t_{23} t_{23}}} - \frac{1}{720} \overline{t_{13} t_{13} \overline{t_{13} t_{23}}} + \frac{1}{720} \overline{t_{14} \overline{t_{14} t_{24} t_{24}}} + \frac{1}{360} \overline{t_{14} \overline{t_{14} t_{34} t_{24}}} + \right. \\ \left. \frac{1}{720} \overline{t_{14} \overline{t_{14} t_{34} t_{34}}} + \frac{1}{360} \overline{t_{14} \overline{t_{24} t_{34} t_{34}}} - \frac{1}{720} \overline{t_{14} t_{14} \overline{t_{14} t_{24}}} - \frac{1}{720} \overline{t_{14} t_{14} \overline{t_{14} t_{34}}} + \right. \\ \left. \frac{\overline{t_{14} t_{14} \overline{t_{24} t_{34}}}}{1440} - \frac{1}{576} \overline{\overline{t_{14} t_{24} t_{24} t_{34}}} + \frac{7 \overline{\overline{t_{24} t_{24} t_{34} t_{34}}}}{5760} - \frac{\overline{\overline{t_{24} t_{24} t_{24} t_{34}}}}{1440}, \dots\right] \end{aligned}$$

$$(\text{lhs} = \text{rhs}) @ \{6\}$$

$$\text{BS}[7 \text{ True}, \dots]$$

$$\text{Bs}[2, 1] = \text{Bs}[3, 1] = 0; \text{Bs}[3, 2] = 1/2; B_1 = \text{DKS}[3, \text{Bs}];$$

$$\text{SeriesSolve}[B_1, B_1^{\sigma[1,2,3]} ** B_1^{\sigma[13,2,4]} ** B_1^{\sigma[1,3,4]} \equiv B_1^{\sigma[12,3,4]} ** B_1^{\sigma[1,2,4]} ** B_1^{\sigma[14,2,3]}];$$

**B<sub>1</sub>@{6}**

SeriesSolve::ArbitrarilySetting : In degree 3 arbitrarily setting {Bs[3, 1, 2, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 5 arbitrarily setting {Bs[3, 1, 1, 1, 2, 2] → 0}.

$$\begin{aligned}
 \text{DKS} \left[ \frac{t_{23}}{2}, \frac{1}{12} t_{13} t_{23}, 0, -\frac{t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{1}{720} t_{13} t_{13} t_{23} t_{23} - \frac{1}{720} t_{13} t_{13} t_{13} t_{23}, \right. \\
 -\frac{t_{13} t_{23} t_{23} t_{23} t_{23}}{7680} + \frac{t_{13} t_{23} t_{13} t_{23} t_{23}}{8640} + \frac{t_{13} t_{13} t_{23} t_{23} t_{23}}{3840}, \\
 \frac{t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{645120} - \frac{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{145152} - \frac{71 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{2903040} - \\
 \frac{23 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{20160} + \frac{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{22680} + \\
 \left. \frac{13 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{161280} - \frac{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{15120} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{30240}, \dots \right]
 \end{aligned}$$

**B<sub>0</sub>@{6}**

$$\begin{aligned}
 \text{DKS} \left[ \frac{t_{23}}{2}, \frac{1}{12} t_{13} t_{23}, 0, -\frac{t_{13} t_{23} t_{23} t_{23}}{5760} + \frac{1}{720} t_{13} t_{13} t_{23} t_{23} - \frac{1}{720} t_{13} t_{13} t_{13} t_{23}, \right. \\
 -\frac{t_{13} t_{23} t_{23} t_{23} t_{23}}{7680} + \frac{t_{13} t_{23} t_{13} t_{23} t_{23}}{8640} + \frac{t_{13} t_{13} t_{23} t_{23} t_{23}}{3840}, \\
 \frac{t_{13} t_{23} t_{23} t_{23} t_{23} t_{23}}{645120} - \frac{t_{13} t_{13} t_{23} t_{23} t_{13} t_{23}}{145152} - \frac{71 t_{13} t_{23} t_{13} t_{23} t_{23} t_{23}}{2903040} - \\
 \frac{23 t_{13} t_{13} t_{23} t_{23} t_{23} t_{23}}{483840} + \frac{t_{13} t_{13} t_{13} t_{23} t_{13} t_{23}}{20160} + \frac{t_{13} t_{13} t_{23} t_{13} t_{23} t_{23}}{22680} + \\
 \left. \frac{13 t_{13} t_{13} t_{13} t_{23} t_{23} t_{23}}{161280} - \frac{t_{13} t_{13} t_{13} t_{13} t_{23} t_{23}}{15120} + \frac{t_{13} t_{13} t_{13} t_{13} t_{13} t_{23}}{30240}, \dots \right]
 \end{aligned}$$

**(B<sub>0</sub> ≡ B<sub>1</sub>)@{7}**

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Φs[3, 1, 1, 1, 1, 1, 2] → 0}.

SeriesSolve::ArbitrarilySetting : In degree 7 arbitrarily setting {Bs[3, 1, 1, 1, 1, 1, 2, 2] → 0}.

BS[8 True, ...]

**$(B_0 \equiv B_1) @ \{8\}$**

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting  $\{\Phi_s[3, 1, 1, 1, 1, 1, 2, 1, 2] \rightarrow 0\}$ .

SeriesSolve::ArbitrarilySetting : In degree 8 arbitrarily setting  $\{Bs[3, 1, 1, 1, 1, 2, 1, 2, 2] \rightarrow 0\}$ .

BS[8 True,

$$\begin{aligned}
& \frac{23 \overbrace{111}^{\overbrace{12}^{\overbrace{122}}}}{12441600} - \frac{23 \overbrace{11}^{\overbrace{12}^{\overbrace{1222}}}^{\overbrace{22}}}{16588800} + \frac{23 \overbrace{111}^{\overbrace{122}^{\overbrace{12}}}}{6220800} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{122}}}^{\overbrace{22}}}}{5529600} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{22}}}^{\overbrace{22}}}}{16588800} + \\
& \frac{23 \overbrace{1}^{\overbrace{11}^{\overbrace{12}^{\overbrace{1}^{\overbrace{122}}}^{\overbrace{22}}}}}{24883200} - \frac{23 \overbrace{11}^{\overbrace{12}^{\overbrace{122}^{\overbrace{122}}}^{\overbrace{22}}}}{5529600} - \frac{23 \overbrace{11}^{\overbrace{12}^{\overbrace{122}^{\overbrace{2}^{\overbrace{12}}}^{\overbrace{22}}}}}{5529600} - \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{122}^{\overbrace{2}^{\overbrace{22}}}}}}}{16588800} - \\
& \frac{23 \overbrace{1}^{\overbrace{11}^{\overbrace{12}^{\overbrace{12}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}{4147200} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{122}^{\overbrace{2}^{\overbrace{22}}}^{\overbrace{22}}}}}}{4147200} - \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{22}^{\overbrace{2}^{\overbrace{22}}}^{\overbrace{22}}}}}}{12441600} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{1}^{\overbrace{122}^{\overbrace{22}}}^{\overbrace{22}}}}}}{16588800} - \\
& \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{6220800} + \frac{23 \overbrace{11}^{\overbrace{12}^{\overbrace{122}^{\overbrace{1}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{12441600} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{122}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{5529600} - \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{2}^{\overbrace{122}}}^{\overbrace{22}}}}}}{3317760} - \\
& \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{2}^{\overbrace{2}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{16588800} - \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{122}^{\overbrace{2}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{4976640} - \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{12}^{\overbrace{122}^{\overbrace{122}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{5529600} + \frac{23 \overbrace{1}^{\overbrace{12}^{\overbrace{122}^{\overbrace{12}^{\overbrace{12}^{\overbrace{22}}}^{\overbrace{22}}}}}}}{2764800} = 0, \dots ]
\end{aligned}$$