Arrow Diagrams and gl(N) - Testing

Joint project of Louis Leung and Dror Bar - Natan.

Testing

```
Place[{ar}, {1, 3}]
```

```
{Diag[ar[1, 3]], Diag[ar[3, 1]]}
```

```
Place[{ar, ar}, {1, 2, 3, 4}]
```

```
{Diag[ar[1, 2], ar[3, 4]], Diag[ar[1, 2], ar[4, 3]], Diag[ar[2, 1], ar[3, 4]],
 Diag[ar[2, 1], ar[4, 3]], Diag[ar[1, 3], ar[2, 4]], Diag[ar[1, 3], ar[4, 2]],
 Diag[ar[2, 4], ar[3, 1]], Diag[ar[3, 1], ar[4, 2]], Diag[ar[1, 4], ar[2, 3]],
 Diag[ar[1, 4], ar[3, 2]], Diag[ar[2, 3], ar[4, 1]], Diag[ar[3, 2], ar[4, 1]]}
```

```
{Place[{ar, ar, ar, ar}, {1, 2, 3, 4, 5, 6, 7, 8}] // Length, 8!/4!}
```

```
{1680, 1680}
```

```
Place[{R6T}, {1, 2, 3}]
```

```
{Diag[R6T[1, 2, 3]], Diag[R6T[1, 3, 2]], Diag[R6T[2, 1, 3]],
 Diag[R6T[2, 3, 1]], Diag[R6T[3, 1, 2]], Diag[R6T[3, 2, 1]]}
```

```
Place[{ar, R6T}, {1, 2, 3, 4, 5}]
```

```
{Diag[ar[1, 2], R6T[3, 4, 5]], Diag[ar[1, 2], R6T[3, 5, 4]], Diag[ar[1, 2], R6T[4, 3, 5]],
 Diag[ar[1, 2], R6T[4, 5, 3]], Diag[ar[1, 2], R6T[5, 3, 4]], Diag[ar[1, 2], R6T[5, 4, 3]],
 Diag[ar[2, 1], R6T[3, 4, 5]], Diag[ar[2, 1], R6T[3, 5, 4]], Diag[ar[2, 1], R6T[4, 3, 5]],
 Diag[ar[2, 1], R6T[4, 5, 3]], Diag[ar[2, 1], R6T[5, 3, 4]], Diag[ar[2, 1], R6T[5, 4, 3]],
 Diag[ar[1, 3], R6T[2, 4, 5]], Diag[ar[1, 3], R6T[2, 5, 4]], Diag[ar[1, 3], R6T[4, 2, 5]],
 Diag[ar[1, 3], R6T[4, 5, 2]], Diag[ar[1, 3], R6T[5, 2, 4]], Diag[ar[1, 3], R6T[5, 4, 2]],
 Diag[ar[3, 1], R6T[2, 4, 5]], Diag[ar[3, 1], R6T[2, 5, 4]], Diag[ar[3, 1], R6T[4, 2, 5]],
 Diag[ar[3, 1], R6T[4, 5, 2]], Diag[ar[3, 1], R6T[5, 2, 4]], Diag[ar[3, 1], R6T[5, 4, 2]],
 Diag[ar[1, 4], R6T[2, 3, 5]], Diag[ar[1, 4], R6T[2, 5, 3]], Diag[ar[1, 4], R6T[3, 2, 5]],
 Diag[ar[1, 4], R6T[3, 5, 2]], Diag[ar[1, 4], R6T[5, 2, 3]], Diag[ar[1, 4], R6T[5, 3, 2]],
 Diag[ar[4, 1], R6T[2, 3, 5]], Diag[ar[4, 1], R6T[2, 5, 3]], Diag[ar[4, 1], R6T[3, 2, 5]],
 Diag[ar[4, 1], R6T[3, 5, 2]], Diag[ar[4, 1], R6T[5, 2, 3]], Diag[ar[4, 1], R6T[5, 3, 2]],
 Diag[ar[1, 5], R6T[2, 3, 4]], Diag[ar[1, 5], R6T[2, 4, 3]], Diag[ar[1, 5], R6T[3, 2, 4]],
 Diag[ar[1, 5], R6T[3, 4, 2]], Diag[ar[1, 5], R6T[4, 2, 3]], Diag[ar[1, 5], R6T[4, 3, 2]],
 Diag[ar[5, 1], R6T[2, 3, 4]], Diag[ar[5, 1], R6T[2, 4, 3]], Diag[ar[5, 1], R6T[3, 2, 4]],
 Diag[ar[5, 1], R6T[3, 4, 2]], Diag[ar[5, 1], R6T[4, 2, 3]], Diag[ar[5, 1], R6T[4, 3, 2]]}
```

```
Place[{R6T, ar}, {1, 2, 3, 4, 5}]
```

```
{Diag[ar[4, 5], R6T[1, 2, 3]], Diag[ar[5, 4], R6T[1, 2, 3]], Diag[ar[4, 5], R6T[1, 3, 2]],
 Diag[ar[5, 4], R6T[1, 3, 2]], Diag[ar[4, 5], R6T[2, 1, 3]], Diag[ar[5, 4], R6T[2, 1, 3]],
 Diag[ar[4, 5], R6T[2, 3, 1]], Diag[ar[5, 4], R6T[2, 3, 1]], Diag[ar[4, 5], R6T[3, 1, 2]],
 Diag[ar[5, 4], R6T[3, 1, 2]], Diag[ar[4, 5], R6T[3, 2, 1]], Diag[ar[5, 4], R6T[3, 2, 1]],
 Diag[ar[3, 5], R6T[1, 2, 4]], Diag[ar[5, 3], R6T[1, 2, 4]], Diag[ar[3, 5], R6T[1, 4, 2]],
 Diag[ar[5, 3], R6T[1, 4, 2]], Diag[ar[3, 5], R6T[2, 1, 4]], Diag[ar[5, 3], R6T[2, 1, 4]],
 Diag[ar[3, 5], R6T[2, 4, 1]], Diag[ar[5, 3], R6T[2, 4, 1]], Diag[ar[3, 5], R6T[4, 1, 2]],
 Diag[ar[5, 3], R6T[4, 1, 2]], Diag[ar[3, 5], R6T[4, 2, 1]], Diag[ar[5, 3], R6T[4, 2, 1]],
 Diag[ar[2, 5], R6T[1, 3, 4]], Diag[ar[5, 2], R6T[1, 3, 4]], Diag[ar[2, 5], R6T[1, 4, 3]],
 Diag[ar[5, 2], R6T[1, 4, 3]], Diag[ar[2, 5], R6T[3, 1, 4]], Diag[ar[5, 2], R6T[3, 1, 4]],
 Diag[ar[2, 5], R6T[3, 4, 1]], Diag[ar[5, 2], R6T[3, 4, 1]], Diag[ar[2, 5], R6T[4, 1, 3]],
 Diag[ar[5, 2], R6T[4, 1, 3]], Diag[ar[2, 5], R6T[4, 3, 1]], Diag[ar[5, 2], R6T[4, 3, 1]],
 Diag[ar[3, 4], R6T[1, 2, 5]], Diag[ar[4, 3], R6T[1, 2, 5]], Diag[ar[3, 4], R6T[1, 5, 2]],
 Diag[ar[4, 3], R6T[1, 5, 2]], Diag[ar[3, 4], R6T[2, 1, 5]], Diag[ar[4, 3], R6T[2, 1, 5]],
 Diag[ar[3, 4], R6T[2, 5, 1]], Diag[ar[4, 3], R6T[2, 5, 1]], Diag[ar[3, 4], R6T[5, 1, 2]],
 Diag[ar[4, 3], R6T[5, 1, 2]], Diag[ar[3, 4], R6T[5, 2, 1]], Diag[ar[4, 3], R6T[5, 2, 1]],
 Diag[ar[2, 4], R6T[1, 3, 5]], Diag[ar[4, 2], R6T[1, 3, 5]], Diag[ar[2, 4], R6T[1, 5, 3]],
 Diag[ar[4, 2], R6T[1, 5, 3]], Diag[ar[2, 4], R6T[3, 1, 5]], Diag[ar[4, 2], R6T[3, 1, 5]],
 Diag[ar[2, 4], R6T[3, 5, 1]], Diag[ar[4, 2], R6T[3, 5, 1]], Diag[ar[2, 4], R6T[5, 1, 3]],
 Diag[ar[4, 2], R6T[5, 1, 3]], Diag[ar[2, 4], R6T[5, 3, 1]], Diag[ar[4, 2], R6T[5, 3, 1]],
 Diag[ar[2, 3], R6T[1, 4, 5]], Diag[ar[3, 2], R6T[1, 4, 5]], Diag[ar[2, 3], R6T[1, 5, 4]],
 Diag[ar[3, 2], R6T[1, 5, 4]], Diag[ar[2, 3], R6T[4, 1, 5]], Diag[ar[3, 2], R6T[4, 1, 5]],
 Diag[ar[2, 3], R6T[4, 5, 1]], Diag[ar[3, 2], R6T[4, 5, 1]], Diag[ar[2, 3], R6T[5, 1, 4]],
 Diag[ar[3, 2], R6T[5, 1, 4]], Diag[ar[2, 3], R6T[5, 4, 1]], Diag[ar[3, 2], R6T[5, 4, 1]]}
```

```
Place[{R6T, ar, ar}, {1, 2, 3, 4, 5, 6, 7}]
```

A very large output was generated. Here is a sample of it:

```
{Diag[ar[4, 5], ar[6, 7], R6T[1, 2, 3]], Diag[ar[4, 5], ar[7, 6], R6T[1, 2, 3]],
 Diag[ar[5, 4], ar[6, 7], R6T[1, 2, 3]], Diag[ar[5, 4], ar[7, 6], R6T[1, 2, 3]], <<1072>>,
 Diag[ar[2, 5], ar[3, 4], R6T[7, 6, 1]], Diag[ar[2, 5], ar[4, 3], R6T[7, 6, 1]],
 Diag[ar[3, 4], ar[5, 2], R6T[7, 6, 1]], Diag[ar[4, 3], ar[5, 2], R6T[7, 6, 1]]}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
R[Diag[R6T[1, 2, 3], ar[4, 5], ar[6, 7]]]
```

```
Diag[ar[1, 2], ar[3, 4], ar[5, 6], ar[7, 8]] +
 Diag[ar[1, 3], ar[2, 4], ar[5, 6], ar[7, 8]] - 2 Diag[ar[1, 4], ar[2, 3], ar[5, 6], ar[7, 8]]
```

```
diag = Diag[ar[1, 2], ar[1.5`, 3], ar[4, 5], ar[6, 7]]
```

```
Diag[ar[1, 2], ar[1.5, 3], ar[4, 5], ar[6, 7]]
```

```
indices = Union @@ (List @@ diag /. ar → List)
```

```
{1, 1.5, 2, 3, 4, 5, 6, 7}
```

```
Thread[indices → Range[Length[indices]]]
```

```
{1 → 1, 1.5 → 2, 2 → 3, 3 → 4, 4 → 5, 5 → 6, 6 → 7, 7 → 8}
```

```
NormalizeDiag[Diag[ar[1, 2], ar[1.5`, 3], ar[4, 5], ar[6, 7]]]
```

```
Diag[ar[1, 3], ar[2, 4], ar[5, 6], ar[7, 8]]
```

```
? Append
```

Append[*expr*, *elem*] gives *expr* with *elem* appended. ≫

```
k = 7;
Table[ar, {k}] ~ Append ~ R6T
```

```
{ar, ar, ar, ar, ar, ar, ar, R6T}
```

```
Permutations[%]
```

```
{{ar, ar, ar, ar, ar, ar, ar, R6T}, {ar, ar, ar, ar, ar, ar, R6T, ar},
 {ar, ar, ar, ar, ar, R6T, ar, ar}, {ar, ar, ar, ar, R6T, ar, ar, ar},
 {ar, ar, ar, R6T, ar, ar, ar, ar}, {ar, ar, R6T, ar, ar, ar, ar, ar},
 {ar, R6T, ar, ar, ar, ar, ar, ar}, {R6T, ar, ar, ar, ar, ar, ar, ar}}
```

```
diags = Diagrams[4 ar]
```

A very large output was generated. Here is a sample of it:

```
{Diag[ar[1, 2], ar[3, 4], ar[5, 6], ar[7, 8]], Diag[ar[1, 2], ar[3, 4], ar[5, 6], ar[8, 7]],
 Diag[ar[1, 2], ar[3, 4], ar[6, 5], ar[7, 8]], Diag[ar[1, 2], ar[3, 4], ar[6, 5], ar[8, 7]],
 ≪1673≫, Diag[ar[3, 6], ar[5, 4], ar[7, 2], ar[8, 1]],
 Diag[ar[4, 5], ar[6, 3], ar[7, 2], ar[8, 1]], Diag[ar[5, 4], ar[6, 3], ar[7, 2], ar[8, 1]]}
```

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```
rels = R /@ Diagrams[R6T + 2 ar]
```

A very large output was generated. Here is a sample of it:

```
{Diag[ar[1, 2], ar[3, 4], ar[5, 6], ar[7, 8]] + Diag[ar[1, 2], ar[3, 4], ar[5, 7], ar[6, 8]] -
   2 Diag[ar[1, 2], ar[3, 4], ar[5, 8], ar[6, 7]], ≪2518≫,
 -Diag[ar[4, 3], ar[5, 2], ar[6, 1], ar[8, 7]] + Diag[ar[4, 3], ar[5, 2], ar[7, 6], ar[8, 1]] -
   Diag[ar[5, 4], ar[6, 3], ar[7, 1], ar[8, 2]] + Diag[ar[5, 4], ar[6, 3], ar[7, 2], ar[8, 1]]}
```

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```
DimAArrow[1]
```

```
2
```

```
DimAArrow[2]
```

```
7
```

```
DimAArrow[3]
```

```
27
```

```
DimAArrow[4] // Timing
```

```
{36.661, 139}
```

```
a > b && b > a // FullForm
```

```
And[Greater[a, b], Greater[b, a]]
```

```
Expand[((a < b) + 1 / 2 (e < f)) * (b < c) * (a > c)]
```

$$(a > c)\ (a < b)\ (b < c) + \frac{1}{2}\ (a > c)\ (b < c)\ (e < f)$$

```
diag = Diag[ar[5, 4], ar[6, 3], ar[7, 1], ar[8, 2]]
```

```
Diag[ar[5, 4], ar[6, 3], ar[7, 1], ar[8, 2]]
```

```
p = Times @@ (diag /. ar[i_, j_] :> eq[i, j - 1] eq[i - 1, j] (lt[i - 1, i] + heq[i - 1, i]));
While[! FreeQ[p, eq],
 p = (p
     /. Cases[p, eq[i_, j_] :> (i → j), Infinity, 1]
    /. {
     eq[i_, i_] → 1,
     lt[i_, i_] → 0
    }
  )
]
```

```
p
```

$$heq[2, 2]^2\ (heq[3, 2] + lt[3, 2])\ (heq[4, 3] + lt[4, 3])$$

```
ineqs = Expand[p /. heq[i_, j_] :> heq[i, j] / 2] /. {e_heq^_ :> e}
```

$$\frac{1}{16} heq[2, 2]\ heq[3, 2]\ heq[4, 3] + \frac{1}{8} heq[2, 2]\ heq[4, 3]\ lt[3, 2] +$$
$$\frac{1}{8} heq[2, 2]\ heq[3, 2]\ lt[4, 3] + \frac{1}{4} heq[2, 2]\ lt[3, 2]\ lt[4, 3]$$

```
l = Length[indices = Union @@ Cases[ineqs, (lt | heq)[i_, j_] :> {i, j}, Infinity]]
```

```
3
```

```
indices
```

```
{2, 3, 4}
```

```
ineqs = ineqs /. {
   e_heq :> (e /. Thread[indices → Range[l]]),
   e_lt :> (e /. Thread[indices → Range[l]])
  }
```

$$\frac{1}{16} heq[1, 1]\ heq[2, 1]\ heq[3, 2] + \frac{1}{8} heq[1, 1]\ heq[3, 2]\ lt[2, 1] +$$
$$\frac{1}{8} heq[1, 1]\ heq[2, 1]\ lt[3, 2] + \frac{1}{4} heq[1, 1]\ lt[2, 1]\ lt[3, 2]$$

```
ineq = heq[1, 1] heq[2, 1] lt[3, 2]
```

```
heq[1, 1] heq[2, 1] lt[3, 2]
```

```
While[! FreeQ[ineq, heq],
 ineq = (ineq
     /. Cases[ineq, heq[i_, j_] :> (i → j), Infinity, 1]
    /. {
     heq[i_, i_] → 1,
     lt[i_, i_] → 0
    }
  )
]
```

```
ineq
```

```
lt[3, 1]
```

```
OrderTypes[1] = {{1}};
OrderTypes[l_List] := Module[{nl, snl},
   (
      snl = Union[nl = Append[l, #]];
      nl /. Thread[snl → Range[Length[snl]]]
     ) & /@ Range[1 / 2, 1 / 2 + Max[l], 1 / 2]
  ];
OrderTypes[n_Integer] := OrderTypes[n] = Join @@ (OrderTypes /@ OrderTypes[n - 1])
```

```
OrderTypes[{2, 1, 3}]
```

```
{{3, 2, 4, 1}, {2, 1, 3, 1}, {3, 1, 4, 2}, {2, 1, 3, 2}, {2, 1, 4, 3}, {2, 1, 3, 3}, {2, 1, 3, 4}}
```

```
OrderTypes[2]
```

```
{{2, 1}, {1, 1}, {1, 2}}
```

```
OrderTypes[3]
```

```
{{3, 2, 1}, {2, 1, 1}, {3, 1, 2}, {2, 1, 2}, {2, 1, 3}, {2, 2, 1},
 {1, 1, 1}, {1, 1, 2}, {2, 3, 1}, {1, 2, 1}, {1, 3, 2}, {1, 2, 2}, {1, 2, 3}}
```

```
OrderTypes[4]
```

```
{{4, 3, 2, 1}, {3, 2, 1, 1}, {4, 3, 1, 2}, {3, 2, 1, 2}, {4, 2, 1, 3}, {3, 2, 1, 3}, {3, 2, 1, 4},
 {3, 2, 2, 1}, {2, 1, 1, 1}, {3, 1, 1, 2}, {2, 1, 1, 2}, {2, 1, 1, 3}, {4, 2, 3, 1}, {3, 1, 2, 1},
 {4, 1, 3, 2}, {3, 1, 2, 2}, {4, 1, 2, 3}, {3, 1, 2, 3}, {3, 1, 2, 4}, {3, 2, 3, 1}, {2, 1, 2, 1},
 {3, 1, 3, 2}, {2, 1, 2, 2}, {2, 1, 2, 3}, {3, 2, 4, 1}, {2, 1, 3, 1}, {3, 1, 4, 2},
 {2, 1, 3, 2}, {2, 1, 4, 3}, {2, 1, 3, 3}, {2, 1, 3, 4}, {3, 3, 2, 1}, {2, 2, 1, 1},
 {3, 3, 1, 2}, {2, 2, 1, 2}, {2, 2, 1, 3}, {2, 2, 2, 1}, {1, 1, 1, 1}, {1, 1, 1, 2},
 {2, 2, 3, 1}, {1, 1, 2, 1}, {1, 1, 3, 2}, {1, 1, 2, 2}, {1, 1, 2, 3}, {3, 4, 2, 1},
 {2, 3, 1, 1}, {3, 4, 1, 2}, {2, 3, 1, 2}, {2, 4, 1, 3}, {2, 3, 1, 3}, {2, 3, 1, 4},
 {2, 3, 2, 1}, {1, 2, 1, 1}, {1, 3, 1, 2}, {1, 2, 1, 2}, {1, 2, 1, 3}, {2, 4, 3, 1},
 {1, 3, 2, 1}, {1, 4, 3, 2}, {1, 3, 2, 2}, {1, 4, 2, 3}, {1, 3, 2, 3}, {1, 3, 2, 4},
 {2, 3, 3, 1}, {1, 2, 2, 1}, {1, 3, 3, 2}, {1, 2, 2, 2}, {1, 2, 2, 3}, {2, 3, 4, 1},
 {1, 2, 3, 1}, {1, 3, 4, 2}, {1, 2, 3, 2}, {1, 2, 4, 3}, {1, 2, 3, 3}, {1, 2, 3, 4}}
```

```
OrderTypes[6]
```

A very large output was generated. Here is a sample of it:

```
{{6, 5, 4, 3, 2, 1}, {5, 4, 3, 2, 1, 1}, {6, 5, 4, 3, 1, 2},
 {5, 4, 3, 2, 1, 2}, {6, 5, 4, 2, 1, 3}, {5, 4, 3, 2, 1, 3},
 {6, 5, 3, 2, 1, 4}, {5, 4, 3, 2, 1, 4}, {6, 4, 3, 2, 1, 5}, {5, 4, 3, 2, 1, 5},
 {5, 4, 3, 2, 1, 6}, <<4662>>, {1, 2, 3, 4, 5, 1}, {1, 3, 4, 5, 6, 2},
 {1, 2, 3, 4, 5, 2}, {1, 2, 4, 5, 6, 3}, {1, 2, 3, 4, 5, 3}, {1, 2, 3, 5, 6, 4},
 {1, 2, 3, 4, 5, 4}, {1, 2, 3, 4, 6, 5}, {1, 2, 3, 4, 5, 5}, {1, 2, 3, 4, 5, 6}}
```

Show Less | Show More | Show Full Output | Set Size Limit...

```
l
```

3

```
ineqs
```

$$\frac{1}{16}\,\text{heq}[1, 1]\,\text{heq}[2, 1]\,\text{heq}[3, 2] + \frac{1}{8}\,\text{heq}[1, 1]\,\text{heq}[3, 2]\,\text{lt}[2, 1] + \frac{1}{8}\,\text{heq}[1, 1]\,\text{heq}[2, 1]\,\text{lt}[3, 2] + \frac{1}{4}\,\text{heq}[1, 1]\,\text{lt}[2, 1]\,\text{lt}[3, 2]$$

```
OrderTypes[l]
```

```
{{3, 2, 1}, {2, 1, 1}, {3, 1, 2}, {2, 1, 2}, {2, 1, 3}, {2, 2, 1},
 {1, 1, 1}, {1, 1, 2}, {2, 3, 1}, {1, 2, 1}, {1, 3, 2}, {1, 2, 2}, {1, 2, 3}}
```

```
((ineqs /. {
       heq[i_, j_] :> If[#[[i]] == #[[j]], 1, 0],
       lt[i_, j_] :> If[#[[i]] < #[[j]], 1, 0]
      }
    ) * n^Max[#]
  ) & /@ OrderTypes[l]
```

$$\left\{\frac{n^3}{4}, \frac{n^2}{8}, 0, 0, 0, \frac{n^2}{8}, \frac{n}{16}, 0, 0, 0, 0, 0, 0\right\}$$

```
n
```

n

```
Expand[
 (R /@ Diagrams[R6T + 2 ar]) /. diag_Diag :> Wgl[diag]
]
```

```
Position[%, 7 n / 64]
```

```
{{15, 1}, {63}}
```

```
Diagrams[R6T + 1 ar][[63]]
```

```
Diag[ar[3, 5], R6T[1, 4, 2]]
```

```
Diag[ar[3, 5], R6T[1, 4, 2]] // R
```

```
Diag[ar[1, 2], ar[4, 6], ar[5, 3]] - Diag[ar[1, 3], ar[2, 5], ar[4, 6]] -
 Diag[ar[1, 3], ar[4, 6], ar[5, 2]] + Diag[ar[1, 4], ar[3, 6], ar[5, 2]] +
 Diag[ar[1, 5], ar[2, 3], ar[4, 6]] - Diag[ar[1, 5], ar[3, 6], ar[4, 2]]
```

```
Wgl /@ Cases[Diag[ar[3, 5], R6T[1, 4, 2]] // R, _Diag, Infinity]
```

$$\left\{\frac{n^2}{8}, -\frac{n}{8}+\frac{n^2}{4}, -\frac{15\,n}{64}+\frac{n^2}{4}, -\frac{3\,n}{8}+\frac{n^2}{2}, \frac{n^2}{8}, -\frac{n}{8}+\frac{n^2}{4}\right\}$$

```
Cases[Diag[ar[3, 5], R6T[1, 4, 2]] // R, _Diag, Infinity]
```

```
{Diag[ar[1, 2], ar[4, 6], ar[5, 3]], Diag[ar[1, 3], ar[2, 5], ar[4, 6]],
 Diag[ar[1, 3], ar[4, 6], ar[5, 2]], Diag[ar[1, 4], ar[3, 6], ar[5, 2]],
 Diag[ar[1, 5], ar[2, 3], ar[4, 6]], Diag[ar[1, 5], ar[3, 6], ar[4, 2]]}
```

```
(R /@ Diagrams[R6T + 0 ar])[[1]]
```

```
Diag[ar[1, 2], ar[3, 4]] + Diag[ar[1, 3], ar[2, 4]] - 2 Diag[ar[1, 4], ar[2, 3]]
```

```
Wgl /@ {Diag[ar[1, 2], ar[3, 4]], Diag[ar[1, 3], ar[2, 4]], Diag[ar[1, 4], ar[2, 3]]}
```

$$\left\{-\frac{n}{12}+\frac{n^3}{3}, \frac{n}{4}, \frac{n}{12}+\frac{n^3}{6}\right\}$$

```
Wgl /@ Diagrams[1 ar]
```

$$\left\{\frac{n^2}{2}, \frac{n^2}{2}\right\}$$

```
WglE[diag_Diag] := Wgl[Join[Diag[ar[-1, 0]], diag]]
```

```
WglEE[diag_Diag] := Wgl[Join[Diag[ar[-3, -2], ar[-1, 0]], diag]]
```

```
WglE /@ Diagrams[1 ar]
```

$$\left\{-\frac{n}{12}+\frac{n^3}{3}, \frac{n}{12}+\frac{n^3}{6}\right\}$$

```
k
```

k

```
Wgl /@ Diagrams[3 ar]
```

```
Expand[
 (R /@ Diagrams[R6T + 2 ar]) /. diag_Diag :> Wgl[diag]
]
```

```
WglEE /@ Diagrams[2 ar]
```

$$\left\{\frac{7\,n}{240}-\frac{n^3}{6}+\frac{n^5}{5}, -\frac{7\,n}{240}+\frac{n^3}{24}+\frac{n^5}{20}, -\frac{7\,n}{240}+\frac{n^3}{24}+\frac{n^5}{20}, \frac{7\,n}{240}+\frac{n^5}{30}, -\frac{n}{48}+\frac{n^3}{12}, -\frac{n}{48}+\frac{n^3}{12},\right.$$
$$\left.-\frac{n}{48}+\frac{n^3}{12}, -\frac{n}{48}+\frac{n^3}{12}, \frac{n}{240}-\frac{n^3}{24}+\frac{n^5}{10}, -\frac{n}{240}-\frac{n^3}{12}+\frac{3\,n^5}{20}, -\frac{n}{240}+\frac{n^5}{15}, \frac{n}{240}+\frac{n^3}{24}+\frac{n^5}{60}\right\}$$

```
mat = {
    Coefficient[Wgl[#], n],
    Coefficient[Wgl[#], n^3],
    Coefficient[WglE[#], n^2],
    Coefficient[WglE[#], n^4],
    Coefficient[WglEE[#], n],
    Coefficient[WglEE[#], n^3],
    Coefficient[WglEE[#], n^5]
   } & /@ Diagrams[2 ar]
```

$$\left\{\left\{-\frac{1}{12}, \frac{1}{3}, -\frac{1}{8}, \frac{1}{4}, \frac{7}{240}, -\frac{1}{6}, \frac{1}{5}\right\}, \left\{\frac{1}{12}, \frac{1}{6}, \frac{1}{24}, \frac{1}{12}, -\frac{7}{240}, \frac{1}{24}, \frac{1}{20}\right\},\right.$$
$$\left\{\frac{1}{12}, \frac{1}{6}, \frac{1}{24}, \frac{1}{12}, -\frac{7}{240}, \frac{1}{24}, \frac{1}{20}\right\}, \left\{-\frac{1}{12}, \frac{1}{3}, \frac{1}{24}, \frac{1}{12}, \frac{7}{240}, 0, \frac{1}{30}\right\}, \left\{\frac{1}{4}, 0, \frac{1}{8}, 0, -\frac{1}{48}, \frac{1}{12}, 0\right\},$$
$$\left\{\frac{1}{4}, 0, \frac{1}{8}, 0, -\frac{1}{48}, \frac{1}{12}, 0\right\}, \left\{\frac{1}{4}, 0, \frac{1}{8}, 0, -\frac{1}{48}, \frac{1}{12}, 0\right\}, \left\{\frac{1}{4}, 0, \frac{1}{8}, 0, -\frac{1}{48}, \frac{1}{12}, 0\right\},$$
$$\left\{\frac{1}{12}, \frac{1}{6}, 0, \frac{1}{8}, \frac{1}{240}, -\frac{1}{24}, \frac{1}{10}\right\}, \left\{-\frac{1}{12}, \frac{1}{3}, -\frac{1}{12}, \frac{5}{24}, -\frac{1}{240}, -\frac{1}{12}, \frac{3}{20}\right\},$$
$$\left.\left\{-\frac{1}{12}, \frac{1}{3}, 0, \frac{1}{8}, -\frac{1}{240}, 0, \frac{1}{15}\right\}, \left\{\frac{1}{12}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \frac{1}{240}, \frac{1}{24}, \frac{1}{60}\right\}\right\}$$

```
MatrixRank[mat]
```

```
4
```

```
diag = Diag[ar[5, 4], ar[6, 3], ar[7, 1], ar[8, 2]]
```

```
Diag[ar[5, 4], ar[6, 3], ar[7, 1], ar[8, 2]]
```

```
p = diag /. ar[i_, j_] :> Sequence[eq[i, j - 1], eq[i - 1, j], ltheq[i - 1, i]]
```

```
Diag[eq[4, 4], eq[5, 3], eq[5, 3], eq[6, 1], eq[6, 2], eq[7, 0],
 eq[7, 2], eq[8, 1], ltheq[4, 5], ltheq[5, 6], ltheq[6, 7], ltheq[7, 8]]
```

```
While[! FreeQ[p, eq],
 p = (p
     /. Cases[p, eq[i_, j_] :> (i → j), Infinity, 1]
    /. {
     eq[i_, i_] → Sequence[]
    }
  )
]; p
```

```
Diag[ltheq[2, 2], ltheq[2, 2], ltheq[3, 2], ltheq[4, 3]]
```

```
diag = Diagrams[2 ar][[4]]
```

```
Diag[ar[2, 1], ar[4, 3]]
```

```
Wgl[diag]
```

$$\frac{1}{4} - k + k^2$$

```
Sum[
 Wgl[diag],
 {k, 1, n}
]
```

$$\frac{1}{12}\left(-n+4\,n^3\right)$$

```
WglOld[diag]
```

$$-\frac{n}{12}+\frac{n^3}{3}$$

```
p = Append[
  diag /. ar[i_, j_] :> Sequence[eq[i, j - 1], eq[i - 1, j], ltheq[i - 1, i]],
  s[0]
 ]
```

```
Diag[eq[1, 1], eq[2, 0], eq[3, 3], eq[4, 2], ltheq[1, 2], ltheq[3, 4], s[0]]
```

```
While[! FreeQ[p, eq],
 p = (p
     /. Cases[p, eq[i_, j_] :> (i → j), Infinity, 1]
    /. {
     eq[i_, i_] → Sequence[]
    }
  )
]; p
```

```
Diag[ltheq[1, 0], ltheq[3, 0], s[0]]
```

```
Wgl[Diag[ar[2, 1], ar[4, 3]]]
```

$$\frac{1}{4}-k+k^2$$

```
(k - 1) ^ 2 + Binomial[k - 1, 1] + 1 / 4 // Expand
```

$$\frac{1}{4}-k+k^2$$

```
l = Length[indices = Union @@ Cases[{p}, ltheq[i_, j_] :> {i, j}, Infinity]];
ineqs = p /. Thread[indices → Range[l]]
```

```
Diag[ltheq[2, 1], ltheq[3, 1], s[1]]
```

```
{i0} = Cases[ineqs, s[i_] :> i, {1}, 1]
```

```
{1}
```

```
ineqs = DeleteCases[ineqs, _s]
```

```
Diag[ltheq[2, 1], ltheq[3, 1]]
```

```
l
```

```
3
```

```
OrderTypes[3]
```

```
{{3, 2, 1}, {2, 1, 1}, {3, 1, 2}, {2, 1, 2}, {2, 1, 3}, {2, 2, 1},
 {1, 1, 1}, {1, 1, 2}, {2, 3, 1}, {1, 2, 1}, {1, 3, 2}, {1, 2, 2}, {1, 2, 3}}
```

```
v = Wgl /@ Diagrams[2 ar]
```

$$\left\{\frac{1}{4} - k + k^2 + n - 2\,k\,n + n^2,\ -\frac{1}{4} + k - k^2 - \frac{n}{2} + k\,n,\ -\frac{1}{4} + k - k^2 - \frac{n}{2} + k\,n,\ \frac{1}{4} - k + k^2,\ \frac{1}{4},\ \frac{1}{4},\right.$$
$$\left.\frac{1}{4},\ \frac{1}{4},\ \frac{1}{4} - \frac{k}{2} + \frac{k^2}{2} + \frac{n}{2} - k\,n + \frac{n^2}{2},\ -\frac{1}{4} + \frac{k}{2} - \frac{k^2}{2} + \frac{n^2}{2},\ -\frac{1}{4} + \frac{k}{2} - \frac{k^2}{2} - \frac{n}{2} + k\,n,\ \frac{1}{4} - \frac{k}{2} + \frac{k^2}{2}\right\}$$

```
MatrixRank[Table[
  v /. {n → Random[], k → Random[]},
  {50}
 ]]
```

```
4
```

```
RankWgl[m_] := Module[{v, n, k},
  v = Wgl[n, k, #] & /@ Diagrams[m * ar];
  MatrixRank[Table[
    v /. {n → Random[], k → Random[]},
    {Length[v]}
   ]]
 ]
```

```
RankWgl[4]
```

```
11
```

```
Wgl[Diag[ar[1, 2], ar[4, 6], ar[5, 3]]]
```

$$\frac{1}{8} + \frac{\#1}{4} - \frac{\#2}{4}\ \&$$

```
Wgl[Diag[ar[1, 2], ar[4, 6], ar[5, 3]]][n, k]
```

$$\frac{1}{8} - \frac{k}{4} + \frac{n}{4}$$