Pensieve header: Computing and playing with \$\rho_1\$ in the language of perturbed Gaussian Integration.

Programs

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];
Once[<< KnotTheory`; << Rot.m];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
Loading Rot.m from http://drorbn.net/APAI to compute rotation numbers.
```

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ; CF[ℰ_EPD] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p)_, ∞] ⋃ {x, p}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)] ];
CF[eqp_EQP] := CF /@ eqp
```

```
In[∘]:= EQP /: c_ * EQP[Q_, P_] := EQP[Q, CF[c P]];
```

```
In[∘]:= {p*, x*} = {π, ξ}; (z_i__)* := (z*)_i; vs_List* := (v ↦ v*) /@ vs;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^d_. ⧴ (D[f, {z*, d}])) /. z* → 0
```

```
In[∘]:= FI[EQP[Q_, P_]] := FI[EQP[Q, P], Union@Cases[Q, p_, ∞], Union@Cases[Q, x_, ∞]];
FI[EQP[Q_, P_], ps_List, xs_List] := Module[{u, v},
  A = Table[∂_{u,v} Q, {u, ps}, {v, xs}];
  Factor[Det[A]^-1 Zip_{ps⋃xs}[P ⅇ^(-xs*.Inverse[A].ps*)]]]
```

ρ_0 Tests

```
In[∘]:= ρ0i[K_] := ρ0i[K, False]; ρ0i[Flip@K_] := ρ0i[K, True];
ρ0i[K_, flip_] := Module[{Cs, φ, n, s, i, j, k, vs, Q, Qp},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   If[flip, Cs = Cs〚All, {1, 3, 2}〛; φ = -φ];
   Q = -p_{2 n+1} x_{2 n+1}; Qp = 0;
   Cases[Cs, {s_, i_, j_} ⧴
     (Q -= x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1}); Qp -= s T^(s-1) x_i (p_{j+1} - p_{i+1}))];
   EQP[Q, -T^((Total[φ]+Total[Cs〚All,1〛])/2) Qp -
     (Total[φ] + Total[Cs〚All, 1〛]) T^((Total[φ]+Total[Cs〚All,1〛])/2-1) / 2]
  ];
```

In[∘]:=
```
K = Knot[8, 17];
Factor[∂_T (Alexander[K][T]^-1)]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\frac{(-1+T)\,T^2\,(1+T)\,\left(1-T+T^2\right)\,\left(3-5\,T+3\,T^2\right)}{\left(1-4\,T+8\,T^2-11\,T^3+8\,T^4-4\,T^5+T^6\right)^2}$$

In[∘]:=
```
K = Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs];
ν = {lν = 0};
writhe = Total@Cs[[All, 1]];
Do[Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
eqp = ρ0i[K];
eqp1 = T^-writhe CF[eqp /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
        Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1} + T^(s+ν[[i]]) p_1,
           x_i → -T^ν[[i]] p_{i+1} + T^ν[[i]] p_1, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
eqp2 = CF[ρ0i[Flip@K] /. T → T^-1];
FI /@ {eqp, eqp1, eqp2}
```

Out[∘]=

$$\left\{-\frac{(-1+T)\,(1+T)}{\left(1-T+T^2\right)^2},\ -\frac{(-1+T)\,(1+T)}{\left(1-T+T^2\right)^2},\ \frac{(-1+T)\,T^2\,(1+T)}{\left(1-T+T^2\right)^2}\right\}$$

In[∘]:=
```
CF[eqp1[[1]] - eqp2[[1]]]
```

Out[∘]=

0

In[∘]:=
```
CF[T^2 eqp1[[2]] + eqp2[[2]]]
```

Out[∘]=

$$3\,T^2+T^3\,p_2\,x_1-T^3\,p_5\,x_1-T\,p_1\,x_2+T\,p_5\,x_2+T^2\,p_1\,x_3-T^2\,p_3\,x_3+T^3\,p_4\,x_3-T^3\,p_7\,x_3-T\,p_1\,x_4+T\,p_7\,x_4+T^2\,p_1\,x_5-T^3\,p_3\,x_5-T^2\,p_5\,x_5+T^3\,p_6\,x_5-T\,p_1\,x_6+T\,p_3\,x_6+T^2\,p_1\,x_7-T^2\,p_7\,x_7$$

In[∘]:=
```
FI@EQP[eqp1[[1]], T^2 eqp1[[2]] + eqp2[[2]]]
```

Out[∘]=

0

ρ_1 Tests

In[∘]:=
```
r₁[s_, i_, j_] :=
  s (-1 + 2 pᵢ xᵢ - 2 pⱼ xᵢ + (-1 + T^s) pᵢ pⱼ xᵢ^2 + (1 - T^s) pⱼ^2 xᵢ^2 - 2 pᵢ pⱼ xᵢ xⱼ + 2 pⱼ^2 xᵢ xⱼ) / 2;
γ₁[φ_, k_] := φ (1 / 2 - p_k x_k);
ρ1i[K_] := ρ1i[K, False]; ρ1i[Flip@K_] := ρ1i[K, True];
ρ1i[K_, flip_] := Module[{Cs, φ, n, s, i, j, k, vs, Q, P},
   {Cs, φ} = Rot[K]; n = Length[Cs];
   If[flip, Cs = Cs[[All, {1, 3, 2}]]; φ = -φ];
   Q = -x_{2 n+1} p_{2 n+1};
   Cases[Cs, {s_, i_, j_} :> (Q -= xᵢ (pᵢ - T^s p_{i+1} + (T^s - 1) p_{j+1}) + xⱼ (pⱼ - p_{j+1}))];
   P = Sum[r₁ @@ Cs[[k]], {k, n}] + Sum[γ₁[φ[[k]], k], {k, 2 n}];
   CF@EQP[Q, P]
  ];
```

In[∘]:=
```
K = Knot[5, 2];
ρ1i[K]
```

Out[∘]=

$$\text{EQP}\Big[-p_1 x_1+p_2 x_1-p_2 x_2+\frac{p_3 x_2}{T}+\frac{(-1+T)\, p_8 x_2}{T}-p_3 x_3+p_4 x_3+\frac{(-1+T)\, p_2 x_4}{T}-p_4 x_4+\frac{p_5 x_4}{T}-p_5 x_5+p_6 x_5-p_6 x_6+\frac{p_7 x_6}{T}+\frac{(-1+T)\, p_{10} x_6}{T}-p_7 x_7+p_8 x_7+\frac{(-1+T)\, p_4 x_8}{T}-p_8 x_8+\frac{p_9 x_8}{T}-p_9 x_9+p_{10} x_9+\frac{(-1+T)\, p_6 x_{10}}{T}-p_{10} x_{10}+\frac{p_{11} x_{10}}{T}-p_{11} x_{11},$$
$$2-p_2 x_2+p_7 x_2+\frac{(-1+T)\, p_2 p_7 x_2^2}{2T}+\frac{(1-T)\, p_7^2 x_2^2}{2T}+p_1 x_4-p_1^2 x_1 x_4+p_1 p_4 x_1 x_4+\frac{(1-T)\, p_1^2 x_4^2}{2T}+\frac{(-1+T)\, p_1 p_4 x_4^2}{2T}-p_6 x_6+p_9 x_6+\frac{(-1+T)\, p_6 p_9 x_6^2}{2T}+\frac{(1-T)\, p_9^2 x_6^2}{2T}+p_2 p_7 x_2 x_7-p_7^2 x_2 x_7+p_3 x_8-p_8 x_8-p_3^2 x_3 x_8+p_3 p_8 x_3 x_8+\frac{(1-T)\, p_3^2 x_8^2}{2T}+\frac{(-1+T)\, p_3 p_8 x_8^2}{2T}-p_9 x_9+p_6 p_9 x_6 x_9-p_9^2 x_6 x_9+p_5 x_{10}-p_5^2 x_5 x_{10}+p_5 p_{10} x_5 x_{10}+\frac{(1-T)\, p_5^2 x_{10}^2}{2T}+\frac{(-1+T)\, p_5 p_{10} x_{10}^2}{2T}\Big]$$

In[∘]:=
```
Factor@Together[FI@ρ1i[K]]
```

Out[∘]=

$$-\frac{(-1+T)^2\, T^4\, (5-4T+5T^2)}{(2-3T+2T^2)^3}$$

In[∘]:=
```
K = Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs];
ν = {lν = 0}; writhe = Total@Cs[[All, 1]];
Do[Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
eqp = ρ1i[K];
eqp1 = CF[eqp /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
      Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1} + T^(s+ν[[i]]) p_1,
         x_i → -T^ν[[i]] p_{i+1} + T^ν[[i]] p_1, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
eqp2 = CF[ρ1i[Flip@K] /. T → T^-1];
FI /@ {eqp, eqp1, eqp2}
```

Out[∘]=

$$\left\{-\frac{(-1+T)^2 T^3 \left(1+T^2\right)}{\left(1-T+T^2\right)^3}, -\frac{(-1+T)^2 \left(1+T^2\right)}{\left(1-T+T^2\right)^3}, -\frac{(-1+T)^2 \left(1+T^2\right)}{\left(1-T+T^2\right)^3}\right\}$$

In[∘]:=
```
CF[eqp1[[1]] - eqp2[[1]]]
```

Out[∘]=

0

In[∘]:=
```
CF[eqp1[[2]] - eqp2[[2]]]
```

Out[∘]=

$$-1 + (1+T)\, p_1 x_1 - p_4 x_1 - T p_5 x_1 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_1^2 + T^2 p_1 p_2 x_1^2 + \frac{1}{2}(-1+T)\, p_1 p_4 x_1^2 +$$
$$\frac{1}{2}(1-T)\, p_4^2 x_1^2 + T p_1 p_5 x_1^2 - T^2 p_2 p_5 x_1^2 + \frac{1}{2}\left(-T+T^2\right) p_5^2 x_1^2 - p_1 x_2 + p_3 x_2 + T p_1 x_3 + p_3 x_3 -$$
$$p_6 x_3 - T p_7 x_3 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_3^2 + T^2 p_1 p_4 x_3^2 + \frac{1}{2}(-1+T)\, p_3 p_6 x_3^2 + \frac{1}{2}(1-T)\, p_6^2 x_3^2 +$$
$$T p_1 p_7 x_3^2 - T^2 p_4 p_7 x_3^2 + \frac{1}{2}\left(-T+T^2\right) p_7^2 x_3^2 + p_4 x_4 + \frac{1}{2}(1+T)\, p_1^2 x_1 x_4 - T p_1 p_2 x_1 x_4 -$$
$$p_1 p_4 x_1 x_4 + p_4^2 x_1 x_4 - p_1 p_5 x_1 x_4 + T p_2 p_5 x_1 x_4 + \frac{1}{2}(1-T)\, p_5^2 x_1 x_4 + T p_1 x_5 - p_2 x_5 -$$
$$T p_3 x_5 + p_5 x_5 + \frac{1}{2}(1+T)\, p_1^2 x_2 x_5 + p_2^2 x_2 x_5 - p_1 p_3 x_2 x_5 + \frac{1}{2}(1-T)\, p_3^2 x_2 x_5 - p_2 p_5 x_2 x_5 -$$
$$T p_1 p_6 x_2 x_5 + T p_3 p_6 x_2 x_5 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_5^2 + \frac{1}{2}(1-T)\, p_2^2 x_5^2 + T p_1 p_3 x_5^2 + \frac{1}{2}\left(-T+T^2\right) p_3^2 x_5^2 +$$
$$\frac{1}{2}(-1+T)\, p_2 p_5 x_5^2 + T^2 p_1 p_6 x_5^2 - T^2 p_3 p_6 x_5^2 - p_1 x_6 + p_7 x_6 + \frac{1}{2}(1+T)\, p_1^2 x_3 x_6 -$$
$$T p_1 p_4 x_3 x_6 - p_3 p_6 x_3 x_6 + p_6^2 x_3 x_6 - p_1 p_7 x_3 x_6 + T p_4 p_7 x_3 x_6 + \frac{1}{2}(1-T)\, p_7^2 x_3 x_6$$

In[∘]:=
```
FI@EQP[eqp1[[1]], eqp1[[2]] - eqp2[[2]]]
```

Out[∘]=

0

In[∘]:=
```
Monitor[sum = 0; Do[
  {Cs, φ} = Rot[K]; n = Length[Cs];
  ν = {lν = 0}; writhe = Total@Cs⟦All, 1⟧;
  Do[
   Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
  eqp1 = CF[ρ1i[K] /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
       Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν⟦i⟧ p_{j+1} + (1 - T^s) T^ν⟦i⟧ p_{i+1} + T^(s+ν⟦i⟧) p_1,
          x_i → -T^ν⟦i⟧ p_{i+1} + T^ν⟦i⟧ p_1, p_i → T^(-ν⟦i⟧) x_i, p_j → T^(-ν⟦i⟧-s) x_j}}];
  eqp2 = CF[ρ1i[Flip@K] /. T → T^-1];
  sum += Simplify[eqp1⟦1⟧ == eqp2⟦1⟧] ∧ FI@eqp1 == FI@eqp2,
  {K, AllKnots[{3, 7}]}
 ], {K, sum}]; sum
```

Out[∘]=

14 True

In[∘]:=
```
Monitor[sum = 0; Do[
  {Cs, φ} = Rot[K]; n = Length[Cs];
  ν = {lν = 0}; writhe = Total@Cs⟦All, 1⟧;
  Do[
   Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
  eqp1 = CF[ρ1i[K] /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
       Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν⟦i⟧ p_{j+1} + (1 - T^s) T^ν⟦i⟧ p_{i+1} + T^(s+ν⟦i⟧) p_1,
          x_i → -T^ν⟦i⟧ p_{i+1} + T^ν⟦i⟧ p_1, p_i → T^(-ν⟦i⟧) x_i, p_j → T^(-ν⟦i⟧-s) x_j}}];
  eqp2 = CF[ρ1i[Flip@K] /. T → T^-1];
  sum += Simplify[eqp1⟦1⟧ == eqp2⟦1⟧] ∧ FI@EQP[eqp1⟦1⟧, eqp1⟦2⟧ - eqp2⟦2⟧] == 0,
  {K, AllKnots[{3, 7}]}
 ], {K, sum}]; sum
```

Out[∘]=

14 True

Palindromicity for ρ_1

In[∘]:=
```
CF[-(x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})) /.
  {x_j → -T^(ν i) p_{j+1} + (1 - T^s) T^(ν i) p_{i+1} + T^(s+ν i) p_1, x_i → -T^(ν i) p_{i+1} + T^(ν i) p_1, p_i → T^(-ν i) x_i, p_j → T^(-ν i-s) x_j}]
```

Out[∘]=

$$T^{s+\nu i} p_1 p_{1+i} - T^{s+\nu i} p_{1+i}^2 + T^{\nu i} p_1 p_{1+j} - T^{\nu i} p_{1+j}^2 - p_1 x_i + p_{1+i} x_i - p_1 x_j + T^{-s} (-1 + T^s) p_{1+i} x_j + T^{-s} p_{1+j} x_j$$

In[∘]:=
```
Q = CF@PowerExpand[-(x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})) /. {i → j, j → i, T → T^-1}]
```

Out[∘]=

$$-p_i x_i + p_{1+i} x_i + T^{-s} (-1 + T^s) p_{1+i} x_j - p_j x_j + T^{-s} p_{1+j} x_j$$

In[∘]:=
```
Clear[s, i, j]; CF[s^-1 r_1[s, i, j]]
```

Out[∘]=

$$-\frac{1}{2} + p_i x_i - p_j x_i + \frac{1}{2} (-1 + T^s) p_i p_j x_i^2 + \frac{1}{2} (1 - T^s) p_j^2 x_i^2 - p_i p_j x_i x_j + p_j^2 x_i x_j$$

In[∘]:=
```
CF@PowerExpand[Plus[
    r1[s, i, j] /. {xj → -T^(νi) pj+1 + (1 - T^s) T^(νi) pi+1 + T^(s+νi) p1,
      xi → -T^(νi) pi+1 + T^(νi) p1, pi → T^(-νi) xi, pj → T^(-νi-s) xj},
    -r1[s, j, i] /. T → T^-1
   ] / s]
```

Out[∘]=

$$p_1 x_i - p_{1+i} x_i - T^{-s} p_1 x_j + p_i x_j + T^{-s} p_{1+i} x_j - p_j x_j + \frac{1}{2} T^{-s} (-1 - T^s) p_1^2 x_i x_j - p_i^2 x_i x_j + p_1 p_{1+i} x_i x_j + \frac{1}{2} T^{-s} (1 - T^s) p_{1+i}^2 x_i x_j + p_i p_j x_i x_j + T^{-s} p_1 p_{1+j} x_i x_j - T^{-s} p_{1+i} p_{1+j} x_i x_j + \frac{1}{2} T^{-2s} (1 + T^s) p_1^2 x_j^2 + \frac{1}{2} T^{-s} (1 - T^s) p_i^2 x_j^2 - T^{-s} p_1 p_{1+i} x_j^2 + \frac{1}{2} T^{-2s} (-1 + T^s) p_{1+i}^2 x_j^2 + \frac{1}{2} T^{-s} (-1 + T^s) p_i p_j x_j^2 - T^{-2s} p_1 p_{1+j} x_j^2 + T^{-2s} p_{1+i} p_{1+j} x_j^2$$

In[∘]:=
```
K = Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs];
ν = {lν = 0}; writhe = Total@Cs[[All, 1]];
Do[Cs /. {{s_, k, j_} :→ AppendTo[ν, lν += s], {s_, i_, k} :→ AppendTo[ν, lν -= s]}, {k, 2 n}];
eqp = ρ1i[K];
eqp1 = CF[eqp /. Flatten@{{x2n+1 → p1, p2n+1 → x2n+1},
     Cs /. {s_Integer, i_, j_} :→ {xj → -T^(ν[[i]]) pj+1 + (1 - T^s) T^(ν[[i]]) pi+1 + T^(s+ν[[i]]) p1,
        xi → -T^(ν[[i]]) pi+1 + T^(ν[[i]]) p1, pi → T^(-ν[[i]]) xi, pj → T^(-ν[[i]]-s) xj}}];
eqp2 = CF[ρ1i[Flip@K] /. T → T^-1];
FI@EQP[eqp1[[1]], eqp1[[2]] - eqp2[[2]]]
CF@EQP[eqp1[[1]], eqp1[[2]] - eqp2[[2]]]
diff = CF@Plus[Sum[
    s (p1 xi - p1+i xi - T^-s p1 xj + pi xj + T^-s p1+i xj - pj xj + 1/2 T^-s (-1 - T^s) p1^2 xi xj -
        pi^2 xi xj + p1 p1+i xi xj + 1/2 T^-s (1 - T^s) p1+i^2 xi xj + pi pj xi xj + T^-s p1 p1+j xi xj -
        T^-s p1+i p1+j xi xj + 1/2 T^-2 s (1 + T^s) p1^2 xj^2 + 1/2 T^-s (1 - T^s) pi^2 xj^2 - T^-s p1 p1+i xj^2 +
        1/2 T^-2 s (-1 + T^s) p1+i^2 xj^2 + 1/2 T^-s (-1 + T^s) pi pj xj^2 - T^-2 s p1 p1+j xj^2 + T^-2 s p1+i p1+j xj^2)
     /. Thread[{s, i, j} → Cs[[k]]],
    {k, n}],
   0 Sum[γ1[φ[[k]], k], {k, 2 n}]
  ]
CF[eqp1[[2]] - eqp2[[2]] - diff /. (p | x)1 → 0]
```

Out[∘]=

0

Out[#]=

$$\mathrm{EQP}\Big[-p_1 x_1 + T p_2 x_1 + (1-T) p_5 x_1 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1-T) p_7 x_3 - p_4 x_4 + p_5 x_4 + (1-T) p_3 x_5 - p_5 x_5 + T p_6 x_5 - p_6 x_6 + p_7 x_6 - p_7 x_7,$$
$$-1 + (1+T) p_1 x_1 - p_4 x_1 - T p_5 x_1 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_1^2 + T^2 p_1 p_2 x_1^2 + \frac{1}{2}(-1+T) p_1 p_4 x_1^2 + \frac{1}{2}(1-T) p_4^2 x_1^2 + T p_1 p_5 x_1^2 - T^2 p_2 p_5 x_1^2 + \frac{1}{2}\left(-T+T^2\right) p_5^2 x_1^2 - p_1 x_2 + p_3 x_2 + T p_1 x_3 + p_3 x_3 - p_6 x_3 - T p_7 x_3 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_3^2 + T^2 p_1 p_4 x_3^2 + \frac{1}{2}(-1+T) p_3 p_6 x_3^2 + \frac{1}{2}(1-T) p_6^2 x_3^2 + T p_1 p_7 x_3^2 - T^2 p_4 p_7 x_3^2 + \frac{1}{2}\left(-T+T^2\right) p_7^2 x_3^2 + p_4 x_4 + \frac{1}{2}(1+T) p_1^2 x_1 x_4 - T p_1 p_2 x_1 x_4 - p_1 p_4 x_1 x_4 + p_4^2 x_1 x_4 - p_1 p_5 x_1 x_4 + T p_2 p_5 x_1 x_4 + \frac{1}{2}(1-T) p_5^2 x_1 x_4 + T p_1 x_5 - p_2 x_5 - T p_3 x_5 + p_5 x_5 + \frac{1}{2}(1+T) p_1^2 x_2 x_5 + p_2^2 x_2 x_5 - p_1 p_3 x_2 x_5 + \frac{1}{2}(1-T) p_3^2 x_2 x_5 - p_2 p_5 x_2 x_5 - T p_1 p_6 x_2 x_5 + T p_3 p_6 x_2 x_5 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_5^2 + \frac{1}{2}(1-T) p_2^2 x_5^2 + T p_1 p_3 x_5^2 + \frac{1}{2}\left(-T+T^2\right) p_3^2 x_5^2 + \frac{1}{2}(-1+T) p_2 p_5 x_5^2 + T^2 p_1 p_6 x_5^2 - T^2 p_3 p_6 x_5^2 - p_1 x_6 + p_7 x_6 + \frac{1}{2}(1+T) p_1^2 x_3 x_6 - T p_1 p_4 x_3 x_6 - p_3 p_6 x_3 x_6 + p_6^2 x_3 x_6 - p_1 p_7 x_3 x_6 + T p_4 p_7 x_3 x_6 + \frac{1}{2}(1-T) p_7^2 x_3 x_6\Big]$$

Out[#]=

$$(1+T) p_1 x_1 - p_4 x_1 - T p_5 x_1 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_1^2 + T^2 p_1 p_2 x_1^2 + \frac{1}{2}(-1+T) p_1 p_4 x_1^2 + \frac{1}{2}(1-T) p_4^2 x_1^2 + T p_1 p_5 x_1^2 - T^2 p_2 p_5 x_1^2 + \frac{1}{2}\left(-T+T^2\right) p_5^2 x_1^2 - p_1 x_2 + p_3 x_2 + T p_1 x_3 + p_3 x_3 - p_6 x_3 - T p_7 x_3 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_3^2 + T^2 p_1 p_4 x_3^2 + \frac{1}{2}(-1+T) p_3 p_6 x_3^2 + \frac{1}{2}(1-T) p_6^2 x_3^2 + T p_1 p_7 x_3^2 - T^2 p_4 p_7 x_3^2 + \frac{1}{2}\left(-T+T^2\right) p_7^2 x_3^2 - p_1 x_4 + p_5 x_4 + \frac{1}{2}(1+T) p_1^2 x_1 x_4 - T p_1 p_2 x_1 x_4 - p_1 p_4 x_1 x_4 + p_4^2 x_1 x_4 - p_1 p_5 x_1 x_4 + T p_2 p_5 x_1 x_4 + \frac{1}{2}(1-T) p_5^2 x_1 x_4 + T p_1 x_5 - p_2 x_5 - T p_3 x_5 + p_5 x_5 + \frac{1}{2}(1+T) p_1^2 x_2 x_5 + p_2^2 x_2 x_5 - p_1 p_3 x_2 x_5 + \frac{1}{2}(1-T) p_3^2 x_2 x_5 - p_2 p_5 x_2 x_5 - T p_1 p_6 x_2 x_5 + T p_3 p_6 x_2 x_5 + \frac{1}{2}\left(-T-T^2\right) p_1^2 x_5^2 + \frac{1}{2}(1-T) p_2^2 x_5^2 + T p_1 p_3 x_5^2 + \frac{1}{2}\left(-T+T^2\right) p_3^2 x_5^2 + \frac{1}{2}(-1+T) p_2 p_5 x_5^2 + T^2 p_1 p_6 x_5^2 - T^2 p_3 p_6 x_5^2 - p_1 x_6 + p_7 x_6 + \frac{1}{2}(1+T) p_1^2 x_3 x_6 - T p_1 p_4 x_3 x_6 - p_3 p_6 x_3 x_6 + p_6^2 x_3 x_6 - p_1 p_7 x_3 x_6 + T p_4 p_7 x_3 x_6 + \frac{1}{2}(1-T) p_7^2 x_3 x_6$$

Out[#]=

$$-1 + p_4 x_4 - p_5 x_4$$

In[∘]:= **Q**

Out[∘]=

$$-p_i x_i + p_{1+i} x_i + T^{-s} (-1 + T^s) p_{1+i} x_j - p_j x_j + T^{-s} p_{1+j} x_j$$

In[∘]:= **Cs**

Out[∘]=

{{-1, 4, 1}, {-1, 6, 3}, {-1, 2, 5}}

In[∘]:= $\mathbf{f0 = x_i\, x_i\, p_j;\ \ f1 = CF[\partial_{x_i} f0 + f0\, \partial_{x_i} Q]}$

$\mathbf{Table[FI[EQP[-p_1 x_1 + T p_2 x_1 + (1 - T) p_5 x_1 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_7 x_3 - p_4 x_4 + p_5 x_4 + (1 - T) p_3 x_5 - p_5 x_5 + T p_6 x_5 - p_6 x_6 + p_7 x_6 - p_7 x_7,\ f1]],\ \{i, 7\},\ \{j, 7\}]\ //\ MatrixForm}$

Out[∘]=

$$2 p_j x_i - p_i p_j x_i^2 + p_{1+i} p_j x_i^2$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2(-1+T)^2}{(1-T+T^2)^2} & -\frac{2(-1+T)^2}{(1-T+T^2)^3} & -\frac{2(-1+T)^2}{(1-T+T^2)^3} & \frac{2(-1+T)^3}{(1-T+T^2)^3} & \frac{2(-1+T)^3}{(1-T+T^2)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2(-1+T)T}{(1-T+T^2)^2} & -\frac{2(-1+T)T^2}{(1-T+T^2)^3} & -\frac{2(-1+T)T^2}{(1-T+T^2)^3} & -\frac{2(-1+T)T}{(1-T+T^2)^3} & -\frac{2(-1+T)T}{(1-T+T^2)^3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[∘]:= $\mathbf{List\ @@\ Expand\left[(x_i + x_j + x_{i+1} + x_{j+1})^2\right]}$

Out[∘]=

$$\{x_i^2,\ 2 x_i x_{1+i},\ x_{1+i}^2,\ 2 x_i x_j,\ 2 x_{1+i} x_j,\ x_j^2,\ 2 x_i x_{1+j},\ 2 x_{1+i} x_{1+j},\ 2 x_j x_{1+j},\ x_{1+j}^2\}$$

In[∘]:= $\mathbf{ders = Flatten@Table\left[\{v, f0, f1\} \to CF[f1\, \partial_v f0 + f1\, f0\, \partial_v Q],\ \{v, \{x_i, x_j\}\},\ \left\{f0,\ List\ @@\ Expand\left[(x_i + x_j)^2\right]\right\},\ \{f1, \{p_i, p_j\}\}\right]}$

Out[∘]=

$$\begin{aligned}
\{&\{x_i, x_i^2, p_i\} \to 2 p_i x_i - p_i^2 x_i^2 + p_i p_{1+i} x_i^2,\ \{x_i, x_i^2, p_j\} \to 2 p_j x_i - p_i p_j x_i^2 + p_{1+i} p_j x_i^2,\\
&\{x_i, 2 x_i x_j, p_i\} \to 2 p_i x_j - 2 p_i^2 x_i x_j + 2 p_i p_{1+i} x_i x_j,\\
&\{x_i, 2 x_i x_j, p_j\} \to 2 p_j x_j - 2 p_i p_j x_i x_j + 2 p_{1+i} p_j x_i x_j,\ \{x_i, x_j^2, p_i\} \to -p_i^2 x_j^2 + p_i p_{1+i} x_j^2,\\
&\{x_i, x_j^2, p_j\} \to -p_i p_j x_j^2 + p_{1+i} p_j x_j^2,\ \{x_j, x_i^2, p_i\} \to T^{-s} (-1 + T^s) p_i p_{1+i} x_i^2 - p_i p_j x_i^2 + T^{-s} p_i p_{1+j} x_i^2,\\
&\{x_j, x_i^2, p_j\} \to T^{-s} (-1 + T^s) p_{1+i} p_j x_i^2 - p_j^2 x_i^2 + T^{-s} p_j p_{1+j} x_i^2,\\
&\{x_j, 2 x_i x_j, p_i\} \to 2 p_i x_i + T^{-s} (-2 + 2 T^s) p_i p_{1+i} x_i x_j - 2 p_i p_j x_i x_j + 2 T^{-s} p_i p_{1+j} x_i x_j,\\
&\{x_j, 2 x_i x_j, p_j\} \to 2 p_j x_i + T^{-s} (-2 + 2 T^s) p_{1+i} p_j x_i x_j - 2 p_j^2 x_i x_j + 2 T^{-s} p_j p_{1+j} x_i x_j,\\
&\{x_j, x_j^2, p_i\} \to 2 p_i x_j + T^{-s} (-1 + T^s) p_i p_{1+i} x_j^2 - p_i p_j x_j^2 + T^{-s} p_i p_{1+j} x_j^2,\\
&\{x_j, x_j^2, p_j\} \to 2 p_j x_j + T^{-s} (-1 + T^s) p_{1+i} p_j x_j^2 - p_j^2 x_j^2 + T^{-s} p_j p_{1+j} x_j^2\}
\end{aligned}$$

In[∘]:= $\mathbf{Table[}$

$\mathbf{Simplify@FI[EQP[-p_1 x_1 + T p_2 x_1 + (1 - T) p_5 x_1 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_7 x_3 - p_4 x_4 + p_5 x_4 + (1 - T) p_3 x_5 - p_5 x_5 + T p_6 x_5 - p_6 x_6 + p_7 x_6 - p_7 x_7,}$

$\mathbf{d\ /.\ \{s \to -1, i \to 2, j \to 5\}]],\ \{d, Last\ /@\ ders\}]}$

Out[∘]=

{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
Last /@ ders
```

Out[∘]=

$$\{2\,p_i\,x_i - p_i^2\,x_i^2 + p_i\,p_{1+i}\,x_i^2,\ 2\,p_j\,x_i - p_i\,p_j\,x_i^2 + p_{1+i}\,p_j\,x_i^2,\ 2\,p_i\,x_j - 2\,p_i^2\,x_i\,x_j + 2\,p_i\,p_{1+i}\,x_i\,x_j,$$
$$2\,p_j\,x_j - 2\,p_i\,p_j\,x_i\,x_j + 2\,p_{1+i}\,p_j\,x_i\,x_j,\ -p_i^2\,x_j^2 + p_i\,p_{1+i}\,x_j^2,\ -p_i\,p_j\,x_j^2 + p_{1+i}\,p_j\,x_j^2,$$
$$T^{-s}\,(-1+T^s)\,p_i\,p_{1+i}\,x_i^2 - p_i\,p_j\,x_i^2 + T^{-s}\,p_i\,p_{1+j}\,x_i^2,\ T^{-s}\,(-1+T^s)\,p_{1+i}\,p_j\,x_i^2 - p_j^2\,x_i^2 + T^{-s}\,p_j\,p_{1+j}\,x_i^2,$$
$$2\,p_i\,x_i + T^{-s}\,(-2+2\,T^s)\,p_i\,p_{1+i}\,x_i\,x_j - 2\,p_i\,p_j\,x_i\,x_j + 2\,T^{-s}\,p_i\,p_{1+j}\,x_i\,x_j,$$
$$2\,p_j\,x_i + T^{-s}\,(-2+2\,T^s)\,p_{1+i}\,p_j\,x_i\,x_j - 2\,p_j^2\,x_i\,x_j + 2\,T^{-s}\,p_j\,p_{1+j}\,x_i\,x_j,$$
$$2\,p_i\,x_j + T^{-s}\,(-1+T^s)\,p_i\,p_{1+i}\,x_j^2 - p_i\,p_j\,x_j^2 + T^{-s}\,p_i\,p_{1+j}\,x_j^2,$$
$$2\,p_j\,x_j + T^{-s}\,(-1+T^s)\,p_{1+i}\,p_j\,x_j^2 - p_j^2\,x_j^2 + T^{-s}\,p_j\,p_{1+j}\,x_j^2\}$$

In[∘]:=
```
Module[{i, j, k},
 AllMonomials[{}, 0] = {1};
 AllMonomials[{}, d_Integer] /; d > 0 := {};
 AllMonomials[{v_, vs___}, d_Integer] :=
  Join @@ Table[v^(d-k) AllMonomials[{vs}, k], {k, 0, d}];
 AllMonomials[vs_List, {d_}] := Join @@ Table[AllMonomials[vs, k], {k, 0, d}];
 Basis[js_List, m_] := Flatten@Outer[Times,
    AllMonomials[Table[p_j, {j, js}], m], AllMonomials[Table[x_j, {j, js}], m]];
 Basis[js_List, {m_}] := Flatten@Table[Basis[js, k], {k, 0, m}];
 GenericCombination[bas_, c_] := bas.Table[c_j, {j, Length@bas}];
 GenericCombination[bas_, c_k_] := bas.Table[c_{k,j}, {j, Length@bas}];
]
```

In[∘]:=
```
Basis[{i, j}, 2]
```

Out[∘]=

$$\{p_i^2\,x_i^2,\ p_i^2\,x_i\,x_j,\ p_i^2\,x_j^2,\ p_i\,p_j\,x_i^2,\ p_i\,p_j\,x_i\,x_j,\ p_i\,p_j\,x_j^2,\ p_j^2\,x_i^2,\ p_j^2\,x_i\,x_j,\ p_j^2\,x_j^2\}$$

In[∘]:=
```
Table[Coefficient[r, b], {r, Last /@ ders}, {b, Basis[{i, j, i + 1, j + 1}, 2]}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T^{-s}\,(-1+T^s) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T^{-s}\,(-2+2\,T^s) & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T^{-s}\,(- \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

In[∘]:=
```
vans = CF /@ (
   RowReduce[
     Table[
      Coefficient[r, b] /. (p | x)_ → 0,
      {r, (Last /@ ders) ⋃ {x_{i+1} p_{i+1} + x_{j+1} p_{j+1} - x_i p_i - x_j p_j, x_{i+1}^2 p_{i+1}^2 + x_{j+1}^2 p_{j+1}^2 - x_i^2 p_i^2 - x_j^2 p_j^2}},
      {b, bas = Basis[{i, j, i + 1, j + 1}, 2] ⋃ Basis[{i, j, i + 1, j + 1}, 1]}
     ]
   ].bas
  )
```

Out[∘]=

$$
\begin{aligned}
\{&p_i x_i + T^{-s}(-1+T^s)\, p_i p_{1+i} x_i x_j - p_{1+i} p_j x_i x_j + T^{-s} p_i p_{1+j} x_i x_j + \frac{1}{2} T^{-s}(-1+T^s)\, p_{1+i} p_j x_j^2 - \\
&\quad \frac{1}{2} p_j^2 x_j^2 + \frac{1}{2} T^{-s} p_j p_{1+j} x_j^2,\ p_j x_i + T^{-s}(-1+T^s)\, p_{1+i} p_j x_i x_j - p_j^2 x_i x_j + T^{-s} p_j p_{1+j} x_i x_j, \\
&p_i^2 x_i^2 - p_{1+i}^2 x_{1+i}^2 + p_j^2 x_j^2 - p_{1+j}^2 x_{1+j}^2,\ p_i p_{1+i} x_i^2 - p_{1+i}^2 x_{1+i}^2 + T^{-s}(2-2T^s)\, p_i p_{1+i} x_i x_j + \\
&\quad 2 p_{1+i} p_j x_i x_j - 2T^{-s} p_i p_{1+j} x_i x_j + T^{-s}(1-T^s)\, p_{1+i} p_j x_j^2 + 2 p_j^2 x_j^2 - T^{-s} p_j p_{1+j} x_j^2 - p_{1+j}^2 x_{1+j}^2, \\
&p_i p_j x_i^2 - T^{-s} p_i p_{1+j} x_i^2 + T^{-s}(1-T^s)\, p_{1+i}^2 x_{1+i}^2 + T^{-2s}\left(-2+4T^s-2T^{2s}\right) p_i p_{1+i} x_i x_j + \\
&\quad T^{-s}(-2+2T^s)\, p_{1+i} p_j x_i x_j + T^{-2s}(2-2T^s)\, p_i p_{1+j} x_i x_j + T^{-2s}\left(-1+2T^s-T^{2s}\right) p_{1+i} p_j x_j^2 + \\
&\quad T^{-s}(-2+2T^s)\, p_j^2 x_j^2 + T^{-2s}(1-T^s)\, p_j p_{1+j} x_j^2 + T^{-s}(1-T^s)\, p_{1+j}^2 x_{1+j}^2, \\
&p_{1+i} p_j x_i^2 - T^{-s} p_i p_{1+j} x_i^2 + T^{-s}(1-T^s)\, p_{1+i}^2 x_{1+i}^2 + T^{-2s}\left(-2+4T^s-2T^{2s}\right) p_i p_{1+i} x_i x_j + \\
&\quad 2 p_j^2 x_i x_j + T^{-2s}(2-2T^s)\, p_i p_{1+j} x_i x_j - 2T^{-s} p_j p_{1+j} x_i x_j + T^{-2s}\left(-1+2T^s-T^{2s}\right) p_{1+i} p_j x_j^2 + \\
&\quad T^{-s}(-2+2T^s)\, p_j^2 x_j^2 + T^{-2s}(1-T^s)\, p_j p_{1+j} x_j^2 + T^{-s}(1-T^s)\, p_{1+j}^2 x_{1+j}^2, \\
&p_j^2 x_i^2 + T^{-2s}(1-T^s)\, p_i p_{1+j} x_i^2 - T^{-s} p_j p_{1+j} x_i^2 + T^{-2s}\left(-1+2T^s-T^{2s}\right) p_{1+i}^2 x_{1+i}^2 + \\
&\quad T^{-3s}\left(2-6T^s+6T^{2s}-2T^{3s}\right) p_i p_{1+i} x_i x_j + T^{-s}(-2+2T^s)\, p_j^2 x_i x_j + \\
&\quad T^{-3s}\left(-2+4T^s-2T^{2s}\right) p_i p_{1+j} x_i x_j + T^{-2s}(2-2T^s)\, p_j p_{1+j} x_i x_j + \\
&\quad T^{-3s}\left(1-3T^s+3T^{2s}-T^{3s}\right) p_{1+i} p_j x_j^2 + T^{-2s}\left(2-4T^s+2T^{2s}\right) p_j^2 x_j^2 + \\
&\quad T^{-3s}\left(-1+2T^s-T^{2s}\right) p_j p_{1+j} x_j^2 + T^{-2s}\left(-1+2T^s-T^{2s}\right) p_{1+j}^2 x_{1+j}^2, \\
&p_{1+i} x_{1+i} + T^{-s}(-1+T^s)\, p_i p_{1+i} x_i x_j - p_{1+i} p_j x_i x_j + T^{-s} p_i p_{1+j} x_i x_j + T^{-s}(-1+T^s)\, p_{1+i} p_j x_j^2 - \\
&\quad p_j^2 x_j^2 + T^{-s} p_j p_{1+j} x_j^2 + p_{1+j} x_{1+j},\ p_i x_j + \frac{1}{2} T^{-s}(-1+T^s)\, p_i p_{1+i} x_j^2 - \frac{1}{2} p_{1+i} p_j x_j^2 + \frac{1}{2} T^{-s} p_i p_{1+j} x_j^2, \\
&p_j x_j + \frac{1}{2} T^{-s}(-1+T^s)\, p_{1+i} p_j x_j^2 - \frac{1}{2} p_j^2 x_j^2 + \frac{1}{2} T^{-s} p_j p_{1+j} x_j^2, \\
&p_i^2 x_i x_j - p_i p_{1+i} x_i x_j + \frac{1}{2} T^{-s}(-1+T^s)\, p_i p_{1+i} x_j^2 - \frac{1}{2} p_{1+i} p_j x_j^2 + \frac{1}{2} T^{-s} p_i p_{1+j} x_j^2, \\
&p_i p_j x_i x_j - p_{1+i} p_j x_i x_j + \frac{1}{2} T^{-s}(-1+T^s)\, p_{1+i} p_j x_j^2 - \frac{1}{2} p_j^2 x_j^2 + \frac{1}{2} T^{-s} p_j p_{1+j} x_j^2, \\
&p_i^2 x_j^2 - p_i p_{1+i} x_j^2,\ p_i p_j x_j^2 - p_{1+i} p_j x_j^2\}
\end{aligned}
$$

In[∘]:=
```
Table[Simplify@
  FI[EQP[-p_1 x_1 + T p_2 x_1 + (1 - T) p_5 x_1 - p_2 x_2 + p_3 x_2 - p_3 x_3 + T p_4 x_3 + (1 - T) p_7 x_3 - p_4 x_4 + p_5 x_4 +
     (1 - T) p_3 x_5 - p_5 x_5 + T p_6 x_5 - p_6 x_6 + p_7 x_6 - p_7 x_7, d /. {s → -1, i → 2, j → 5}]], {d, vans}]
```

Out[∘]=

```
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}
```