

Pensieve header: Palindromicity by flipping and manipulating, in matrix language.

## Programs

(Alt) In[ ]:=

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];  
Once[  
  << KnotTheory` ;  
  << Rot.m  
];
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.

Read more at <http://katlas.org/wiki/KnotTheory>.

Loading Rot.m from <http://drorbn.net/APAI> to compute rotation numbers.

(Alt) In[ ]:=

```

AlexanderMatrices[K_] := Module[{ },
  {Cs, ρ} = Rot[K]; n = Length[Cs];
  v = {lv = 0};
  For[k = 1, k ≤ 2 n, ++k,
    Cs /. {{s_, k, j_} => AppendTo[v, lv += s], {s_, i_, k} => AppendTo[v, lv -= s]}}];
  v = DiagonalMatrix[T^v]; vi = Inverse@v;
  A0 = Table[0, {2 n + 1, 2 n + 1}]; A0[[2 n + 1, 2 n + 1]] = 1;
  C2 = C1 = Aπ = A1 = A2 = A3 = A4 = A0;
  P1 = RotateLeft[IdentityMatrix[2 n + 1]];
  (*P1=Table[0,2n+1,2n+1]; Do[P1[[i,i+1]]=1,{i,2n}];*)
  P2 = IdentityMatrix[2 n + 1]; Do[P2[[i, 1]] = -1, {i, 2, 2 n + 1}];
  E_{i_,j_} := ReplacePart[Table[0, {2 n + 1, 2 n + 1}], {i, j} -> 1];

  Cases[Cs, {s_Integer, i_, j_} => {
    Aπ[[{i, j}, {i, j, i + 1, j + 1}]] =  $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & T^s & -1 \end{pmatrix}$ ;

    A0[[{i, j}, {i, j, i + 1, j + 1}]] =  $\begin{pmatrix} 1 & 0 & -T^s & T^s - 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$ ;

    C1[[{i, j}, {i, j}]] =  $\begin{pmatrix} -T^{-s} & T^{-s} - 1 \\ 0 & -1 \end{pmatrix}$ ; C2[[{i, j}, {i, j}]] =  $\begin{pmatrix} -T^{-s} & 0 \\ T^{-s} - 1 & -1 \end{pmatrix}$ ;

    A1[[{i, j}, {i, j, i + 1, j + 1}]] =  $\begin{pmatrix} -T^{-s} & T^{-s} - 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$ ;

    A2[[{i, j, i + 1, j + 1}, {i, j}]] =  $\begin{pmatrix} -T^{-s} & 0 \\ T^{-s} - 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ ;

    A3[[{i, j}, {i, j, i + 1, j + 1}]] =  $\begin{pmatrix} 1 & 0 & -T^{-s} & 0 \\ 0 & 1 & T^{-s} - 1 & -1 \end{pmatrix}$ ;

    A4[[{i, j}, {i, j, i + 1, j + 1}]] =  $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & T^{-s} - 1 & -T^{-s} \end{pmatrix}$ ;
  }];

AlexanderMatrices[Knot[10, 165]]
Det /@ {Aπ, A0, A1, A2, A3, A4}
Simplify[{C1.A0 == A1, (C1.A0)^T == A2, A0^T.C2.P1 + E_{1,1} == A3,
  vi.A0^T.C2.P1.v + E_{1,1} == A4, vi.A0^T.C2.P1.v.P2 == (Aπ /. T -> T^-1)}]

```

 KnotTheory: Loading precomputed data in PD4Knots`.

(Alt) Out[ ]=

$$\left\{ \frac{-2 T^3 + 10 T^4 - 15 T^5 + 10 T^6 - 2 T^7}{T}, \frac{-2 T^2 + 10 T^3 - 15 T^4 + 10 T^5 - 2 T^6}{T^2}, \right. \\ \left. - \frac{2 T^2 - 10 T^3 + 15 T^4 - 10 T^5 + 2 T^6}{T^8}, - \frac{2 T^9 - 10 T^{10} + 15 T^{11} - 10 T^{12} + 2 T^{13}}{T^{15}}, \right. \\ \left. - \frac{-2 T^9 + 10 T^{10} - 15 T^{11} + 10 T^{12} - 2 T^{13}}{T^{15}}, - \frac{-2 T^2 + 10 T^3 - 15 T^4 + 10 T^5 - 2 T^6}{T^8} \right\}$$

(Alt) Out[ ]=

{True, True, False, False, True}

```
In[ ]:= Total@Table[AlexanderMatrices[K];
  Simplify[vi.A0^T.C2.P1.v.P2 == (Aπ /. T -> T^-1) ^ (C2 /. T -> T^-1) == Inverse[C2]],
  {K, AllKnots[{3, 10}]}]
```

Out[ ]=

249 True

```
In[ ]:= AlexanderMatrices[Knot[3, 1]];
{lhs, rhs} = Simplify@{vi.A0^T.C2.P1.v.P2, Aπ /. T -> T^-1};
MatrixForm /@ {C2, C2.P1, P1.v.P2, Inverse[P1.v.P2], v, lhs, rhs}
Det /@ {lhs, rhs}
Simplify[lhs == rhs]
```

Out[ ]=

$$\left\{ \begin{pmatrix} -1 & 0 & 0 & -1+T & 0 & 0 & 0 \\ 0 & -T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1+T & 0 \\ 0 & 0 & 0 & -T & 0 & 0 & 0 \\ 0 & -1+T & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & -1+T & 0 & 0 \\ 0 & 0 & -T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1+T \\ 0 & 0 & 0 & 0 & -T & 0 & 0 \\ 0 & 0 & -1+T & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -T \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\begin{pmatrix} -T & T & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -T & 0 & 0 & T & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -T & 0 & 0 & 0 & 0 & T & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{T} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{T} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left. \begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

Out[ ]=

$$\{1 - T + T^2, 1 - T + T^2\}$$

Out[ ]=

True

```
In[ ]:= AlexanderMatrices[Knot[5, 2]];
{lhs, rhs} = Simplify@{vi.A0^T.C2.P1.v.P2, Aπ /. T -> T^-1};
MatrixForm /@ {C2, Inverse[C2], v, P1, P2, A0, Aπ}
Simplify[(C2 /. T -> T^-1) == Inverse[C2]]
lhs == rhs
```

Out[ ]=

$$\left\{ \begin{pmatrix} -1 & 0 & 0 & -1+T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1+T & 0 & 0 \\ 0 & 0 & 0 & -T & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1+T & 0 \\ 0 & 0 & 0 & 0 & 0 & -T & 0 & 0 & 0 & 0 \\ 0 & -1+T & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1+T & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \right.$$



$$\left( \begin{array}{cccccccccccc} 1 & -\frac{1}{T} & 0 & 0 & -1 + \frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & -1 + \frac{1}{T} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & 0 & -1 + \frac{1}{T} \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 + \frac{1}{T} & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 + \frac{1}{T} & 0 & 1 & -\frac{1}{T} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right)$$

Out[\*]=  
True

Out[\*]=  
True

```
(Alt) In[*]:=
QF[m_?MatrixQ] := Sum[Factor[m[[i, j]]] x_i p_j, {i, 2 n + 1}, {j, 2 n + 1}]
```

(Alt) In[\*]:=
 AlexanderMatrices[Mirror[Knot[3, 1]]]

(Alt) In[\*]:=
 Simplify[vi.A0^T.C2.P1.v.P2 == (Aπ /. T -> T^-1)]

(Alt) Out[\*]=  
True

(Alt) In[\*]:=
 QF[A0] == p1 x1 - T p2 x1 + (-1 + T) p5 x1 + p2 x2 - p3 x2 + p3 x3 - T p4 x3 + (-1 + T) p7 x3 + p4 x4 - p5 x4 + (-1 + T) p3 x5 + p5 x5 - T p6 x5 + p6 x6 - p7 x6 + p7 x7

(Alt) Out[\*]=  
True

(Alt) In[\*]:=
 QF[C2^T.A0]

(Alt) Out[\*]=
 -  $\frac{p_1 x_1}{T}$  + p2 x1 -  $\frac{(-1 + T) p_4 x_1}{T}$  - p2 x2 + p3 x2 -  $\frac{p_3 x_3}{T}$  + p4 x3 -  $\frac{(-1 + T) p_6 x_3}{T}$  - p4 x4 + p5 x4 -  $\frac{(-1 + T) p_2 x_5}{T}$  -  $\frac{p_5 x_5}{T}$  + p6 x5 - p6 x6 + p7 x6 + p7 x7

(Alt) In[\*]:=
 Simplify[QF[C2^T.A0] == -  $\frac{p_1 x_1}{T}$  + p2 x1 -  $\frac{(-1 + T) p_4 x_1}{T}$  - p2 x2 + p3 x2 -  $\frac{p_3 x_3}{T}$  + p4 x3 -  $\frac{(-1 + T) p_6 x_3}{T}$  - p4 x4 + p5 x4 -  $\frac{(-1 + T) p_2 x_5}{T}$  -  $\frac{p_5 x_5}{T}$  + p6 x5 - p6 x6 + p7 x6 + p7 x7]

(Alt) Out[\*]=  
True

(Alt) In[ ]:=

**QF[A0<sup>T</sup>.C2]**

(Alt) Out[ ]:=

$$-\frac{p_1 x_1}{T} + p_1 x_2 - p_2 x_2 - \frac{(-1+T) p_5 x_2}{T} + p_2 x_3 - \frac{p_3 x_3}{T} - \frac{(-1+T) p_1 x_4}{T} +$$

$$p_3 x_4 - p_4 x_4 + p_4 x_5 - \frac{p_5 x_5}{T} - \frac{(-1+T) p_3 x_6}{T} + p_5 x_6 - p_6 x_6 + p_6 x_7 + p_7 x_7$$

(Alt) In[ ]:=

$$\mathbf{QF[A0^T.C2]} = -\frac{p_1 x_1}{T} + p_1 x_2 - p_2 x_2 - \frac{(-1+T) p_5 x_2}{T} + p_2 x_3 - \frac{p_3 x_3}{T} -$$

$$\frac{(-1+T) p_1 x_4}{T} + p_3 x_4 - p_4 x_4 + p_4 x_5 - \frac{p_5 x_5}{T} - \frac{(-1+T) p_3 x_6}{T} + p_5 x_6 - p_6 x_6 + p_6 x_7 + p_7 x_7$$

(Alt) Out[ ]:=

True

(Alt) In[ ]:=

**QF[A0<sup>T</sup>.C2.P1]**

(Alt) Out[ ]:=

$$-\frac{p_2 x_1}{T} + p_2 x_2 - p_3 x_2 - \frac{(-1+T) p_6 x_2}{T} + p_3 x_3 - \frac{p_4 x_3}{T} - \frac{(-1+T) p_2 x_4}{T} +$$

$$p_4 x_4 - p_5 x_4 + p_5 x_5 - \frac{p_6 x_5}{T} - \frac{(-1+T) p_4 x_6}{T} + p_6 x_6 - p_7 x_6 + p_1 x_7 + p_7 x_7$$

(Alt) In[ ]:=

$$\mathbf{QF[A0^T.C2.P1]} = -\frac{p_2 x_1}{T} + p_2 x_2 - p_3 x_2 - \frac{(-1+T) p_6 x_2}{T} + p_3 x_3 - \frac{p_4 x_3}{T} -$$

$$\frac{(-1+T) p_2 x_4}{T} + p_4 x_4 - p_5 x_4 + p_5 x_5 - \frac{p_6 x_5}{T} - \frac{(-1+T) p_4 x_6}{T} + p_6 x_6 - p_7 x_6 + p_1 x_7 + p_7 x_7$$

(Alt) Out[ ]:=

True

(Alt) In[ ]:=

**vi.A0<sup>T</sup>.C2.P1.v // QF**

(Alt) Out[ ]:=

$$-p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1+T) p_6 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1+T) p_2 x_4}{T} +$$

$$p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - \frac{(-1+T) p_4 x_6}{T} + p_6 x_6 - \frac{p_7 x_6}{T} + p_1 x_7 + p_7 x_7$$

(Alt) In[ ]:=

$$\mathbf{QF[vi.A0^T.C2.P1.v]} = -p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1+T) p_6 x_2}{T} + p_3 x_3 - p_4 x_3 -$$

$$\frac{(-1+T) p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - \frac{(-1+T) p_4 x_6}{T} + p_6 x_6 - \frac{p_7 x_6}{T} + p_1 x_7 + p_7 x_7$$

(Alt) Out[ ]:=

True

(Alt) In[ ]:=

**QF [v1.A0T.C2.P1.v.P2]**

(Alt) Out[ ]:=

$$p_1 x_1 - p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1 + T) p_6 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1 + T) p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - \frac{(-1 + T) p_4 x_6}{T} + p_6 x_6 - \frac{p_7 x_6}{T} + p_7 x_7$$

(Alt) In[ ]:=

$$\mathbf{QF [v1.A0T.C2.P1.v.P2]} = p_1 x_1 - p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1 + T) p_6 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1 + T) p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - \frac{(-1 + T) p_4 x_6}{T} + p_6 x_6 - \frac{p_7 x_6}{T} + p_7 x_7$$

(Alt) Out[ ]:=

True