Pensieve header: Palindromicity by flipping and manipulating, in Gaussian integration language.

Initialization and Programs

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];
Once[
  << KnotTheory`;
  << Rot.m
 ];
CF[ℰ_] := Sum[Factor[∂_{x_i,p_j} ℰ] x_i p_j, {i, 0, 2 n + 2}, {j, 0, 2 n + 2}];
```

In[∘]:=
```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} := {g_{iβ_} ⧴ δ_{iβ} + T^s g_{i+1,β} + (1 - T^s) g_{j+1,β}, g_{jβ_} ⧴ δ_{jβ} + g_{j+1,β},
   g_{α_,i} ⧴ T^-s (g_{α,i+1} - δ_{α,i+1}), g_{α_j} ⧴ g_{α,j+1} - (1 - T^s) g_{αi} - δ_{α,j+1}}
```

pdf

In[∘]:=
```
{p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄};  (z_{i__})* := (z*)_i;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^d_. ⧴ (D[f, {z*, d}])) /. z* → 0
```

pdf

In[∘]:=
```
gPair[ℰ_, w_] := Collect[Zip_{Join@@Table[{p_α,p̄_α,x_α,x̄_α},{α,w}]}[
    ℰ Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]], g__, Factor]
```

Playing with a single knot

Initialization

In[∘]:=
```
K = Knot[5, 2]; {Cs, ρ} = Rot[K];  n = Length[Cs]; ν = {lν = 0};
Do[Cs /. {{s_, k, j_} ⧴ AppendTo[ν, lν += s], {s_, i_, k} ⧴ AppendTo[ν, lν -= s]}, {k, 2 n}];
{Cs, ν}
```

Out[∘]=
```
{{{-1, 4, 1}, {-1, 8, 3}, {-1, 10, 5}, {-1, 6, 9}, {-1, 2, 7}},
 {0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0}}
```

The quadratic of *K*:

In[∘]:=
```
Q0 = Echo@CF[Total[
     Cs /. {s_Integer, i_, j_} ⧴ x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2n+1} p_{2n+1}];
```

» $$p_1 x_1 - p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1+T)\, p_8 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1+T)\, p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 + p_6 x_6 - \frac{p_7 x_6}{T} - \frac{(-1+T)\, p_{10} x_6}{T} + p_7 x_7 - p_8 x_7 - \frac{(-1+T)\, p_4 x_8}{T} + p_8 x_8 - \frac{p_9 x_8}{T} + p_9 x_9 - p_{10} x_9 - \frac{(-1+T)\, p_6 x_{10}}{T} + p_{10} x_{10} - \frac{p_{11} x_{10}}{T} + p_{11} x_{11}$$

Applying $x_j \to -x_j + (T^{-s} - 1)\, x_i$, $x_i \to -T^{-s}\, x_i$ and splitting off the edge terms: (Jacobian is $T^{-\text{writhe}}$)

In[∘]:=
```
Echo@CF[Q0 /. Join @@ (Cs /. {s_Integer, i_, j_} :> {x_j → -x_j + (T^-s - 1) x_i, x_i → -T^-s x_i})] ==
  (Q1 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s p_i x_i - p_j x_i + T^-s p_j x_i - p_j x_j] +
      Sum[x_k p_(k+1), {k, 1, 2 n}] + x_(2 n+1) p_(2 n+1)])
```

» $-p_1 x_1 + p_2 x_1 - T p_2 x_2 + p_3 x_2 + (-1+T)\, p_7 x_2 - p_3 x_3 + p_4 x_3 + (-1+T)\, p_1 x_4 - T p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - T p_6 x_6 + p_7 x_6 + (-1+T)\, p_9 x_6 - p_7 x_7 + p_8 x_7 + (-1+T)\, p_3 x_8 - T p_8 x_8 + p_9 x_8 - p_9 x_9 + p_{10} x_9 + (-1+T)\, p_5 x_{10} - T p_{10} x_{10} + p_{11} x_{10} + p_{11} x_{11}$

Out[∘]=
True

Transposing, shifting from forward edges to backwards edges: (Jacobian is (−1)).

In[∘]:=
```
(Q1 /. {p → x, x → p}) ==
 Echo@(Q2 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_i - x_j p_i + T^-s x_j p_i - x_j p_j] +
      Sum[p_k x_(k+1), {k, 1, 2 n}] + p_(2 n+1) x_(2 n+1)]) ==
 CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_i - x_j p_i + T^-s x_j p_i - x_j p_j] +
   Sum[p_(k-1) x_k, {k, 1, 2 n}] - p_0 x_1 + p_(2 n) x_(2 n+1) + p_(2 n+1) x_(2 n+1)]
```

» $-p_1 x_1 + (-1+T)\, p_4 x_1 + p_1 x_2 - T p_2 x_2 + p_2 x_3 - p_3 x_3 + (-1+T)\, p_8 x_3 + p_3 x_4 - T p_4 x_4 + p_4 x_5 - p_5 x_5 + (-1+T)\, p_{10} x_5 + p_5 x_6 - T p_6 x_6 + (-1+T)\, p_2 x_7 + p_6 x_7 - p_7 x_7 + p_7 x_8 - T p_8 x_8 + (-1+T)\, p_6 x_9 + p_8 x_9 - p_9 x_9 + p_9 x_{10} - T p_{10} x_{10} + p_{10} x_{11} + p_{11} x_{11}$

Out[∘]=
True

Permuting the p variables and re-absorbing the edge terms into the crossings: (Jacobian is 1).

In[∘]:=
```
(Q2 /. {p_(2 n+1) → p_1, p_i_ :> p_(i+1)}) ==
 Echo@(Q3 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_(i+1) - x_j p_(i+1) + T^-s x_j p_(i+1) - x_j p_(j+1)] +
      Sum[p_k x_k, {k, 1, 2 n}] - p_1 x_1 + p_(2 n+1) x_(2 n+1) + p_1 x_(2 n+1)]) ==
 CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_(i+1) + (T^-s - 1) x_j p_(i+1) - x_j p_(j+1) + p_i x_i + p_j x_j] -
   p_1 x_1 + p_(2 n+1) x_(2 n+1) + p_1 x_(2 n+1)]
```

» $-p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - T p_3 x_2 + p_3 x_3 - p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - T p_5 x_4 + p_5 x_5 - p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - T p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - p_8 x_7 + p_8 x_8 - T p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - p_{10} x_9 + p_{10} x_{10} - T p_{11} x_{10} + p_1 x_{11} + p_{11} x_{11}$

Out[∘]=
True

Rescaling by T^{ν}: (Jacobian is 1).

In[∘]:=
```
(Q4 = Echo@CF[Q3 /. {p_i_ :> T^ν[[i]] p_i, x_i_ :> T^-ν[[i]] x_i}]) ==
 CF[Total[Cs /. {s_Integer, i_, j_} :> -x_i p_(i+1) + (T^-s - 1) x_j p_(i+1) - T^-s x_j p_(j+1) + p_i x_i + p_j x_j] -
   p_1 x_1 + p_(2 n+1) x_(2 n+1) + p_1 x_(2 n+1)]
```

» $-T p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_1 x_{11} + p_{11} x_{11}$

Out[∘]=
True

Using “col-sum = 0”: (Jacobian is 1, and so the overall Jacobian is $-T^{-\text{writhe}}$).

In[∘]:=
```
Simplify[Echo@CF[Q4 /. p_k_ /; k > 1 :→ p_k - p_1] ==
  (Q5 = CF[Total[Cs /. {s_Integer, i_, j_} :→ x_j (p_j - T^-s p_(j+1) + (T^-s - 1) p_(i+1)) + x_i (p_i - p_(i+1))] +
      p_(2 n+1) x_(2 n+1)]])]
```

» $p_1 x_1 - T p_2 x_1 + (-1 + T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1 + T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1 + T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1 + T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1 + T) p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11}$

Out[∘]=

True

The conjugate quadratic of the flip of *K*:

In[∘]:=
```
Q̄π = Echo@CF[Total[
     Cs /. {s_Integer, j_, i_} :→ x_i (p_i - T^-s p_(i+1) + (T^-s - 1) p_(j+1)) + x_j (p_j - p_(j+1))] + x_(2 n+1) p_(2 n+1)];
```

» $p_1 x_1 - T p_2 x_1 + (-1 + T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1 + T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1 + T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1 + T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1 + T) p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11}$

In[∘]:=
```
Q̄π == Q5
```

Out[∘]=

True

Playing with a single knot - all in one shot

In[∘]:=

```
K = Knot[5, 2]; {Cs, ρ} = Rot[K];  n = Length[Cs]; ν = {lν = 0};
Do[
  Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
Echo@{Cs, ν};
Q0p = Echo@CF[Total[
      Cs /. {s_Integer, i_, j_} :> x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2 n+1} p_{2 n+1}];
Echo[Q0p == Q0];
Q1p =
  Echo@CF[Q0p /. Join @@ (Cs /. {s_Integer, i_, j_} :> {x_j → -x_j + (T^-s - 1) x_i, x_i → -T^-s x_i})];
Echo[Q1p == Q1];
Q2p =
  Echo@CF[Q0p /. Join @@ (Cs /. {s_Integer, i_, j_} :> {x_j → -x_j + (T^-s - 1) x_i, x_i → -T^-s x_i}) /.
     {p → x, x → p}];
Echo[Q2p == Q2];
Q3p = Echo@CF[Q0p /. Flatten@{{x_{2 n+1} → p_{2 n+1}, p_{2 n+1} → x_{2 n+1}},
       Cs /. {s_Integer, i_, j_} :> {x_j → -p_j + (T^-s - 1) p_i, x_i → -T^-s p_i, p_i → x_i, p_j → x_j}}];
Print["Testing Q3p: ", Simplify[Q3p - Q2]];
Q4p = Echo@CF[Q0p /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}}, Cs /.
        {s_Integer, i_, j_} :> {x_j → -p_{j+1} + (T^-s - 1) p_{i+1}, x_i → -T^-s p_{i+1}, p_i → x_i, p_j → x_j}}];
Print["Testing Q4p: ", Simplify[Q4p - Q3]];
Q5p = Echo@CF[
    Q0p /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}}, Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[j+1]] p_{j+1} +
            (T^-s - 1) T^ν[[i+1]] p_{i+1}, x_i → -T^-s T^ν[[i+1]] p_{i+1}, p_i → T^-ν[[i]] x_i, p_j → T^-ν[[j]] x_j}}];
Print["Testing Q5p: ", Simplify[Q5p - Q4]];
(* Q4=Echo@CF[Q3/.{p_i_:>T^ν[[i]] p_i,x_i_:>T^-ν[[i]] x_i}] *)
Q6p = Echo@CF[Q0p /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}}, Cs /. {s_Integer, i_, j_} :>
        {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1}, x_i → -T^ν[[i]] p_{i+1}, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
Print["Testing Q6p: ", Simplify[Q6p - Q4]];
Q7p = Echo@CF[Q0p /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
        Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1},
           x_i → -T^ν[[i]] p_{i+1}, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}} /. p_k_ /; k > 1 :> p_k - p_1];
Print["Testing Q7p: ", Simplify[Q7p - Q5]];
Q8p = Echo@CF[Q0p /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
       Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1} + T^(s+ν[[i]]) p_1,
          x_i → -T^ν[[i]] p_{i+1} + T^ν[[i]] p_1, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
Print["Testing Q8p: ", Simplify[Q8p - Q5]];
```

» {{{-1, 4, 1}, {-1, 8, 3}, {-1, 10, 5}, {-1, 6, 9}, {-1, 2, 7}}, {0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0}}

» $p_1 x_1 - p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1+T)\, p_8 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1+T)\, p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 + p_6 x_6 - \frac{p_7 x_6}{T} - \frac{(-1+T)\, p_{10} x_6}{T} + p_7 x_7 - p_8 x_7 - \frac{(-1+T)\, p_4 x_8}{T} + p_8 x_8 - \frac{p_9 x_8}{T} + p_9 x_9 - p_{10} x_9 - \frac{(-1+T)\, p_6 x_{10}}{T} + p_{10} x_{10} - \frac{p_{11} x_{10}}{T} + p_{11} x_{11}$

» True

» $-p_1 x_1 + p_2 x_1 - T p_2 x_2 + p_3 x_2 + (-1+T)\, p_7 x_2 - p_3 x_3 + p_4 x_3 + (-1+T)\, p_1 x_4 - T p_4 x_4 + p_5 x_4 - p_5 x_5 + p_6 x_5 - T p_6 x_6 + p_7 x_6 + (-1+T)\, p_9 x_6 - p_7 x_7 + p_8 x_7 + (-1+T)\, p_3 x_8 - T p_8 x_8 + p_9 x_8 - p_9 x_9 + p_{10} x_9 + (-1+T)\, p_5 x_{10} - T p_{10} x_{10} + p_{11} x_{10} + p_{11} x_{11}$

» True

» $-p_1 x_1 + (-1+T)\, p_4 x_1 + p_1 x_2 - T p_2 x_2 + p_2 x_3 - p_3 x_3 + (-1+T)\, p_8 x_3 + p_3 x_4 - T p_4 x_4 + p_4 x_5 - p_5 x_5 + (-1+T)\, p_{10} x_5 + p_5 x_6 - T p_6 x_6 + (-1+T)\, p_2 x_7 + p_6 x_7 - p_7 x_7 + p_7 x_8 - T p_8 x_8 + (-1+T)\, p_6 x_9 + p_8 x_9 - p_9 x_9 + p_9 x_{10} - T p_{10} x_{10} + p_{10} x_{11} + p_{11} x_{11}$

» True

» $-p_1 x_1 + (-1+T)\, p_4 x_1 + p_1 x_2 - T p_2 x_2 + p_2 x_3 - p_3 x_3 + (-1+T)\, p_8 x_3 + p_3 x_4 - T p_4 x_4 + p_4 x_5 - p_5 x_5 + (-1+T)\, p_{10} x_5 + p_5 x_6 - T p_6 x_6 + (-1+T)\, p_2 x_7 + p_6 x_7 - p_7 x_7 + p_7 x_8 - T p_8 x_8 + (-1+T)\, p_6 x_9 + p_8 x_9 - p_9 x_9 + p_9 x_{10} - T p_{10} x_{10} + p_{10} x_{11} + p_{11} x_{11}$

Testing Q3p: 0

» $-p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - T p_3 x_2 + p_3 x_3 - p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - T p_5 x_4 + p_5 x_5 - p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - T p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - p_8 x_7 + p_8 x_8 - T p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - p_{10} x_9 + p_{10} x_{10} - T p_{11} x_{10} + p_1 x_{11} + p_{11} x_{11}$

Testing Q4p: 0

» $-T p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_1 x_{11} + p_{11} x_{11}$

Testing Q5p: 0

» $-T p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_1 x_{11} + p_{11} x_{11}$

Testing Q6p: 0

» $p_1 x_1 - T p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11}$

Testing Q7p: 0

» $p_1 x_1 - T p_2 x_1 + (-1+T)\, p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T)\, p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T)\, p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T)\, p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T)\, p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11}$

Testing Q8p: 0

The conjugate quadratic of the flip of *K*:

In[∘]:=
```
Q̅π = Echo@CF[Total[
    Cs /. {s_Integer, j_, i_} :> x_i (p_i - T^-s p_{i+1} + (T^-s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2 n+1} p_{2 n+1}];
```

» $p_1 x_1 - p_2 x_1 + p_2 x_2 - \frac{p_3 x_2}{T} - \frac{(-1+T)\, p_6 x_2}{T} + p_3 x_3 - p_4 x_3 - \frac{(-1+T)\, p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - \frac{(-1+T)\, p_4 x_6}{T} + p_6 x_6 - \frac{p_7 x_6}{T} + p_7 x_7$

In[∘]:=
```
Q̅π == CF[Q /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}}, Cs /.
    {s_Integer, i_, j_} :> {x_j → T^s p_1 - p_{j+1} + (1 - T^s) p_{i+1}, x_i → p_1 - p_{i+1}, p_i → x_i, p_j → T^-s x_j}}]
```

Out[∘]=

True

In[∘]:=
```
Total@Table[{Cs, ρ} = Rot[K]; n = Length[Cs];
  Q = CF[Total[
      Cs /. {s_Integer, i_, j_} :→ x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2n+1} p_{2n+1}];
  \overline{Qπ} = CF[Total[
      Cs /. {s_Integer, j_, i_} :→ x_i (p_i - T^-s p_{i+1} + (T^-s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2n+1} p_{2n+1}];
  \overline{Qπ} == CF[Q /. Flatten@{{x_{2n+1} → p_1, p_{2n+1} → x_{2n+1}}, Cs /. {s_Integer, i_, j_} :→
        {x_j → T^s p_1 - p_{j+1} + (1 - T^s) p_{i+1}, x_i → p_1 - p_{i+1}, p_i → x_i, p_j → T^-s x_j}}],
  {K, AllKnots[{3, 5}]}
 ]
```

Out[∘]=

$$\Big(p_1 x_1 - T p_2 x_1 + (-1+T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_7 x_3 + p_4 x_4 - p_5 x_4 + (-1+T) p_3 x_5 + p_5 x_5 - T p_6 x_5 + p_6 x_6 - p_7 x_6 + p_7 x_7 == p_1 x_1 - T p_2 x_1 + (-1+T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_7 x_3 + p_4 x_4 - p_5 x_4 + (-1+T) p_3 x_5 + p_5 x_5 - T p_6 x_5 + p_6 x_6 - p_7 x_6 + \frac{(-1+T) p_1 x_7}{T} + \frac{p_7 x_7}{T}\Big) +$$

$$\Big(p_1 x_1 - p_2 x_1 + p_2 x_2 - T p_3 x_2 + (-1+T) p_8 x_2 + p_3 x_3 - p_4 x_3 - \frac{(-1+T) p_2 x_4}{T} + p_4 x_4 - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 + (-1+T) p_4 x_6 + p_6 x_6 - T p_7 x_6 + p_7 x_7 - p_8 x_7 - \frac{(-1+T) p_6 x_8}{T} + p_8 x_8 - \frac{p_9 x_8}{T} + p_9 x_9 == p_1 x_1 - p_2 x_1 - (-1+T)(1+T) p_1 x_2 + T^2 p_2 x_2 - T p_3 x_2 + (-1+T) p_8 x_2 + p_3 x_3 - p_4 x_3 + \frac{(-1+T)(1+T) p_1 x_4}{T^2} - \frac{(-1+T) p_2 x_4}{T} + \frac{p_4 x_4}{T^2} - \frac{p_5 x_4}{T} + p_5 x_5 - p_6 x_5 - (-1+T)(1+T) p_1 x_6 + (-1+T) p_4 x_6 + T^2 p_6 x_6 - T p_7 x_6 + p_7 x_7 - p_8 x_7 + \frac{(-1+T)(1+T) p_1 x_8}{T^2} - \frac{(-1+T) p_6 x_8}{T} + \frac{p_8 x_8}{T^2} - \frac{p_9 x_8}{T} + p_9 x_9\Big) +$$

$$\Big(p_1 x_1 - T p_2 x_1 + (-1+T) p_7 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T) p_5 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11} == p_1 x_1 - T p_2 x_1 + (-1+T) p_7 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T) p_5 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + \frac{(-1+T) p_1 x_{11}}{T} + \frac{p_{11} x_{11}}{T}\Big) +$$

$$\Big(p_1 x_1 - T p_2 x_1 + (-1+T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T) p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + p_{11} x_{11} == p_1 x_1 - T p_2 x_1 + (-1+T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1+T) p_9 x_3 + p_4 x_4 - p_5 x_4 + p_5 x_5 - T p_6 x_5 + (-1+T) p_{11} x_5 + p_6 x_6 - p_7 x_6 + (-1+T) p_3 x_7 + p_7 x_7 - T p_8 x_7 + p_8 x_8 - p_9 x_8 + (-1+T) p_7 x_9 + p_9 x_9 - T p_{10} x_9 + p_{10} x_{10} - p_{11} x_{10} + \frac{(-1+T) p_1 x_{11}}{T} + \frac{p_{11} x_{11}}{T}\Big)$$

Playing with R_1

r_1 is taken from Talks/Oaxaca-2210/Rho.nb

In[∘]:= $\text{P0} = s\left(-1 + 2\,p_i\,x_i - 2\,p_j\,x_i + \left(-1 + T^s\right) p_i\,p_j\,x_i^2 + \left(1 - T^s\right) p_j^2\,x_i^2 - 2\,p_i\,p_j\,x_i\,x_j + 2\,p_j^2\,x_i\,x_j\right) / 2$

Out[∘]=

$$\frac{1}{2}\,s\left(-1 + 2\,p_i\,x_i - 2\,p_j\,x_i + \left(-1 + T^s\right) p_i\,p_j\,x_i^2 + \left(1 - T^s\right) p_j^2\,x_i^2 - 2\,p_i\,p_j\,x_i\,x_j + 2\,p_j^2\,x_i\,x_j\right)$$

In[∘]:= $\text{lhs} = \text{P0}\ /.\ \left\{i \to j,\ j \to i,\ T^s \to T^{-s}\right\}$

Out[∘]=

$$\frac{1}{2}\,s\left(-1 - 2\,p_i\,x_j + 2\,p_j\,x_j + 2\,p_i^2\,x_i\,x_j - 2\,p_i\,p_j\,x_i\,x_j + \left(1 - T^{-s}\right) p_i^2\,x_j^2 + \left(-1 + T^{-s}\right) p_i\,p_j\,x_j^2\right)$$

In[∘]:= $\text{rhs} = \text{Expand}\left[\text{P0}\ /.\ \left\{x_j \to T^s\,p_1 - p_{j+1} + \left(1 - T^s\right) p_{i+1},\ x_i \to p_1 - p_{i+1},\ p_i \to x_i,\ p_j \to T^{-s}\,x_j\right\}\right]$

Out[∘]=

$$-\frac{s}{2} + s\,p_1\,x_i - s\,p_{1+i}\,x_i - s\,T^{-s}\,p_1\,x_j + s\,T^{-s}\,p_{1+i}\,x_j - \frac{1}{2}\,s\,p_1^2\,x_i\,x_j - \frac{1}{2}\,s\,T^{-s}\,p_1^2\,x_i\,x_j + s\,p_1\,p_{1+i}\,x_i\,x_j -$$
$$\frac{1}{2}\,s\,p_{1+i}^2\,x_i\,x_j + \frac{1}{2}\,s\,T^{-s}\,p_{1+i}^2\,x_i\,x_j + s\,T^{-s}\,p_1\,p_{1+j}\,x_i\,x_j - s\,T^{-s}\,p_{1+i}\,p_{1+j}\,x_i\,x_j + \frac{1}{2}\,s\,T^{-2\,s}\,p_1^2\,x_j^2 +$$
$$\frac{1}{2}\,s\,T^{-s}\,p_1^2\,x_j^2 - s\,T^{-s}\,p_1\,p_{1+i}\,x_j^2 - \frac{1}{2}\,s\,T^{-2\,s}\,p_{1+i}^2\,x_j^2 + \frac{1}{2}\,s\,T^{-s}\,p_{1+i}^2\,x_j^2 - s\,T^{-2\,s}\,p_1\,p_{1+j}\,x_j^2 + s\,T^{-2\,s}\,p_{1+i}\,p_{1+j}\,x_j^2$$

In[∘]:= $\text{Simplify}\left[\text{gPair}[\text{lhs} - \text{rhs},\ \{1, i, j, i+1, j+1\}]\ /.\right.$
$\left\{g_{i\beta\_} \mapsto \delta_{i\beta} + T^s\,g_{i+1,\beta} + \left(1 - T^s\right) g_{j+1,\beta},\ g_{j\beta\_} \mapsto \delta_{j\beta} + g_{j+1,\beta},\right.$
$\left.\left. g_{\alpha\_,i} \mapsto T^{-s}\left(g_{\alpha,i+1} - \delta_{\alpha,i+1}\right),\ g_{\alpha\_j} \mapsto g_{\alpha,j+1} - \left(1 - T^s\right) g_{\alpha i} - \delta_{\alpha,j+1}\right\}\right]$

Out[∘]=

$$s\left(1 - T^{-2\,s}\left(1 + T^s\right)\left(\left(-1 + T^s\right) g_{1,i} + g_{1,1+j}\right)^2 -\right.$$
$$T^{-2\,s}\left(-1 + T^s\right)\left(\left(-1 + T^s\right) g_{1+i,i} + g_{1+i,1+j}\right)^2 + g_{1+j,j} - \left(1 + T^s\,g_{1+i,i} - \left(-1 + T^s\right) g_{1+j,i}\right)\left(1 + g_{1+j,j}\right) +$$
$$T^{-s}\left(1 + 2\,T^{2\,s}\,g_{1+i,i} + \left(T^s - 2\,T^{2\,s}\right) g_{1+j,i} + g_{1+j,j} - T^s\,g_{1+j,j}\right)\left(T^s\,g_{1+i,j} - \left(-1 + T^s\right) g_{1+j,j}\right) +$$
$$T^{-s}\left(-1 + T^s\right)\left(T^s\,g_{1+i,j} - \left(-1 + T^s\right) g_{1+j,j}\right)^2 + T^{-2\,s}\,g_{1,1+i}\left(1 - T^s + \left(-1 + T^{2\,s}\right) g_{1,i} +\right.$$
$$\left.\left(1 + T^s\right) g_{1,1+j} + T^s\,g_{1+i,i} - T^{2\,s}\,g_{1+i,i} - T^s\,g_{1+i,1+j} + g_{1+j,i} - T^s\,g_{1+j,i} - g_{1+j,1+j}\right) +$$
$$T^{-2\,s}\left(-1 + g_{1+i,1+i}\right)\left(-1 + T^s + \left(-1 + T^s\right)^2 g_{1+i,i} + \left(-1 + T^s\right) g_{1+i,1+j} - g_{1+j,i} + T^s\,g_{1+j,i} + g_{1+j,1+j}\right) +$$
$$T^{-2\,s}\left(\left(-1 + T^s\right) g_{1,i} + g_{1,1+j}\right)\left(-2 + 2\,T^s + 2\,T^s\left(-1 + T^s\right) g_{1+i,i} -\right.$$
$$\left.T^s\,g_{1+i,1+i} + 2\,T^s\,g_{1+i,1+j} - 2\,g_{1+j,i} + 2\,T^s\,g_{1+j,i} - g_{1+j,1+i} + 2\,g_{1+j,1+j}\right) -$$
$$\left.T^{-2\,s}\left(\left(-1 + T^s\right) g_{1+i,i} + g_{1+i,1+j}\right)\left(-2 + T^s + 2\left(-1 + T^s\right) g_{1+j,i} - g_{1+j,1+i} + 2\,g_{1+j,1+j}\right)\right)$$

In[∘]:= $\left\{g_{i\beta\_} \mapsto \delta_{i\beta} + T^s\,g_{i+1,\beta} + \left(1 - T^s\right) g_{j+1,\beta},\ g_{j\beta\_} \mapsto \delta_{j\beta} + g_{j+1,\beta},\ g_{\alpha\_,i} \mapsto T^{-s}\left(g_{\alpha,i+1} - \delta_{\alpha,i+1}\right),\right.$
$\left.g_{\alpha\_j} \mapsto g_{\alpha,j+1} - \left(1 - T^s\right) g_{\alpha i} - \delta_{\alpha,j+1}\right\}\ /.\ \left\{i \to j,\ j \to i,\ T^s \to T^{-s},\ T^{-s} \to T^s\right\}$

Out[∘]=

$$\left\{g_{j,\beta\_} \mapsto \delta_{j,\beta} + T^{-s}\,g_{j+1,\beta} + \left(1 - T^{-s}\right) g_{i+1,\beta},\ g_{i,\beta\_} \mapsto \delta_{i,\beta} + g_{i+1,\beta},\right.$$
$$\left.g_{\alpha\_,j} \mapsto T^s\left(g_{\alpha,j+1} - \delta_{\alpha,j+1}\right),\ g_{\alpha\_,i} \mapsto g_{\alpha,i+1} - \left(1 - T^{-s}\right) g_{\alpha,j} - \delta_{\alpha,i+1}\right\}$$

In[∘]:= **Simplify[gPair[lhs − rhs, {1, i, j, i + 1, j + 1}] //. $\{g_{j,\beta\_} \Rightarrow \delta_{j,\beta} + T^{-s} g_{j+1,\beta} + (1 - T^{-s})\, g_{i+1,\beta},\ g_{i,\beta\_} \Rightarrow \delta_{i,\beta} + g_{i+1,\beta},\ g_{\alpha\_,j} \Rightarrow T^{s} (g_{\alpha,j+1} - \delta_{\alpha,j+1}),\ g_{\alpha\_,i} \Rightarrow g_{\alpha,i+1} - (1 - T^{-s})\, g_{\alpha,j} - \delta_{\alpha,i+1}\}$]**

Out[∘]=

$$-s \left(T^{s} (1 + T^{s})\, g^{2}_{1,1+j} + g_{1,1+i} \left(-\left((1 + T^{s})\, g_{1,1+j}\right) + T^{s} g_{1+i,1+j} + g_{1+j,1+j}\right) + (1 + T^{s})\, g_{1+i,1+j} \left(-g_{1+i,1+i} + (-1 + T^{s})\, g_{1+i,1+j} + g_{1+j,1+j}\right) + g_{1,1+j} \left(T^{s} g_{1+i,1+i} - 2 T^{2 s} g_{1+i,1+j} + g_{1+j,1+i} - 2 T^{s} g_{1+j,1+j}\right)\right)$$

In[∘]:= **e1 = Simplify[gPair[lhs − rhs, {i, j, i + 1, j + 1}] //. $\{g_{j,\beta\_} \Rightarrow \delta_{j,\beta} + T^{-s} g_{j+1,\beta} + (1 - T^{-s})\, g_{i+1,\beta},\ g_{i,\beta\_} \Rightarrow \delta_{i,\beta} + g_{i+1,\beta},\ g_{\alpha\_,j} \Rightarrow T^{s} (g_{\alpha,j+1} - \delta_{\alpha,j+1}),\ g_{\alpha\_,i} \Rightarrow g_{\alpha,i+1} - (1 - T^{-s})\, g_{\alpha,j} - \delta_{\alpha,i+1}\}$]**

Out[∘]=

$$-s (1 + T^{s})\, g_{1+i,1+j} \left(-g_{1+i,1+i} + (-1 + T^{s})\, g_{1+i,1+j} + g_{1+j,1+j}\right)$$

In[∘]:= **c1 = Simplify[$s \left(g^{2}_{i,i} + g^{2}_{j,j} - g^{2}_{i+1,i+1} - g^{2}_{j+1,j+1}\right)$ //. $\{g_{j,\beta\_} \Rightarrow \delta_{j,\beta} + T^{-s} g_{j+1,\beta} + (1 - T^{-s})\, g_{i+1,\beta},\ g_{i,\beta\_} \Rightarrow \delta_{i,\beta} + g_{i+1,\beta},\ g_{\alpha\_,j} \Rightarrow T^{s} (g_{\alpha,j+1} - \delta_{\alpha,j+1}),\ g_{\alpha\_,i} \Rightarrow g_{\alpha,i+1} - (1 - T^{-s})\, g_{\alpha,j} - \delta_{\alpha,i+1}\}$]**

Out[∘]=

$$2 s (-1 + T^{s})\, g_{1+i,1+j} \left(-g_{1+i,1+i} + (-1 + T^{s})\, g_{1+i,1+j} + g_{1+j,1+j}\right)$$

In[∘]:= **c2 = Simplify[$\left(g^{2}_{j,j} - g^{2}_{i+1,i+1}\right)$ //. $\{g_{j,\beta\_} \Rightarrow \delta_{j,\beta} + T^{-s} g_{j+1,\beta} + (1 - T^{-s})\, g_{i+1,\beta},\ g_{i,\beta\_} \Rightarrow \delta_{i,\beta} + g_{i+1,\beta},\ g_{\alpha\_,j} \Rightarrow T^{s} (g_{\alpha,j+1} - \delta_{\alpha,j+1}),\ g_{\alpha\_,i} \Rightarrow g_{\alpha,i+1} - (1 - T^{-s})\, g_{\alpha,j} - \delta_{\alpha,i+1}\}$]**

Out[∘]=

$$-g^{2}_{1+i,1+i} + \left((-1 + T^{s})\, g_{1+i,1+j} + g_{1+j,1+j}\right)^{2}$$

In[∘]:= **Simplify[e1 − a1 c1 − a2 c2]**

Out[∘]=

$$\left(g_{1+i,1+i} - (-1 + T^{s})\, g_{1+i,1+j} - g_{1+j,1+j}\right) \left(a2\, g_{1+i,1+i} + \left(a2 (-1 + T^{s}) + s \left(1 + T^{s} + 2\, a1 (-1 + T^{s})\right)\right) g_{1+i,1+j} + a2\, g_{1+j,1+j}\right)$$

In[∘]:= **Simplify[e1 − c1 / 2]**

Out[∘]=

$$-2 s T^{s} g_{1+i,1+j} \left(-g_{1+i,1+i} + (-1 + T^{s})\, g_{1+i,1+j} + g_{1+j,1+j}\right)$$