

Pensieve header: Other near-Wirtinger matrices to produce the Alexander polynomial.

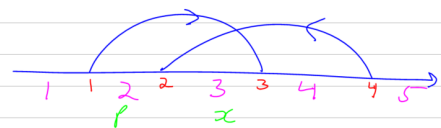
Programs

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In[ ]:= Once[<< KnotTheory`];

In[ ]:= PD[GST48] = PD[X[1, 15, 2, 14], X[29, 2, 30, 3], X[40, 4, 41, 3],
  X[4, 44, 5, 43], X[5, 26, 6, 27], X[95, 7, 96, 6], X[7, 1, 8, 96], X[8, 14, 9, 13],
  X[28, 9, 29, 10], X[41, 11, 42, 10], X[11, 43, 12, 42], X[12, 27, 13, 28],
  X[15, 31, 16, 30], X[61, 16, 62, 17], X[72, 17, 73, 18], X[83, 18, 84, 19],
  X[34, 20, 35, 19], X[20, 89, 21, 90], X[92, 21, 93, 22], X[22, 79, 23, 80],
  X[23, 68, 24, 69], X[24, 57, 25, 58], X[56, 25, 57, 26], X[31, 63, 32, 62],
  X[32, 74, 33, 73], X[33, 85, 34, 84], X[35, 50, 36, 51], X[81, 37, 82, 36],
  X[70, 38, 71, 37], X[59, 39, 60, 38], X[54, 39, 55, 40], X[55, 45, 56, 44],
  X[45, 59, 46, 58], X[46, 70, 47, 69], X[47, 81, 48, 80], X[91, 49, 92, 48],
  X[49, 91, 50, 90], X[82, 52, 83, 51], X[71, 53, 72, 52], X[60, 54, 61, 53],
  X[74, 63, 75, 64], X[85, 64, 86, 65], X[65, 76, 66, 77], X[66, 87, 67, 88],
  X[94, 67, 95, 68], X[86, 75, 87, 76], X[77, 88, 78, 89], X[93, 78, 94, 79]];
  
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Multi-Alexanders



$R_{ij}^s = T^{s/2} p^{(T^s - 1)(p_i - p_j)} X_{ij}$
 $Xp = px - 1 \downarrow$
 $G_{\alpha\beta} = \langle R_{\alpha} X_{\beta} \rangle$ & with effort: $\tilde{G}_{\alpha\beta} = \langle X_{\alpha} X_{\beta} \rangle = G_{\alpha\beta} - \delta_{\alpha\beta}$
 $G_{1,1\beta} = 0$
 $G_{1,1\beta} = 0$

X_{ij}^s make $B \in M_{2n \times (2n+1)}$

$$\begin{cases} \text{row } i & \tilde{G}_{i,1\beta} - G_{i+1,\beta} = 0 \Leftrightarrow G_{i\beta} - G_{i+1,\beta} = \delta_{i\beta} \\ \text{row } j & \tilde{G}_{j\beta} - G_{j+1,\beta} - (T^s - 1)(G_{i+1,\beta} - \tilde{G}_{i\beta}) = 0 \\ & \Leftrightarrow T^s G_{j\beta} - G_{j+1,\beta} + (1 - T^s)G_{i+1,\beta} = T^s \delta_{j\beta} \end{cases}$$

$B = (\phi | A) \quad G = \begin{pmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad BG = \begin{pmatrix} I_{2n \times 2n} & 0 \end{pmatrix}$
 $AD = I$

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In[ ]:= Als[K_Knot | K_PD] := Als[GD@@Cases[PD@K, x_X -> {
  Xp[x[[4]], x[[1]] PositiveQ@x
  Xm[x[[2]], x[[1]] True
}]];

Als[gd_GD] := (gd;
  n = Length[gd];
  B1 = B2 = B3 = B4 = B5 = Table[0, 2 n, 2 n + 1];
  Do[s = If[Head[c] === Xp, 1, -1]; {i, j} = List@@c;

  B1[{i, j}, {i, i + 1, j, j + 1}] =  $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & T^s & -1 & 1 & -T^s \end{pmatrix}$ ; (* The Original B *)

  B2[{i, j}, {i, i + 1, j, j + 1}] =  $\begin{pmatrix} 1 & -1/\alpha & 0 & 0 \\ 0 & T^s & -1 & 1 & -\alpha T^s \end{pmatrix}$ ; (* Depth-twisting by  $\alpha^d$  *)

  B3[{i, j}, {i, i + 1, j, j + 1}] =  $\begin{pmatrix} 1 & -T^s & 0 & 0 \\ 0 & T^s & -1 & 1 & -1 \end{pmatrix}$ ; (* Depth-twisting with  $\alpha=T^{-s}$  *)

  B4[{i, j}, {i, i + 1, j, j + 1}] =  $\begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & T^{-s} & -1 & 1 & -T^{-s} \end{pmatrix}$ ; (* B/.T->1/T *)

  B5[{i, j}, {i, i + 1, j, j + 1}] =  $\begin{pmatrix} 1 & -T^s & 0 & T^s & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$ ,

  (* Vertical mirror then horizontal mirror *)
  {c, List@@gd}];

(*Echo[MatrixForm/@{B3,B4}];*)
 $\Delta_1 = \text{Det}[B1[\text{All}, 2 ;;]]$ ;
 $\Delta_2 = \text{Det}[B2[\text{All}, 2 ;;]]$ ;
 $\Delta_3 = \text{Det}[B3[\text{All}, 2 ;;]]$ ;
 $\Delta_4 = \text{Det}[B4[\text{All}, 2 ;;]]$ ;
 $\Delta_5 = \text{Det}[B5[\text{All}, 2 ;;]]$ ;

Factor@{ $\Delta_1, \Delta_2, \Delta_3, \Delta_4, \Delta_5$ } )

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In[*]:= **Als** /@AllKnots[{3, 7}]

$$\begin{aligned}
 \text{Out[*]} = & \left\{ \frac{1-T+T^2}{T^2}, \frac{1-T+T^2}{T^2}, \frac{1-T+T^2}{T^2}, 1-T+T^2, \frac{1-T+T^2}{T^3} \right\}, \\
 & \left\{ -1+3T-T^2, -1+3T-T^2, -1+3T-T^2, -\frac{1-3T+T^2}{T^2}, -\frac{1-3T+T^2}{T^2} \right\}, \\
 & \left\{ \frac{1-T+T^2-T^3+T^4}{T^4}, \frac{1-T+T^2-T^3+T^4}{T^4}, \frac{1-T+T^2-T^3+T^4}{T^4}, 1-T+T^2-T^3+T^4, \frac{1-T+T^2-T^3+T^4}{T^5} \right\}, \\
 & \left\{ \frac{2-3T+2T^2}{T^3}, \frac{2-3T+2T^2}{T^3}, \frac{2-3T+2T^2}{T^3}, T(2-3T+2T^2), \frac{2-3T+2T^2}{T^4} \right\}, \\
 & \left\{ -\frac{(-2+T)(-1+2T)}{T}, -\frac{(-2+T)(-1+2T)}{T}, -\frac{(-2+T)(-1+2T)}{T}, -\frac{(-2+T)(-1+2T)}{T}, \right. \\
 & \left. -\frac{(-2+T)(-1+2T)}{T^3} \right\}, \left\{ -\frac{1-3T+3T^2-3T^3+T^4}{T^2}, -\frac{1-3T+3T^2-3T^3+T^4}{T^2}, \right. \\
 & \left. -\frac{1-3T+3T^2-3T^3+T^4}{T^2}, -\frac{1-3T+3T^2-3T^3+T^4}{T^2}, -\frac{1-3T+3T^2-3T^3+T^4}{T^4} \right\}, \\
 & \left\{ \frac{1-3T+5T^2-3T^3+T^4}{T^2}, \frac{1-3T+5T^2-3T^3+T^4}{T^2}, \frac{1-3T+5T^2-3T^3+T^4}{T^2}, \frac{1-3T+5T^2-3T^3+T^4}{T^2}, \right. \\
 & \left. \frac{1-3T+5T^2-3T^3+T^4}{T^2} \right\}, \left\{ \frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^6}, \frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^6}, \right. \\
 & \left. \frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^6}, 1-T+T^2-T^3+T^4-T^5+T^6, \frac{1-T+T^2-T^3+T^4-T^5+T^6}{T^7} \right\}, \\
 & \left\{ \frac{3-5T+3T^2}{T^4}, \frac{3-5T+3T^2}{T^4}, \frac{3-5T+3T^2}{T^4}, T^2(3-5T+3T^2), \frac{3-5T+3T^2}{T^5} \right\}, \\
 & \left\{ T^2(2-3T+3T^2-3T^3+2T^4), T^2(2-3T+3T^2-3T^3+2T^4), \right. \\
 & \left. T^2(2-3T+3T^2-3T^3+2T^4), \frac{2-3T+3T^2-3T^3+2T^4}{T^6}, T(2-3T+3T^2-3T^3+2T^4) \right\}, \\
 & \left\{ T^3(4-7T+4T^2), T^3(4-7T+4T^2), T^3(4-7T+4T^2), \frac{4-7T+4T^2}{T^5}, T^2(4-7T+4T^2) \right\}, \\
 & \left\{ \frac{2-4T+5T^2-4T^3+2T^4}{T^5}, \frac{2-4T+5T^2-4T^3+2T^4}{T^5}, \right. \\
 & \left. \frac{2-4T+5T^2-4T^3+2T^4}{T^5}, T(2-4T+5T^2-4T^3+2T^4), \frac{2-4T+5T^2-4T^3+2T^4}{T^6} \right\}, \\
 & \left\{ -\frac{1-5T+7T^2-5T^3+T^4}{T^3}, -\frac{1-5T+7T^2-5T^3+T^4}{T^3}, -\frac{1-5T+7T^2-5T^3+T^4}{T^3}, \right. \\
 & \left. -\frac{1-5T+7T^2-5T^3+T^4}{T}, -\frac{1-5T+7T^2-5T^3+T^4}{T^4} \right\}, \left\{ 1-5T+9T^2-5T^3+T^4, \right. \\
 & \left. 1-5T+9T^2-5T^3+T^4, 1-5T+9T^2-5T^3+T^4, \frac{1-5T+9T^2-5T^3+T^4}{T^4}, \frac{1-5T+9T^2-5T^3+T^4}{T^3} \right\}
 \end{aligned}$$

In[]:= **Als**[**PD**[**X**[1, 4, 2, 3], **X**[2, 1, 3, 4]]]

$$\text{Out[]} = \left\{ 1, -\frac{-1 + T - T^2 \alpha - T \alpha^2 + T^2 \alpha^2}{\alpha}, 1, 1, T^2 \right\}$$

In[]:= **Als**[**GD**[**Xp**[1, 3], **Xp**[4, 2]]]

$$\text{Out[]} = \left\{ 1 - T + T^2, \frac{1 - T + T^2 \alpha}{\alpha}, T, \frac{1 - T + T^2}{T^2}, T \right\}$$

In[]:= **Als**[**GD**[**Xm**[4, 1], **Xm**[6, 3], **Xm**[2, 5]]]

$$\text{Out[]} = \left\{ \frac{1 - T + T^2}{T^2}, \frac{1 - T + T^2}{T^2}, \frac{1 - T + T^2}{T^2}, 1 - T + T^2, \frac{1 - T + T^2}{T^3} \right\}$$

In[]:= **Als**[**GD**[**Xp**[4, 1], **Xm**[6, 3], **Xm**[2, 5]]]

$$\text{Out[]} = \left\{ \frac{1 - T + T^2}{T^2}, \frac{1 - T + T^2}{T^2}, 2 - T, 1 - T + T^2, \frac{-1 + 2 T}{T^2} \right\}$$

In[]:= **Als**[**GD**[**Xp**[4, 1], **Xp**[6, 3], **Xm**[2, 5]]]

$$\text{Out[]} = \left\{ 2 - T, 2 - T, 2 - T, \frac{-1 + 2 T}{T}, -1 + 2 T \right\}$$

In[]:= **Als**[**GD**[**Xp**[4, 1], **Xp**[6, 3], **Xp**[2, 5]]]

$$\text{Out[]} = \left\{ 1 - T + T^2, 1 - T + T^2, 1 - T + T^2, \frac{1 - T + T^2}{T^2}, T (1 - T + T^2) \right\}$$

Conjugacy Tests

In[]:= **Als**[**Knot**[7, 7]]

$$\text{Out[]} = \left\{ 1 - 5 T + 9 T^2 - 5 T^3 + T^4, 1 - 5 T + 9 T^2 - 5 T^3 + T^4, \right. \\ \left. 1 - 5 T + 9 T^2 - 5 T^3 + T^4, \frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^4}, \frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^3} \right\}$$

In[]:= **Collect**[**Det**[**B1**[[**All**, 2 ;;]] - λ **IdentityMatrix**[**Length**[**B3**]]], λ , **Factor**]

$$\begin{aligned} \text{Out[]}= & 1 - 5 T + 9 T^2 - 5 T^3 + T^4 + \frac{(1 + T) (4 - 15 T + 12 T^2 + 12 T^3 - 8 T^4 + 2 T^5) \lambda}{T^2} + \\ & \frac{(1 - 32 T + 47 T^2 + 35 T^3 + 49 T^4 - 13 T^5 + 4 T^6) \lambda^2}{T^2} + \\ & \frac{(1 + T) (-3 - 18 T + 22 T^2 + 80 T^3 + 84 T^4 + 14 T^5 + 3 T^6) \lambda^3}{T^3} + \\ & \frac{(-12 - 25 T + 95 T^2 + 260 T^3 + 383 T^4 + 216 T^5 + 77 T^6 + 7 T^7) \lambda^4}{T^3} + \\ & \frac{(1 + T) (-16 + 42 T + 191 T^2 + 356 T^3 + 283 T^4 + 124 T^5 + 21 T^6) \lambda^5}{T^3} + \\ & \frac{(107 + 409 T + 821 T^2 + 868 T^3 + 566 T^4 + 197 T^5 + 35 T^6) \lambda^6}{T^2} + \\ & \frac{(1 + T) (25 + 140 T + 395 T^2 + 530 T^3 + 421 T^4 + 170 T^5 + 35 T^6) \lambda^7}{T^3} + \\ & \frac{(33 + 171 T + 512 T^2 + 814 T^3 + 820 T^4 + 471 T^5 + 161 T^6 + 21 T^7) \lambda^8}{T^3} + \\ & \frac{(1 + T) (21 + 108 T + 249 T^2 + 322 T^3 + 214 T^4 + 80 T^5 + 7 T^6) \lambda^9}{T^3} + \\ & \frac{(7 + 67 T + 187 T^2 + 307 T^3 + 266 T^4 + 138 T^5 + 28 T^6 + T^7) \lambda^{10}}{T^3} + \\ & \frac{(1 + T) (1 + 20 T + 55 T^2 + 64 T^3 + 38 T^4 + 4 T^5) \lambda^{11}}{T^3} + \\ & \frac{(3 + 21 T + 33 T^2 + 28 T^3 + 6 T^4) \lambda^{12}}{T^2} + \frac{(1 + T) (3 + 4 T) \lambda^{13}}{T} + \lambda^{14} \end{aligned}$$

In[]:= **Collect**[**Det**[**B2**[[**All**, 2 ;;]] - λ **IdentityMatrix**[**Length**[**B3**]]], λ , **Factor**]

$$\begin{aligned} \text{Out[]}= & 1 - 5 T + 9 T^2 - 5 T^3 + T^4 + \\ & \frac{(4 - 18 T + 29 T^2 - 24 T^3 + 24 T^4 - 10 T^5 + 2 T^6 + 7 T \alpha^2 - 32 T^2 \alpha^2 + 48 T^3 \alpha^2 - 20 T^4 \alpha^2 + 4 T^5 \alpha^2) \lambda}{T^2 \alpha} + \\ & \frac{1}{T^2 \alpha^2} (-5 + 13 T - 18 T^2 + 35 T^3 - 9 T^4 + 5 T^5 + 6 \alpha^2 - 46 T \alpha^2 + 87 T^2 \alpha^2 - 50 T^3 \alpha^2 + \\ & 68 T^4 \alpha^2 - 20 T^5 \alpha^2 + 4 T^6 \alpha^2 + T \alpha^4 - 22 T^2 \alpha^4 + 50 T^3 \alpha^4 - 10 T^4 \alpha^4 + 2 T^5 \alpha^4) \lambda^2 + \frac{1}{T^3 \alpha^3} \\ & (-2 + 5 T - 8 T^2 + 25 T^3 + 14 T^5 + T^7 - \alpha^2 - 24 T \alpha^2 + 65 T^2 \alpha^2 - 52 T^3 \alpha^2 + 150 T^4 \alpha^2 - 18 T^5 \alpha^2 + 27 T^6 \alpha^2 - \\ & 2 T \alpha^4 - 50 T^2 \alpha^4 + 139 T^3 \alpha^4 - 34 T^4 \alpha^4 + 102 T^5 \alpha^4 - 10 T^6 \alpha^4 + 2 T^7 \alpha^4 - 3 T^2 \alpha^6 - 10 T^3 \alpha^6 + 48 T^4 \alpha^6) \lambda^3 + \\ & \frac{1}{T^3 \alpha^4} (1 - 4 T + 15 T^2 - 3 T^3 + 22 T^4 + 4 T^6 - 10 \alpha^2 + 33 T \alpha^2 - 42 T^2 \alpha^2 + 165 T^3 \alpha^2 - 6 T^4 \alpha^2 + \end{aligned}$$

$$\begin{aligned}
& 98 T^5 \alpha^2 + 7 T^7 \alpha^2 - 3 \alpha^4 - 48 T \alpha^4 + 163 T^2 \alpha^4 - 74 T^3 \alpha^4 + 342 T^4 \alpha^4 - 12 T^5 \alpha^4 + 73 T^6 \alpha^4 - \\
& 6 T \alpha^6 - 40 T^2 \alpha^6 + 179 T^3 \alpha^6 - 18 T^4 \alpha^6 + 130 T^5 \alpha^6 - T^2 \alpha^8 - 7 T^3 \alpha^8 + 43 T^4 \alpha^8) \lambda^4 + \\
& \frac{1}{T^3 \alpha^5} (3 T - 2 T^2 + 14 T^3 + 6 T^5 + 7 \alpha^2 - 20 T \alpha^2 + 100 T^2 \alpha^2 - 18 T^3 \alpha^2 + 148 T^4 \alpha^2 + 28 T^6 \alpha^2 - 20 \alpha^4 + 95 T \alpha^4 - \\
& 90 T^2 \alpha^4 + 465 T^3 \alpha^4 - 24 T^4 \alpha^4 + 288 T^5 \alpha^4 + 21 T^7 \alpha^4 - 3 \alpha^6 - 50 T \alpha^6 + 249 T^2 \alpha^6 - 76 T^3 \alpha^6 + 506 T^4 \alpha^6 - \\
& 8 T^5 \alpha^6 + 117 T^6 \alpha^6 - 2 T \alpha^8 - 24 T^2 \alpha^8 + 164 T^3 \alpha^8 - 14 T^4 \alpha^8 + 121 T^5 \alpha^8 - 2 T^3 \alpha^{10} + 23 T^4 \alpha^{10}) \lambda^5 + \\
& \frac{1}{T^3 \alpha^6} (3 T^2 + 4 T^4 + 21 T \alpha^2 - 10 T^2 \alpha^2 + 94 T^3 \alpha^2 + 42 T^5 \alpha^2 + 21 \alpha^4 - 40 T \alpha^4 + 286 T^2 \alpha^4 - \\
& 42 T^3 \alpha^4 + 426 T^4 \alpha^4 + 84 T^6 \alpha^4 - 20 \alpha^6 + 153 T \alpha^6 - 98 T^2 \alpha^6 + 727 T^3 \alpha^6 - 36 T^4 \alpha^6 + \\
& 464 T^5 \alpha^6 + 35 T^7 \alpha^6 - \alpha^8 - 27 T \alpha^8 + 234 T^2 \alpha^8 - 48 T^3 \alpha^8 + 471 T^4 \alpha^8 - 7 T^5 \alpha^8 + \\
& 113 T^6 \alpha^8 - 6 T^2 \alpha^{10} + 90 T^3 \alpha^{10} - 4 T^4 \alpha^{10} + 67 T^5 \alpha^{10} + 7 T^4 \alpha^{12}) \lambda^6 + \frac{1}{T^3 \alpha^7} \\
& (T^3 + 21 T^2 \alpha^2 + 28 T^4 \alpha^2 + 63 T \alpha^4 - 20 T^2 \alpha^4 + 272 T^3 \alpha^4 + 126 T^5 \alpha^4 + 35 \alpha^6 - 40 T \alpha^6 + 456 T^2 \alpha^6 - 48 T^3 \alpha^6 + \\
& 682 T^4 \alpha^6 + 140 T^6 \alpha^6 - 10 \alpha^8 + 148 T \alpha^8 - 54 T^2 \alpha^8 + 684 T^3 \alpha^8 - 24 T^4 \alpha^8 + 446 T^5 \alpha^8 + 35 T^7 \alpha^8 - \\
& 6 T \alpha^{10} + 132 T^2 \alpha^{10} - 12 T^3 \alpha^{10} + 264 T^4 \alpha^{10} - 2 T^5 \alpha^{10} + 65 T^6 \alpha^{10} + 28 T^3 \alpha^{12} + 21 T^5 \alpha^{12} + T^4 \alpha^{14}) \lambda^7 + \\
& \frac{1}{T^3 \alpha^6} (7 T^3 + 63 T^2 \alpha^2 + 84 T^4 \alpha^2 + 105 T \alpha^4 - 20 T^2 \alpha^4 + 440 T^3 \alpha^4 + 210 T^5 \alpha^4 + 35 \alpha^6 - 20 T \alpha^6 + \\
& 439 T^2 \alpha^6 - 27 T^3 \alpha^6 + 658 T^4 \alpha^6 + 140 T^6 \alpha^6 - 2 \alpha^8 + 86 T \alpha^8 - 12 T^2 \alpha^8 + 390 T^3 \alpha^8 - 6 T^4 \alpha^8 + \\
& 258 T^5 \alpha^8 + 21 T^7 \alpha^8 + 42 T^2 \alpha^{10} + 84 T^4 \alpha^{10} + 21 T^6 \alpha^{10} + 4 T^3 \alpha^{12} + 3 T^5 \alpha^{12}) \lambda^8 + \frac{1}{T^3 \alpha^5} \\
& (21 T^3 + 105 T^2 \alpha^2 + 140 T^4 \alpha^2 + 105 T \alpha^4 - 10 T^2 \alpha^4 + 430 T^3 \alpha^4 + 210 T^5 \alpha^4 + 21 \alpha^6 - 4 T \alpha^6 + 256 T^2 \alpha^6 - \\
& 6 T^3 \alpha^6 + 384 T^4 \alpha^6 + 84 T^6 \alpha^6 + 28 T \alpha^8 + 126 T^3 \alpha^8 + 84 T^5 \alpha^8 + 7 T^7 \alpha^8 + 6 T^2 \alpha^{10} + 12 T^4 \alpha^{10} + 3 T^6 \alpha^{10}) \\
& \lambda^9 + \frac{1}{T^3 \alpha^4} (35 T^3 + 105 T^2 \alpha^2 + 140 T^4 \alpha^2 + 63 T \alpha^4 - 2 T^2 \alpha^4 + 254 T^3 \alpha^4 + 126 T^5 \alpha^4 + \\
& 7 \alpha^6 + 84 T^2 \alpha^6 + 126 T^4 \alpha^6 + 28 T^6 \alpha^6 + 4 T \alpha^8 + 18 T^3 \alpha^8 + 12 T^5 \alpha^8 + T^7 \alpha^8) \lambda^{10} + \\
& (35 T^3 + 63 T^2 \alpha^2 + 84 T^4 \alpha^2 + 21 T \alpha^4 + 84 T^3 \alpha^4 + 42 T^5 \alpha^4 + \alpha^6 + 12 T^2 \alpha^6 + 18 T^4 \alpha^6 + 4 T^6 \alpha^6) \lambda^{11} \\
& \frac{1}{T^3 \alpha^3} + \\
& \frac{(21 T^2 + 21 T \alpha^2 + 28 T^3 \alpha^2 + 3 \alpha^4 + 12 T^2 \alpha^4 + 6 T^4 \alpha^4) \lambda^{12}}{T^2 \alpha^2} + \\
& \frac{(7 T + 3 \alpha^2 + 4 T^2 \alpha^2) \lambda^{13}}{T \alpha} + \lambda^{14}
\end{aligned}$$

In[]:= Collect[Det[B3[All, 2 ;]] - λ IdentityMatrix[Length[B3]]], λ, Factor]

$$\begin{aligned}
\text{Out[*]} = & 1 - 5 T + 9 T^2 - 5 T^3 + T^4 + \frac{(1 + T) (4 - 15 T + 12 T^2 + 12 T^3 - 8 T^4 + 2 T^5) \lambda}{T^2} + \\
& \frac{(1 - 32 T + 47 T^2 + 35 T^3 + 49 T^4 - 13 T^5 + 4 T^6) \lambda^2}{T^2} + \\
& \frac{(1 + T) (-3 - 18 T + 22 T^2 + 80 T^3 + 84 T^4 + 14 T^5 + 3 T^6) \lambda^3}{T^3} + \\
& \frac{(-12 - 25 T + 95 T^2 + 260 T^3 + 383 T^4 + 216 T^5 + 77 T^6 + 7 T^7) \lambda^4}{T^3} + \\
& \frac{(1 + T) (-16 + 42 T + 191 T^2 + 356 T^3 + 283 T^4 + 124 T^5 + 21 T^6) \lambda^5}{T^3} + \\
& \frac{(107 + 409 T + 821 T^2 + 868 T^3 + 566 T^4 + 197 T^5 + 35 T^6) \lambda^6}{T^2} + \\
& \frac{(1 + T) (25 + 140 T + 395 T^2 + 530 T^3 + 421 T^4 + 170 T^5 + 35 T^6) \lambda^7}{T^3} + \\
& \frac{(33 + 171 T + 512 T^2 + 814 T^3 + 820 T^4 + 471 T^5 + 161 T^6 + 21 T^7) \lambda^8}{T^3} + \\
& \frac{(1 + T) (21 + 108 T + 249 T^2 + 322 T^3 + 214 T^4 + 80 T^5 + 7 T^6) \lambda^9}{T^3} + \\
& \frac{(7 + 67 T + 187 T^2 + 307 T^3 + 266 T^4 + 138 T^5 + 28 T^6 + T^7) \lambda^{10}}{T^3} + \\
& \frac{(1 + T) (1 + 20 T + 55 T^2 + 64 T^3 + 38 T^4 + 4 T^5) \lambda^{11}}{T^3} + \\
& \frac{(3 + 21 T + 33 T^2 + 28 T^3 + 6 T^4) \lambda^{12}}{T^2} + \frac{(1 + T) (3 + 4 T) \lambda^{13}}{T} + \lambda^{14}
\end{aligned}$$

In[]:= **Collect**[**Det**[**B4**[[**All**, **2** ; ;]] - λ **IdentityMatrix**[**Length**[**B3**]]], λ , **Factor**]

$$\begin{aligned}
 \text{Out[]:=} & \frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^4} + \frac{(1 + T) (2 - 8 T + 12 T^2 + 12 T^3 - 15 T^4 + 4 T^5) \lambda}{T^4} + \\
 & \frac{(4 - 13 T + 49 T^2 + 35 T^3 + 47 T^4 - 32 T^5 + T^6) \lambda^2}{T^4} - \\
 & \frac{(1 + T) (-3 - 14 T - 84 T^2 - 80 T^3 - 22 T^4 + 18 T^5 + 3 T^6) \lambda^3}{T^4} - \\
 & \frac{(-7 - 77 T - 216 T^2 - 383 T^3 - 260 T^4 - 95 T^5 + 25 T^6 + 12 T^7) \lambda^4}{T^4} - \\
 & \frac{(1 + T) (-21 - 124 T - 283 T^2 - 356 T^3 - 191 T^4 - 42 T^5 + 16 T^6) \lambda^5}{T^4} + \\
 & \frac{(35 + 197 T + 566 T^2 + 868 T^3 + 821 T^4 + 409 T^5 + 107 T^6) \lambda^6}{T^4} + \\
 & \frac{(1 + T) (35 + 170 T + 421 T^2 + 530 T^3 + 395 T^4 + 140 T^5 + 25 T^6) \lambda^7}{T^4} + \\
 & \frac{(21 + 161 T + 471 T^2 + 820 T^3 + 814 T^4 + 512 T^5 + 171 T^6 + 33 T^7) \lambda^8}{T^4} + \\
 & \frac{(1 + T) (7 + 80 T + 214 T^2 + 322 T^3 + 249 T^4 + 108 T^5 + 21 T^6) \lambda^9}{T^4} + \\
 & \frac{(1 + 28 T + 138 T^2 + 266 T^3 + 307 T^4 + 187 T^5 + 67 T^6 + 7 T^7) \lambda^{10}}{T^4} + \\
 & \frac{(1 + T) (4 + 38 T + 64 T^2 + 55 T^3 + 20 T^4 + T^5) \lambda^{11}}{T^3} + \\
 & \frac{(6 + 28 T + 33 T^2 + 21 T^3 + 3 T^4) \lambda^{12}}{T^2} + \frac{(1 + T) (4 + 3 T) \lambda^{13}}{T} + \lambda^{14}
 \end{aligned}$$

In[*]:= **Collect**[**Det**[**B5**[[**All**, **2** ; ;]] - λ **IdentityMatrix**[**Length**[**B3**]]], λ , **Factor**]

$$\begin{aligned}
 \text{Out[*]} = & \frac{1 - 5 T + 9 T^2 - 5 T^3 + T^4}{T^3} + \frac{(1 + T) (3 - 15 T + 25 T^2 - 7 T^3 + T^4) \lambda}{T^3} - \frac{(-1 + 13 T - 77 T^3 - 27 T^4 + T^5) \lambda^2}{T^3} + \\
 & \frac{(1 + T) (-9 - 2 T + 33 T^2 + 126 T^3 + 34 T^4) \lambda^3}{T^3} + \frac{(-19 + 3 T + 129 T^2 + 331 T^3 + 371 T^4 + 166 T^5 + 20 T^6) \lambda^4}{T^3} + \\
 & \frac{(1 + T) (-13 + 60 T + 229 T^2 + 366 T^3 + 275 T^4 + 80 T^5 + 4 T^6) \lambda^5}{T^3} + \\
 & \frac{(11 + 117 T + 463 T^2 + 841 T^3 + 885 T^4 + 515 T^5 + 155 T^6 + 16 T^7) \lambda^6}{T^3} + \\
 & \frac{(1 + T) (33 + 144 T + 409 T^2 + 536 T^3 + 417 T^4 + 152 T^5 + 25 T^6) \lambda^7}{T^3} + \\
 & \frac{(35 + 183 T + 507 T^2 + 831 T^3 + 812 T^4 + 471 T^5 + 145 T^6 + 19 T^7) \lambda^8}{T^3} + \\
 & \frac{(1 + T) (21 + 112 T + 245 T^2 + 324 T^3 + 216 T^4 + 76 T^5 + 7 T^6) \lambda^9}{T^3} + \\
 & \frac{(7 + 67 T + 189 T^2 + 303 T^3 + 270 T^4 + 136 T^5 + 28 T^6 + T^7) \lambda^{10}}{T^3} + \\
 & \frac{(1 + T) (1 + 20 T + 55 T^2 + 64 T^3 + 38 T^4 + 4 T^5) \lambda^{11}}{T^3} + \\
 & \frac{(3 + 21 T + 33 T^2 + 28 T^3 + 6 T^4) \lambda^{12}}{T^2} + \frac{(1 + T) (3 + 4 T) \lambda^{13}}{T} + \lambda^{14}
 \end{aligned}$$