

Pensieve Header: A version of ρ_1 for links with some testing, by Roland.

Roland says: The determinant that gives Alexander for knots now computes the Alexander-specialization of the HOMFLYPT polynomial (normalized univariate Alexander for links). However this HOMFLY-specialization sometimes is 0 so that the matrix A of which we are computing the inverse for ρ_1 becomes singular. I suspect we should still be able to compute ρ_1 when A is singular like this but this requires a little more care.

APAI for links.

```
In[ ]:= Once [ << KnotTheory` ]
```

Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at <http://katlas.org/wiki/KnotTheory>.

A version of Dror's Rot program that computes the rotation numbers from PD code of a link (the component containing edge 1 is assumed to be opened)

The output is a list of crossings each written as $\{\text{sign}, i, j, i^+, j^+\}$ and a list of rotation numbers for the edges. There are $2cr+1$ edges where cr is the number of crossings.

```

In[*]:= NextHead[fr_] := Min[Select[fr, (Abs[#] > 1 && IntegerQ[#]) &]]
FindNextCross[todo_, Nh_] :=
  First@Select[todo, If[PositiveQ[#], #[[1]] == Nh || #[[4]] == Nh, #[[1]] == Nh || #[[2]] == Nh] &]
Bend[X[a_, b_, c_, d_], h_] := If[PositiveQ[X[a, b, c, d]],
  If[h == a, {I d, c, b}, {c, b, -I a}], If[h == a, {d, c, -I b}, {I a, d, c}]]
CheckCap[fr_, rots_] := Module[{i, newrots = rots, newfront = fr},
  Do[If[Abs[fr[[i]]] === Abs[fr[[i + 1]]],
    If[IntegerQ[fr[[i]]], newrots[[Abs[fr[[i]]]] = (-I + fr[[i + 1]] / fr[[i]]) / (2 I),
      newrots[[Abs[fr[[i]]]] = (I + fr[[i]] / fr[[i + 1]]) / (2 I)];
    newfront = Delete[newfront, {{i}, {i + 1}}]; Break[];
  ], {i, Length[fr] - 1}];
  Return[{newfront, newrots}]
]
RotLink[pd_] :=
  Module[{safety = 0, front = {1}, Nh = 1, rots = ConstantArray[0, 2 Length[pd] + 1]},
  While[safety < 1000 && ! (front === {1} && Nh == 1),
    safety++;
    If[Length[SequenceCases[Abs /@ front, {a_, a_}]] > 0,
      {front, rots} = CheckCap[front, rots];
      front = Flatten[front /. Nh -> Bend[FindNextCross[pd, Nh], Nh]]
    ];
    Nh = NextHead[front];
  ];
  Return[
    {Cases[List@@pd, X[a_, b_, c_, d_] -> If[PositiveQ[X[a, b, c, d]], {1, d, a, b, c},
      {-1, b, a, d, c}]] /. {{s_, k_, l_, 1, m_} -> {s, k, l, 2 Length[pd] + 1, m},
      {s_, k_, l_, m_, 1} -> {s, k, l, m, 2 Length[pd] + 1}}, rots}];
  ]

```

```
In[*]:= PD[Link[8, NonAlternating, 8]]
```

 KnotTheory: Loading precomputed data in PD4Links`.

```
Out[*]=
```

```
PD[X[6, 1, 7, 2], X[2, 5, 3, 6], X[16, 11, 13, 12], X[3, 11, 4, 10],
  X[9, 1, 10, 4], X[7, 15, 8, 14], X[13, 5, 14, 8], X[12, 15, 9, 16]]
```

```
In[*]:= RotLink[PD[Link[8, NonAlternating, 8]]]
```

```
Out[*]=
```

```
{{{-1, 1, 6, 2, 7}, {-1, 5, 2, 6, 3}, {-1, 11, 16, 12, 13}, {1, 10, 3, 11, 4},
  {1, 4, 9, 17, 10}, {1, 14, 7, 15, 8}, {1, 8, 13, 5, 14}, {-1, 15, 12, 16, 9}},
  {{0, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, -1, 1, 0, 1, 0}}
```

```
In[*]:= RotLink[PD[Knot[3, 1]]]
```

 KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[*]=
```

```
{{{-1, 4, 1, 5, 2}, {-1, 6, 3, 7, 4}, {-1, 2, 5, 3, 6}}, {{0, 0, 0, -1, 0, 0, 0}}
```

A slight modification of the usual APAI program to compute for rotational link diagrams.

```
In[*]:= R1[s_, i_, j_, ip_, jp_] := S (gji (gjp,j + gj,jp - gij) - gii (gj,jp - 1) - 1 / 2);
rho[Cs_, phi_] := Module[{n, A, s, i, j, k, Delta, G, rho1},
  n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_, ip_, jp_} -> (A[[{i, j}], {ip, jp}] += ( -T^s T^s - 1 ))];
  Delta = T^(-Total[phi] - Total[Cs[[All,1]]]) / 2 Det[A];
  G = Inverse[A];
  rho1 = Sum_{k=1}^n R1 @@ Cs[[k]] - Sum_{k=1}^{2^n} phi[[k]] (gkk - 1 / 2);
  Factor@{Delta, Delta^2 rho1 /. alpha_+ -> alpha + 1 /. g_alpha_beta -> G[[alpha, beta]]};
];
```

Except for the links Link[8,NonAlternating,8] and Link[9,NonAlternating,27]

The computation runs smoothly in all links in the table up to 9 crossings, see below.

We also interpreted the “Alexander” part of the output of rho in terms of the univariate Alexander for links computed below as a specialization of the Homfly polynomial (sometimes up to a multiplicative factor of T^{+-1/2})

Interestingly the above specialization of the Homfly polynomial vanishes precisely on the two problematic links

Link[8,NonAlternating,8] and Link[9,NonAlternating,27]. This could help explaining why our rho fails to compute in these cases.

```
In[*]:= rhoLink[L_] := {L, rho @@ RotLink[PD[L]]}
```

```
In[*]:= rhoLink /@ AllLinks[{3, 9}] // Column
```

Inverse: Matrix

$$\left\{ \left\{ 1, -\frac{1}{T}, 0, 0, 0, 0, -1 + \frac{1}{T}, 0, 0, 0, \ll 7 \gg \right\}, \{0, 1, -1, 0, 0, 0, 0, 0, 0, 0, \ll 7 \gg\}, \{0, 0, 1, -1, 0, 0, 0, 0, 0, 0, \ll 7 \gg\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, -1, 0, 0, 0, -T, 0, 0, 1, 0, 0, \ll 7 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1, -1, \ll 7 \gg\}, \{0, 0, 0, -1 + T, 0, 0, 0, 0, 0, 1, \ll 7 \gg\}, \ll 7 \gg \right\}$$

is singular.

Inverse: Matrix

$$\left\{ \left\{ 1, -\frac{1}{T}, 0, 0, 0, 0, -1 + \frac{1}{T}, 0, 0, 0, \ll 9 \gg \right\}, \{0, 1, -1, 0, 0, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 1, -T, 0, 0, 0, 0, -1 + T, 0, \ll 9 \gg\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 1, -1, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 1, -1, 0, 0, 0, \ll 9 \gg\}, \left\{ 0, 0, 0, 0, 0, 1, -\frac{1}{T}, 0, 0, \ll 9 \gg \right\}, \{0, 0, 0, 0, 0, 0, 1, -1, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 1, -1, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1, \ll 9 \gg\}, \ll 9 \gg \right\}$$

is singular.

Out[*]=

$$\left\{ \text{Link}[4, \text{Alternating}, 1], \left\{ \frac{2(-1+\sqrt{T})(1+\sqrt{T})}{\sqrt{T}}, \frac{3(-1+T)^2}{T} \right\} \right\}$$

$$\left\{ \text{Link}[5, \text{Alternating}, 1], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, \frac{(-1+T)^2(1+T^2)(1-4T+T^2)}{2T^3} \right\} \right\}$$

$$\begin{aligned}
 & \left\{ \text{Link}[6, \text{Alternating}, 1], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-4T+T^2)}{T^{3/2}}, \frac{(-1+T)^2(1-8T+16T^2-8T^3+T^4)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[6, \text{Alternating}, 2], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-T+2T^2)}{T^{3/2}}, \frac{(-1+T)^2(14-12T+23T^2-12T^3+14T^4)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[6, \text{Alternating}, 3], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-\sqrt{T}+T)(1+\sqrt{T}+T)(1-T+T^2)}{T^{5/2}}, \frac{(-1+T)^2(5+8T^2+9T^4+8T^6+5T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[6, \text{Alternating}, 4], \left\{ \frac{(-1+T)^4}{T^2}, \theta \right\} \right\} \\
 & \left\{ \text{Link}[6, \text{Alternating}, 5], \left\{ \frac{3(-1+T)^2}{T}, \frac{2(-1+T)^2(6-11T+6T^2)}{T^2} \right\} \right\} \\
 & \left\{ \text{Link}[6, \text{NonAlternating}, 1], \left\{ -\frac{(-1+T)^2}{T}, \frac{2(-1+T)^2}{T} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 1], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2)}{T^{5/2}}, -\frac{(-1+T)^2(1-6T+15T^2-18T^3+20T^4-18T^5+15T^6-6T^7+T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 2], \left\{ \frac{(-1+T)(3-4T+3T^2)}{T^2}, \frac{(-1+T)^2(9-18T+18T^2-11T^3+2T^4+3T^5)}{T^4} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 3], \left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1+T^2)}{T^{5/2}}, -\frac{(-1+T)^2(3-12T+18T^2-24T^3+22T^4-24T^5+18T^6-12T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 4], \left\{ \frac{2(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, -\frac{(-1+T)^2(3-12T+14T^2-12T^3+3T^4)}{T^3} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 5], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-2+T)(-1+2T)}{T^{3/2}}, \frac{(-1+T)^2(2-6T-9T^2+18T^3-16T^4+4T^5)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 6], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T+T^2-2T^3+T^4)}{T^{5/2}}, -\frac{(-1+T)^2(3-12T+14T^2-16T^3+15T^4-16T^5+14T^6-12T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{Alternating}, 7], \left\{ \frac{(-1+T)^2(1-3T+T^2)}{T^2}, \frac{2(-1+T)^2(1-3T+T^2)}{T^2} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{NonAlternating}, 1], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T^4)}{T^{5/2}}, \frac{(-1+T)^2(5+4T^4+5T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[7, \text{NonAlternating}, 2], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, \frac{(-1+T)^2(1-8T+10T^2-8T^3+T^4)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 1], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-\sqrt{T}+T)(-1+\sqrt{T}+T)}{T^{5/2}}, \frac{(-1+T)^2(1-3T+T^2)(1-7T+19T^2-22T^3+19T^4-7T^5+T^6)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 2], \left\{ -\frac{(-1+T)^5}{T^2}, -\frac{(-1+T)^4(-1+2T-4T^2+10T^3-6T^4+T^5)}{T^3} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 3], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(3-8T+3T^2)}{T^{3/2}}, \frac{(-1+T)(2-3T+2T^2)(3-17T+26T^2-11T^4+3T^5)}{2T^4} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 4], \left\{ \frac{(-1+\sqrt{T})^5(1+\sqrt{T})^5}{T^{5/2}}, -\frac{(-1+T)^4(-1+4T-3T^2-8T^3+9T^4-6T^5+T^6)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 5], \right. \\
 & \quad \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-4T+4T^2-4T^3+T^4)}{T^{5/2}}, -\frac{(-1+T)^2(1-8T+20T^2-16T^3+8T^4-16T^5+20T^6-8T^7+T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 6], \left\{ -\frac{2(-1+\sqrt{T})(1+\sqrt{T})(-1-\sqrt{T}+T)(-1+\sqrt{T}+T)}{T^{3/2}}, \frac{(-1+T)^2(1-3T+T^2)(1-T+T^2)}{T^3} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 7], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(5-8T+5T^2)}{T^{3/2}}, \frac{9(-1+T)^2(11-32T+44T^2-32T^3+11T^4)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 8], \right. \\
 & \quad \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-1-\sqrt{T}+T)(-1+\sqrt{T}+T)(1-T+T^2)}{T^{5/2}}, -\frac{(-1+T)^2(1-3T+T^2)(1-5T+4T^2+T^3+4T^4-5T^5+T^6)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[8, \text{Alternating}, 9], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-4T+7T^2-4T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(1-8T+32T^2-64T^3+79T^4-64T^5+32T^6-8T^7+T^8)}{2T^5} \right\} \right\}
 \end{aligned}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 10], \left\{ \frac{(-1+T)(4-5T+4T^2)}{T^2}, -\frac{(-1+T)^2(8-50T+100T^2-122T^3+91T^4-41T^5+4T^6)}{T^5} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 11], \left\{ \frac{(-1+T)(2-2T+3T^2-2T^3+2T^4)}{T^3}, -\frac{(-1+T)^2(2-17T+35T^2-60T^3+73T^4-80T^5+70T^6-55T^7+31T^8-15T^9+2T^{10})}{T^7} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 12], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-T+2T^2-T^3+2T^4)}{T^{5/2}}, \frac{(-1+T)^2(22-20T+41T^2-32T^3+50T^4-32T^5+41T^6-20T^7+22T^8)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 13], \left\{ \frac{4(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2)}{T^{3/2}}, \frac{2(-1+T)^2(16-28T+39T^2-28T^3+16T^4)}{T^3} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 14], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T^2)(1+T^4)}{T^{7/2}}, \frac{(-1+T)^2(7+12T^2+15T^4+16T^6+15T^8+12T^{10}+7T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 15], \left\{ -\frac{(-1+T)^2(1-5T+T^2)}{T^2}, \frac{(-1+T)^2(1-12T+45T^2-66T^3+45T^4-12T^5+T^6)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 16], \left\{ \frac{(-1+T)^4(1+T^2)}{T^3}, -\frac{(-1+T)^4(1-4T+7T^2-6T^3+8T^4-6T^5+7T^6-4T^7+T^8)}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 17], \left\{ \frac{(-1+T)^2(3-T+3T^2)}{T^2}, \frac{(-1+T)^2(21-52T+78T^2-90T^3+78T^4-52T^5+21T^6)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 18], \left\{ -\frac{(-1+T)^2(1-T+T^2+T^4)}{T^3}, -\frac{(-1+T)^2(1+T^2)(2-8T+11T^2-10T^3+8T^4-10T^5+11T^6-8T^7+2T^8)}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 19], \left\{ -\frac{(-1+T)^2(1-T+T^2)^2}{T^3}, \{0\} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 20], \left\{ \frac{(-1+\sqrt{T})^2(1+\sqrt{T})^2(-1-2\sqrt{T}+T)(-1+2\sqrt{T}+T)}{T^{5/2}}, -\frac{(-1+T)^4(1-6T+T^2)(1-10T+5T^2)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{Alternating}, 21], \left\{ \frac{4(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, \frac{2(-1+T)^4(15-26T+15T^2)}{T^3} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 1], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T-2T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(3-12T+10T^2-4T^3+12T^4-4T^5+10T^6-12T^7+3T^8)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 2], \left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, -\frac{(-1+T)^2(-1-2T+T^2)(-1+2T+T^2)}{2T^3} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 3], \left\{ \frac{(-1+T)^2(1+T+T^2+T^3+T^4)}{T^3}, \frac{(-1+T)^2(3-2T^5+3T^{10})}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 4], \left\{ -\frac{(-1+T)^2(1-T+T^2)}{T^2}, \frac{(-1+T)^2(1+T^2)(1-6T+8T^2-6T^3+T^4)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 5], \left\{ -\frac{(-1+T)^4}{T^2}, \frac{(-1+T)^4(1-6T+6T^2-6T^3+T^4)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 6], \left\{ \frac{(-1+\sqrt{T})^2(1+\sqrt{T})^2(1+T)^2}{T^{5/2}}, \frac{(-1+T)^4(1+T)^2(3+2T+7T^2)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 7], \left\{ \frac{2(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, -\frac{(-3+T)(-1+T)^4(-1+3T)}{T^3} \right\} \right\}$$

$$\left\{ \text{Link}[8, \text{NonAlternating}, 8], \{0, 0\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 1], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2-3T+2T^2)}{T^{3/2}}, \frac{(-1+T)^2(-18+142T-465T^2+884T^3-1112T^4+958T^5-561T^6+208T^7-44T^8+4T^9)}{2T^3} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 2], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2-T^3+T^4)}{T^{7/2}}, -\frac{(-1+T)^2(1+T^2)(3-18T+46T^2-62T^3+54T^4-44T^5+54T^6-62T^7+46T^8-18T^9+3T^{10})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 3], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2-3T+2T^2)}{T^{5/2}}, -\frac{(-1+T)^2(6-40T+127T^2-232T^3+282T^4-232T^5+127T^6-40T^7+6T^8)}{2T^5} \right\} \right\}$$

- { Link [9, Alternating, 4],

$$\left\{ \frac{2(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2)}{T^{5/2}}, \frac{(-1+T)^2(2-19T+80T^2-189T^3+306T^4-362T^5+324T^6-211T^7+94T^8-23T^9+2T^{10})}{T^6} \right\}$$
- { Link [9, Alternating, 5],

$$\left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-7T+10T^2-7T^3+T^4)}{T^{3/2}}, -\frac{(-1+T)^2(-1+4T+15T^2-60T^3+120T^4-174T^5+131T^6-46T^7+5T^8)}{2T^5} \right\}$$
- { Link [9, Alternating, 6],

$$\left\{ \frac{(-1+T)(1+T^2)(3-5T+3T^2)}{T^3}, \frac{(-1+T)^2(18-53T+94T^2-125T^3+132T^4-117T^5+83T^6-42T^7+10T^8+3T^9)}{T^6} \right\}$$
- { Link [9, Alternating, 7],

$$\left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(5-8T+5T^2)}{T^{3/2}}, -\frac{(-1+T)(-50+267T-655T^2+946T^3-884T^4+515T^5-153T^6+10T^7)}{2T^4} \right\}$$
- { Link [9, Alternating, 8],

$$\left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-4T+T^2)}{T^{3/2}}, -\frac{(-1+T)^2(2-15T+36T^2-30T^3-16T^4+92T^5-138T^6+76T^7-16T^8+T^9)}{2T^4} \right\}$$
- { Link [9, Alternating, 9],

$$\left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1+T^2)(1-T+T^2)}{T^{7/2}}, \frac{(-1+T)^2(1-6T+19T^2-38T^3+49T^4-56T^5+54T^6-56T^7+49T^8-38T^9+19T^{10}-6T^{11}+T^{12})}{2T^7} \right\}$$
- { Link [9, Alternating, 10], {

$$-\frac{2(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2)}{T^{5/2}}, -\frac{(-1+T)^2(1-3T+T^2)(1-3T-T^2+2T^3-T^4-3T^5+T^6)}{T^5}$$
- { Link [9, Alternating, 11],

$$\left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-6T+12T^2-6T^3+T^4)}{T^{5/2}}, -\frac{(-1+T)^2(-3+30T-128T^2+278T^3-296T^4+122T^5-8T^6-6T^7+T^8)}{2T^5} \right\}$$
- { Link [9, Alternating, 12],

$$\left\{ \frac{(-1+T)(3-4T+4T^2-4T^3+3T^4)}{T^3}, \frac{(-1+T)^2(18-42T+60T^2-70T^3+70T^4-63T^5+52T^6-34T^7+11T^8+3T^9)}{T^6} \right\}$$
- { Link [9, Alternating, 13], {

$$\frac{2(-1+T)(3-5T+3T^2)}{T^2}, \frac{(-1+T)^2(45-122T+143T^2-80T^3+7T^4+12T^5)}{T^4}$$
- { Link [9, Alternating, 14], {

$$\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-\sqrt{T}+T)(1+\sqrt{T}+T)(1-T+T^2)}{T^{7/2}},$$

$$-\frac{(-1+T)^2(5-20T+34T^2-48T^3+54T^4-60T^5+58T^6-60T^7+54T^8-48T^9+34T^{10}-20T^{11}+5T^{12})}{2T^7}$$
- { Link [9, Alternating, 15],

$$\left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2-T+2T^2)}{T^{5/2}}, -\frac{(-1+T)^2(14-68T+139T^2-200T^3+218T^4-200T^5+139T^6-68T^7+14T^8)}{2T^5} \right\}$$
- { Link [9, Alternating, 16], {

$$-\frac{(-1+T)(1-6T+8T^2-6T^3+T^4)}{T^3}, -\frac{(-1+T)^2(-2+21T-76T^2+133T^3-145T^4+107T^5-48T^6+7T^7)}{T^6}$$
- { Link [9, Alternating, 17],

$$\left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2-T+2T^2)}{T^{5/2}}, -\frac{(-1+T)^2(14-68T+147T^2-220T^3+242T^4-220T^5+147T^6-68T^7+14T^8)}{2T^5} \right\}$$
- { Link [9, Alternating, 18], {

$$\frac{3(-1+\sqrt{T})^3(1+\sqrt{T})^3}{T^{3/2}}, -\frac{(-1+T)^2(19-76T+102T^2-76T^3+19T^4)}{2T^3}$$
- { Link [9, Alternating, 19],

$$\left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-7T+12T^2-7T^3+2T^4)}{T^{5/2}}, \frac{(-1+T)^2(6-36T+123T^2-244T^3+312T^4-244T^5+123T^6-36T^7+6T^8)}{2T^5} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 20], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2)(1-3T+3T^2-3T^3+T^4)}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(-1+8T-28T^2+62T^3-105T^4+126T^5-101T^6+40T^7+17T^8-36T^9+24T^{10}-8T^{11}+T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 21], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-1-\sqrt{T}+T^{3/2})(1-\sqrt{T}+T^{3/2})(1-T+T^{3/2})(-1+T+T^{3/2})}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(1-6T+19T^2-40T^3+53T^4-62T^5+63T^6-62T^7+53T^8-40T^9+19T^{10}-6T^{11}+T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 22], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-3T+5T^2-5T^3+5T^4-3T^5+T^6)}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(3-18T+53T^2-96T^3+135T^4-154T^5+163T^6-154T^7+135T^8-96T^9+53T^{10}-18T^{11}+3T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 23], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-5T+9T^2-5T^3+2T^4)}{T^{5/2}}, \frac{(-1+T)^2(2-6T+9T^2+14T^3-45T^4+80T^5-67T^6+44T^7-16T^8+4T^9)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 24], \right. \\ \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(2-5T+5T^2-5T^3+2T^4)}{T^{5/2}}, \frac{(-1+T)^2(10-46T+75T^2-82T^3+61T^4-52T^5+35T^6-12T^7-8T^8+4T^9)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 25], \left\{ -\frac{(-1+T)(4-9T+4T^2)}{T^2}, -\frac{(-1+T)^2(8-58T+160T^2-207T^3+147T^4-45T^5+4T^6)}{T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 26], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-6T+7T^2-6T^3+2T^4)}{T^{5/2}}, -\frac{(-1+T)^2(14-80T+186T^2-284T^3+321T^4-284T^5+186T^6-80T^7+14T^8)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 27], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-1-\sqrt{T}+T)^2(-1+\sqrt{T}+T)^2}{T^{5/2}}, -\frac{(-1+T)^2(1-3T+T^2)^2(1-6T+3T^2-6T^3+T^4)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 28], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T+4T^2-3T^3+4T^4-2T^5+T^6)}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(3-12T+34T^2-54T^3+76T^4-76T^5+85T^6-76T^7+76T^8-54T^9+34T^{10}-12T^{11}+3T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 29], \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T+2T^2-3T^3+2T^4-2T^5+T^6)}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(5-20T+34T^2-54T^3+66T^4-76T^5+75T^6-76T^7+66T^8-54T^9+34T^{10}-20T^{11}+5T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 30], \right. \\ \left. \left\{ -\frac{(-1+T)(2-4T+3T^2-4T^3+2T^4)}{T^3}, -\frac{(-1+T)^2(2-19T+61T^2-101T^3+129T^4-135T^5+122T^6-86T^7+51T^8-17T^9+2T^{10})}{T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 31], \right. \\ \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2)^3}{T^{7/2}}, \frac{(-1+T)^2(1-3T+T^2)(1-T+T^2)^2(1+T+T^2)(1-2T+5T^2-2T^3+T^4)}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 32], \left\{ \frac{(-1+T)(7-11T+7T^2)}{T^2}, \frac{(-1+T)^2(7+27T-78T^2+83T^3-31T^4-16T^5+21T^6)}{T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 33], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-6T+13T^2-6T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(1-12T+70T^2-208T^3+309T^4-208T^5+70T^6-12T^7+8T^8)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 34], \right. \\ \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-2+T)(-1+2T)(1+T^2)}{T^{5/2}}, \frac{(-1+T)^2(10-46T+69T^2-72T^3+60T^4-48T^5+43T^6-18T^7-8T^8+4T^9)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 35], \right. \\ \left. \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2-T+2T^2)}{T^{5/2}}, \frac{(-1+T)^2(-6+34T-83T^2+116T^3-146T^4+156T^5-137T^6+74T^7-24T^8+4T^9)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 36], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T+T^2-2T^3+T^4-2T^5+T^6)}{T^{7/2}}, \right. \right. \\ \left. \left. - \frac{(-1+T)^2(5-20T+26T^2-32T^3+37T^4-36T^5+38T^6-36T^7+37T^8-32T^9+26T^{10}-20T^{11}+5T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 37], \right. \\ \left. \left\{ \frac{2(-1+\sqrt{T})(1+\sqrt{T})(1-3T+3T^2-3T^3+T^4)}{T^{5/2}}, - \frac{(-1+T)^2(7-40T+89T^2-128T^3+143T^4-128T^5+89T^6-40T^7+7T^8)}{T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 38], \left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1+T^2)^2}{T^{7/2}}, - \frac{(-1+T)^2(1+T^2)(1+T^4)(1-4T+9T^2-16T^3+9T^4-4T^5+T^6)}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 39], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-2T+3T^2-2T^3+3T^4-2T^5+T^6)}{T^{7/2}}, \right. \right. \\ \left. \left. \frac{(-1+T)^2(3-12T+30T^2-40T^3+49T^4-44T^5+50T^6-44T^7+49T^8-40T^9+30T^{10}-12T^{11}+3T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 40], \right. \\ \left. \left\{ - \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(2+2T^2)}{T^{5/2}}, \frac{(-1+T)^2(10-28T+29T^2-32T^3+22T^4-32T^5+29T^6-28T^7+10T^8)}{2T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 41], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T^2)(1-T+T^2)^2}{T^{7/2}}, \frac{(-1+T)^2(1-T+T^2)^2(3-6T+13T^2-6T^3+18T^4-6T^5+13T^6-6T^7+3T^8)}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 42], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+3T^2-T^3+T^4)}{T^{7/2}}, - \frac{(-1+T)^2(1-6T+19T^2-44T^3+66T^4-86T^5+88T^6-86T^7+66T^8-44T^9+19T^{10}-6T^{11}+T^{12})}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 43], \left\{ \frac{(-1+T)^2(4-5T+4T^2)}{T^2}, \frac{2(-1+T)^2(19-80T+160T^2-197T^3+160T^4-80T^5+19T^6)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 44], \right. \\ \left. \left\{ \frac{(-1+T)^2(1-3T+3T^2-3T^3+T^4)}{T^3}, - \frac{(-1+T)^2(1-8T+24T^2-40T^3+47T^4-46T^5+47T^6-40T^7+24T^8-8T^9+T^{10})}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 45], \left\{ \frac{(-2+T)(-1+T)^2(-1+2T)}{T^2}, - \frac{(-1+T)^2(1-8T+19T^2-22T^3+19T^4-8T^5+T^6)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 46], \left\{ - \frac{(-1+T)^6}{T^3}, \frac{2(-1+T)^6(1-3T+T^2)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 47], \left\{ - \frac{(-1+T)^2(3-7T+3T^2)}{T^2}, \frac{(-1+T)^4(12-52T+67T^2-52T^3+12T^4)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 48], \right. \\ \left. \left\{ \frac{(-1+T)^2(1-3T+T^2-3T^3+T^4)}{T^3}, - \frac{(-1+T)^2(1-8T+22T^2-28T^3+26T^4-22T^5+26T^6-28T^7+22T^8-8T^9+T^{10})}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 49], \left\{ \frac{(-1+T)^2(2-7T+2T^2)}{T^2}, - \frac{(-1+T)^2(1-8T+22T^2-26T^3+22T^4-8T^5+T^6)}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 50], \right. \\ \left. \left\{ \frac{(-1+\sqrt{T})^2(1+\sqrt{T})^2(1-3T+3T^2-3T^3+T^4)}{T^{7/2}}, - \frac{(-1+T)^4(3-18T+37T^2-44T^3+49T^4-44T^5+37T^6-18T^7+3T^8)}{2T^7} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 51], \right. \\ \left. \left\{ \frac{(-1+T)^2(1-3T+T^2)(1-T+T^2)}{T^3}, - \frac{(-1+T)^2(1-3T+T^2)(1-7T+20T^2-29T^3+34T^4-29T^5+20T^6-7T^7+T^8)}{T^6} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 52], \left\{ - \frac{3(-1+T)^4}{T^2}, \frac{(-1+T)^4(-3+18T-34T^2+23T^3-16T^4-3T^5+3T^6)}{T^5} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 53], \left\{ - \frac{(-1+T)^6}{T^3}, \frac{2(-1+T)^8}{T^4} \right\} \right\}$$

$$\left\{ \text{Link}[9, \text{Alternating}, 54], \left\{ \frac{(-1+\sqrt{T})^4(1+\sqrt{T})^4(1-T+T^2)}{T^{5/2}}, - \frac{(-1+T)^6(1-6T+6T^2-12T^3+12T^4-12T^5+3T^6)}{2T^5} \right\} \right\}$$

$$\begin{aligned}
 & \left\{ \text{Link}[9, \text{Alternating}, 55], \left\{ \frac{(-1+\sqrt{T})^3 (1+\sqrt{T})^3 (1-4T+T^2)}{T^{5/2}}, \frac{(-1+T)^4 (1-10T+37T^2-64T^3+37T^4-10T^5+T^6)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 1], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(2-T-T^3+2T^4)}{T^{3/2}}, -\frac{(-1+T)(-4+28T-60T^2+59T^3-59T^4+88T^5-100T^6+67T^7-63T^8+76T^9-44T^{10}+4T^{11})}{2T^4} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 2], \left\{ -\frac{2(-1+\sqrt{T})^3(1+\sqrt{T})^3}{\sqrt{T}}, \frac{(-1+T)^2(6-18T+31T^2-52T^3+60T^4-38T^5+7T^6)}{T^3} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 3], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3}{\sqrt{T}}, \frac{(-1+T)^2(6-22T+43T^2-60T^3+52T^4-26T^5+3T^6)}{2T^3} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 4], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T^2)(1-T^2+T^4)}{T^{7/2}}, \frac{(-1+T)^2(7+4T^6+7T^{12})}{2T^7} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 5], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1+T^2)}{T^{5/2}}, \frac{(-1+T)^2(3-16T+26T^2-36T^3+38T^4-36T^5+26T^6-16T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 6], \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2)}{T^{5/2}}, \frac{(-1+T)^2(1+T^2)(3-4T+3T^2)(1-6T+9T^2-6T^3+T^4)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 7], \right. \\
 & \quad \left. \left\{ \frac{2(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2-T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(11-20T+27T^2-32T^3+35T^4-32T^5+27T^6-20T^7+11T^8)}{T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 8], \right. \\
 & \quad \left. \left\{ -\frac{(-1+\sqrt{T})^3(1+\sqrt{T})^3(1-T+T^2)}{T^{5/2}}, \frac{(-1+T)^2(3-18T+41T^2-62T^3+68T^4-62T^5+41T^6-18T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 9], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T-2T^2+T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(5+6T-3T^2-6T^3-8T^4+30T^5-19T^6-4T^7+11T^8+2T^9)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 10], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-3T+6T^2-3T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(1-2T-7T^2+54T^3-116T^4+150T^5-103T^6+48T^7-13T^8+2T^9)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 11], \right. \\
 & \quad \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(-1-\sqrt{T}+T)(-1+\sqrt{T}+T)(1+T^2)}{T^{5/2}}, \frac{(-1+T)^2(5-28T+53T^2-64T^3+68T^4-58T^5+49T^6-22T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 12], \left\{ \frac{(-1+T)(1+T-2T^2+T^3+T^4)}{T^3}, \frac{(-1+T)^2(-3-T+5T^2-T^3-2T^4-T^5+8T^6-4T^7-3T^8+T^9)}{T^6} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 13], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2)^2}{T^{5/2}}, \frac{(-1+T)^2(1-T+T^2)^2(1-6T+17T^2-8T^3+3T^4)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 14], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T+T^2-T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(-1+2T-9T^2+22T^3-21T^4+20T^5-9T^6+2T^7+T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 15], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T^{3/2}+T^3)(1+T^{3/2}+T^3)}{T^{7/2}}, -\frac{(-1+T)^2(-7-10T^3+2T^4-2T^5-9T^6+4T^7-2T^8-10T^9+4T^{10}-2T^{11}-7T^{12}+2T^{13})}{2T^7} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 16], \right. \\
 & \quad \left. \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1-T+3T^2-T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(-1+2T+5T^2-10T^3+25T^4-20T^5+15T^6-4T^7+T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 17], \right. \\
 & \quad \left. \left\{ -\frac{(-1+\sqrt{T})(1+\sqrt{T})(1-3T+3T^2-3T^3+T^4)}{T^{5/2}}, \frac{(-1+T)^2(3-22T+49T^2-76T^3+85T^4-76T^5+49T^6-22T^7+3T^8)}{2T^5} \right\} \right\} \\
 & \left\{ \text{Link}[9, \text{NonAlternating}, 18], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T)^2(1-T+T^2)^2}{T^{7/2}}, \frac{(-1+T)^2(1+T)^2(1-T+T^2)^2(7+6T^3+7T^6)}{2T^7} \right\} \right\}
 \end{aligned}$$

$$\begin{aligned} & \{ \text{Link}[9, \text{NonAlternating}, 19], \left\{ \frac{(-1+\sqrt{T})(1+\sqrt{T})(1+T)^2(1-T+T^2)}{T^{5/2}}, \frac{(-1+T)^2(1+T)^2(5-2T^2+10T^3-2T^4+5T^6)}{2T^5} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 20], \left\{ -\frac{(-1+T)^2(2-3T+2T^2)}{T^2}, -\frac{(-1+T)^2(5-28T+61T^2-78T^3+61T^4-28T^5+5T^6)}{T^4} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 21], \left\{ -\frac{(-1+T)^2}{T}, \frac{2(-1+T)^2(1-T+T^2)}{T^2} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 22], \left\{ \frac{(-1+T)^2(2-T+2T^2)}{T^2}, \frac{(-1+T)^2(1-4T+17T^2-26T^3+17T^4-4T^5+T^6)}{T^4} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 23], \left\{ -\frac{(-1+T)^2(1-T-T^2-T^3+T^4)}{T^3}, \frac{(-1+T)^4(2-4T-3T^2-2T^3+T^4-2T^5-3T^6-4T^7+2T^8)}{T^6} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 24], \left\{ \frac{(-1+T)^2(1-3T+T^2)}{T^2}, -\frac{2(-1+T)^4(1-3T+T^2)}{T^3} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 25], \left\{ \frac{(-1+T)^4}{T^2}, -\frac{2(-1+T)^6}{T^3} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 26], \left\{ \frac{(-1+T)^4}{T^2}, \frac{(-1+T)^4(-1+2T-3T^2+11T^3-6T^4+T^5)}{T^4} \right\} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 27], \{ \mathbf{0}, \mathbf{0} \} \} \\ & \{ \text{Link}[9, \text{NonAlternating}, 28], \left\{ \frac{(-1+T)^4(1+T^2)}{T^3}, -\frac{(-1+T)^4(1-4T+7T^2-4T^3+8T^4-4T^5+7T^6-4T^7+T^8)}{T^6} \right\} \} \end{aligned}$$

```
In[*]:= HOMFLYPT[Link[9, NonAlternating, 27]][1, z] /. {z -> T^-1/2 - T^1/2}
```

KnotTheory: The HOMFLYPT program was written by Scott Morrison.

Out[*]=

0

```
In[*]:= rhoLinkHPT[L_] :=
  {L, Factor[First[rho@@RotLink[PD[L]]] / (HOMFLYPT[L][1, z] /. {z -> T^-1/2 - T^1/2})]}
```

```
In[*]:= rhoLinkHPT[Link[9, NonAlternating, 27]]
```

Inverse: Matrix

$\left\{ \left\{ 1, -\frac{1}{T}, 0, 0, 0, 0, -1 + \frac{1}{T}, 0, 0, 0, \ll 9 \gg \right\}, \{0, 1, -1, 0, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 1, -T, 0, 0, 0, 0, -1 + T, 0, \ll 9 \gg\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 1, -1, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 1, -1, 0, 0, \ll 9 \gg\}, \left\{ 0, 0, 0, 0, 0, 0, 1, -\frac{1}{T}, 0, 0, \ll 9 \gg \right\}, \{0, 0, 0, 0, 0, 0, 1, -1, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1, -1, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 1, \ll 9 \gg\} \right\}$ is singular.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Out[*]=

{Link[9, NonAlternating, 27], Indeterminate}

```
In[*]:= rhoLinkHPT /@ AllLinks[{3, 9}] // Column
```

Inverse: Matrix

$\left\{ \left\{ 1, -\frac{1}{T}, 0, 0, 0, 0, -1 + \frac{1}{T}, 0, 0, 0, \ll 7 \gg \right\}, \{0, 1, -1, 0, 0, 0, 0, 0, 0, \ll 7 \gg\}, \{0, 0, 1, -1, 0, 0, 0, 0, 0, \ll 7 \gg\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, -T, 0, 0, 1, 0, 0, \ll 7 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1, -1, \ll 7 \gg\}, \{0, 0, 0, -1 + T, 0, 0, 0, 0, 0, 1, \ll 7 \gg\}, \ll 7 \gg \right\}$ is singular.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Inverse: Matrix

$\left\{ \left\{ 1, -\frac{1}{T}, 0, 0, 0, 0, -1 + \frac{1}{T}, 0, 0, 0, \ll 9 \gg \right\}, \{0, 1, -1, 0, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 1, -T, 0, 0, 0, 0, -1 + T, 0, \ll 9 \gg\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 1, -1, 0, 0, 0, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 1, -1, 0, 0, 0, \ll 9 \gg\}, \left\{ 0, 0, 0, 0, 0, 1, -\frac{1}{T}, 0, 0, \ll 9 \gg \right\}, \{0, 0, 0, 0, 0, 1, -1, 0, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 1, -1, \ll 9 \gg\}, \{0, 0, 0, 0, 0, 0, 0, 1, \ll 9 \gg\}, \ll 9 \gg \right\}$ is singular.

Power: Infinite expression $\frac{1}{0}$ encountered.

Infinity: Indeterminate expression 0 ComplexInfinity encountered.

Out[]=

- {Link[4, Alternating, 1], 1}
- {Link[5, Alternating, 1], 1}
- {Link[6, Alternating, 1], 1}
- {Link[6, Alternating, 2], 1}
- {Link[6, Alternating, 3], 1}
- {Link[6, Alternating, 4], 1}
- {Link[6, Alternating, 5], 1}
- {Link[6, NonAlternating, 1], 1}
- {Link[7, Alternating, 1], 1}
- {Link[7, Alternating, 2], $\frac{1}{\sqrt{T}}$ }
- {Link[7, Alternating, 3], 1}
- {Link[7, Alternating, 4], 1}
- {Link[7, Alternating, 5], 1}
- {Link[7, Alternating, 6], 1}
- {Link[7, Alternating, 7], 1}
- {Link[7, NonAlternating, 1], 1}
- {Link[7, NonAlternating, 2], 1}
- {Link[8, Alternating, 1], 1}
- {Link[8, Alternating, 2], \sqrt{T} }
- {Link[8, Alternating, 3], 1}
- {Link[8, Alternating, 4], 1}
- {Link[8, Alternating, 5], 1}
- {Link[8, Alternating, 6], 1}
- {Link[8, Alternating, 7], 1}
- {Link[8, Alternating, 8], 1}
- {Link[8, Alternating, 9], 1}
- {Link[8, Alternating, 10], $\frac{1}{\sqrt{T}}$ }
- {Link[8, Alternating, 11], $\frac{1}{\sqrt{T}}$ }
- {Link[8, Alternating, 12], 1}
- {Link[8, Alternating, 13], 1}
- {Link[8, Alternating, 14], 1}
- {Link[8, Alternating, 15], 1}

```

{Link[8, Alternating, 16], 1}
{Link[8, Alternating, 17], 1}
{Link[8, Alternating, 18], 1}
{Link[8, Alternating, 19], 1}
{Link[8, Alternating, 20],  $\frac{1}{\sqrt{r}}$ }
{Link[8, Alternating, 21], 1}
{Link[8, NonAlternating, 1], 1}
{Link[8, NonAlternating, 2], 1}
{Link[8, NonAlternating, 3], 1}
{Link[8, NonAlternating, 4], 1}
{Link[8, NonAlternating, 5], 1}
{Link[8, NonAlternating, 6],  $\frac{1}{\sqrt{r}}$ }
{Link[8, NonAlternating, 7], 1}
{Link[8, NonAlternating, 8], Indeterminate}
{Link[9, Alternating, 1], T}
{Link[9, Alternating, 2], 1}
{Link[9, Alternating, 3], 1}
{Link[9, Alternating, 4], 1}
{Link[9, Alternating, 5], 1}
{Link[9, Alternating, 6],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 7], 1}
{Link[9, Alternating, 8], T}
{Link[9, Alternating, 9], 1}
{Link[9, Alternating, 10], 1}
{Link[9, Alternating, 11], 1}
{Link[9, Alternating, 12],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 13],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 14], 1}
{Link[9, Alternating, 15], 1}
{Link[9, Alternating, 16],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 17], 1}
{Link[9, Alternating, 18], 1}
{Link[9, Alternating, 19], 1}
{Link[9, Alternating, 20], 1}
{Link[9, Alternating, 21], 1}
{Link[9, Alternating, 22], 1}
{Link[9, Alternating, 23], 1}
{Link[9, Alternating, 24], 1}
{Link[9, Alternating, 25],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 26], 1}
{Link[9, Alternating, 27], 1}
{Link[9, Alternating, 28], 1}
{Link[9, Alternating, 29], 1}
{Link[9, Alternating, 30],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 31], 1}
{Link[9, Alternating, 32],  $\frac{1}{\sqrt{r}}$ }
{Link[9, Alternating, 33], 1}

```

```

{Link[9, Alternating, 34], 1}
{Link[9, Alternating, 35], 1}
{Link[9, Alternating, 36], 1}
{Link[9, Alternating, 37], 1}
{Link[9, Alternating, 38], 1}
{Link[9, Alternating, 39], 1}
{Link[9, Alternating, 40], 1}
{Link[9, Alternating, 41], 1}
{Link[9, Alternating, 42], 1}
{Link[9, Alternating, 43], 1}
{Link[9, Alternating, 44], 1}
{Link[9, Alternating, 45], 1}
{Link[9, Alternating, 46], 1}
{Link[9, Alternating, 47], 1}
{Link[9, Alternating, 48], 1}
{Link[9, Alternating, 49], 1}
{Link[9, Alternating, 50],  $\frac{1}{\sqrt{T}}$ }
{Link[9, Alternating, 51], 1}
{Link[9, Alternating, 52], 1}
{Link[9, Alternating, 53], 1}
{Link[9, Alternating, 54],  $\sqrt{T}$ }
{Link[9, Alternating, 55], 1}
{Link[9, NonAlternating, 1], T}
{Link[9, NonAlternating, 2], T}
{Link[9, NonAlternating, 3], T}
{Link[9, NonAlternating, 4], 1}
{Link[9, NonAlternating, 5], 1}
{Link[9, NonAlternating, 6], 1}
{Link[9, NonAlternating, 7], 1}
{Link[9, NonAlternating, 8], 1}
{Link[9, NonAlternating, 9], 1}
{Link[9, NonAlternating, 10], 1}
{Link[9, NonAlternating, 11], 1}
{Link[9, NonAlternating, 12],  $\frac{1}{\sqrt{T}}$ }
{Link[9, NonAlternating, 13], 1}
{Link[9, NonAlternating, 14], 1}
{Link[9, NonAlternating, 15], 1}
{Link[9, NonAlternating, 16], 1}
{Link[9, NonAlternating, 17], 1}
{Link[9, NonAlternating, 18], 1}
{Link[9, NonAlternating, 19], 1}
{Link[9, NonAlternating, 20], 1}
{Link[9, NonAlternating, 21], 1}
{Link[9, NonAlternating, 22], 1}
{Link[9, NonAlternating, 23], 1}
{Link[9, NonAlternating, 24], 1}
{Link[9, NonAlternating, 25], 1}
{Link[9, NonAlternating, 26], 1}
{Link[9, NonAlternating, 27], Indeterminate}
{Link[9, NonAlternating, 28], 1}

```

It remains to verify that this version of ρ does not depend on the component that was opened up. Here are some preliminary tests suggesting that this is indeed the case:

```
In[*]:= (*Test trefoil*)
\rho[{{1, 1, 4, 2, 5}, {1, 5, 2, 6, 3}, {1, 3, 6, 4, 7}}, {0, 0, 0, -1, 0, 0, 0}]

Out[*]=

$$\left\{ \frac{1 - T + T^2}{T}, -\frac{(-1 + T)^2 (1 + T^2)}{T^2} \right\}$$


In[*]:=
(*Hopf link opened up in two ways*)
\rho[{{1, 1, 4, 2, 3}, {1, 3, 2, 4, 5}}, {0, 0, 0, -1, 0}] // Expand
\rho[{{1, 4, 1, 5, 2}, {1, 2, 5, 3, 4}}, {0, 0, 0, 1, 0}] // Expand

Out[*]=

$$\left\{ \frac{1}{\sqrt{T}} - \sqrt{T}, 1 - \frac{1}{2T} - \frac{T}{2} \right\}$$


Out[*]=

$$\left\{ \frac{1}{\sqrt{T}} - \sqrt{T}, 1 - \frac{1}{2T} - \frac{T}{2} \right\}$$


In[*]:= (*L4,1 opened up in two ways, and its mirror image?!*)
\rho[{{1, 1, 6, 2, 7}, {1, 7, 2, 8, 3}, {1, 3, 8, 4, 9}, {1, 9, 4, 6, 5}},
{0, 0, 0, 0, 0, -1, 0, 0, 0}] // Expand
\rho[{{1, 6, 1, 7, 2}, {1, 2, 7, 3, 8}, {1, 8, 3, 9, 4}, {1, 4, 9, 5, 6}},
{0, 0, 0, 0, 0, 1, 0, 0, 0}] // Expand
\rho@@RotLink[PD[Link[4, Alternating, 1]]] // Expand

Out[*]=

$$\left\{ \frac{1}{T^{3/2}} - \frac{1}{\sqrt{T}} + \sqrt{T} - T^{3/2}, 4 - \frac{3}{2T^3} + \frac{3}{T^2} - \frac{7}{2T} - \frac{7T}{2} + 3T^2 - \frac{3T^3}{2} \right\}$$


Out[*]=

$$\left\{ \frac{1}{T^{3/2}} - \frac{1}{\sqrt{T}} + \sqrt{T} - T^{3/2}, 4 - \frac{3}{2T^3} + \frac{3}{T^2} - \frac{7}{2T} - \frac{7T}{2} + 3T^2 - \frac{3T^3}{2} \right\}$$


Out[*]=

$$\left\{ -\frac{2}{\sqrt{T}} + 2\sqrt{T}, -6 + \frac{3}{T} + 3T \right\}$$


In[*]:= \rho[{{1, 3, 10, 4, 11}, {-1, 9, 6, 10, 7}, {1, 5, 2, 6, 3}, {1, 11, 4, 8, 5}, {-1, 1, 8, 2, 9}},
{0, 0, 0, 1, 0, 0, 0, 0, 0, -1}] // Expand

Out[*]=

$$\left\{ -\frac{1}{\sqrt{T}} + 3\sqrt{T} - 3T^{3/2} + T^{5/2}, 10 - \frac{2}{T} - 20T + 23T^2 - 17T^3 + 7T^4 - T^5 \right\}$$

```