Pensieve header: Computing and playing with ρ_1 in the language of perturbed Gaussian Integration.

```
In[∘]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];
Once[<< KnotTheory`; << Rot.m];
```

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[ℰ];
CF[ℰ_List] := CF /@ ℰ; CF[ℰ_EPD] := CF /@ ℰ;
CF[ℰ_] := Module[{vs = Cases[ℰ, (x | p)_, ∞] ⋃ {x, p}, ps, c},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)] ];
```

```
In[∘]:= EQP /: c_ * EQP[Q_, P_] := EQP[Q, CF[c P]];
```

```
In[∘]:= r1[s_, i_, j_] :=
   s (-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j) / 2;
γ1[φ_, k_] := φ (1 / 2 - p_k x_k);
ρ0i[K_] := ρ0i[K, False]; ρ0i[Flip@K_] := ρ0i[K, True];
ρ0i[K_, flip_] := Module[{Cs, φ, n, s, i, j, k, vs, Q, Qp},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    If[flip, Cs = Cs〚All, {1, 3, 2}〛; φ = -φ];
    Q = -p_{2 n+1} x_{2 n+1}; Qp = 0;
    Cases[Cs, {s_, i_, j_} :>
      (Q -= x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1}); Qp -= s T^(s-1) x_i (p_{j+1} - p_{i+1}))];
    EQP[Q, -T^((Total[φ]+Total[Cs〚All,1〛])/2) Qp -
      (Total[φ] + Total[Cs〚All, 1〛]) T^((Total[φ]+Total[Cs〚All,1〛])/2-1) / 2]
   ];
ρ1i[K_] := ρ1i[K, False]; ρ1i[Flip@K_] := ρ1i[K, True];
ρ1i[K_, flip_] := Module[{Cs, φ, n, s, i, j, k, vs, Q, P},
    {Cs, φ} = Rot[K]; n = Length[Cs];
    If[flip, Cs = Cs〚All, {1, 3, 2}〛; φ = -φ];
    Q = -x_{2 n+1} p_{2 n+1};
    Cases[Cs, {s_, i_, j_} :> (Q -= x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1}))];
    P = Sum[r1 @@ Cs〚k〛, {k, n}] + Sum[γ1[φ〚k〛, k], {k, 2 n}];
    CF@EQP[Q, T^((Total[φ]+Total[Cs〚All,1〛])/2) P]
   ];
```

```
In[∘]:= {p*, x*} = {π, ξ}; (z_i__)* := (z*)_i; vs_List* := (v ↦ v*) /@ vs;
Zip_{}[ℰ_] := ℰ;
Zip_{z_,zs___}[ℰ_] := (Collect[ℰ // Zip_{zs}, z] /. f_. z^d_. :> (D[f, {z*, d}])) /. z* → 0
```

In[•]:=
```
FI[EQP[Q_, P_]] := FI[EQP[Q, P], Union@Cases[Q, p_, ∞], Union@Cases[Q, x_, ∞]];
FI[EQP[Q_, P_], ps_List, xs_List] := Module[{u, v},
  A = Table[∂_{u,v} Q, {u, ps}, {v, xs}];
  Factor[Det[A]^-1 Zip_{ps⋃xs}[P ⅇ^(-xs*.Inverse[A].ps*)]]]
```

In[•]:=
```
K = Knot[8, 17];
Factor[∂_T (Alexander[K][T]^-1)]
```

Out[•]=

$$\frac{(-1+T)\,T^2\,(1+T)\,\left(1-T+T^2\right)\,\left(3-5T+3T^2\right)}{\left(1-4T+8T^2-11T^3+8T^4-4T^5+T^6\right)^2}$$

In[•]:=
```
K = Knot[5, 2];
ρ1i[K]
```

Out[•]=

$$\mathrm{EQP}\Big[-((p_1-p_2)\,x_1)-\left(p_2-\frac{p_3}{T}+\left(-1+\frac{1}{T}\right)p_8\right)x_2-(p_3-p_4)\,x_3-\left(\left(-1+\frac{1}{T}\right)p_2+p_4-\frac{p_5}{T}\right)x_4-(p_5-p_6)\,x_5-\left(p_6-\frac{p_7}{T}+\left(-1+\frac{1}{T}\right)p_{10}\right)x_6-(p_7-p_8)\,x_7-\left(\left(-1+\frac{1}{T}\right)p_4+p_8-\frac{p_9}{T}\right)x_8-(p_9-p_{10})\,x_9-\left(\left(-1+\frac{1}{T}\right)p_6+p_{10}-\frac{p_{11}}{T}\right)x_{10}-p_{11}\,x_{11},$$

$$\frac{1}{T^3}\left(-\frac{1}{2}+p_4x_4+\frac{1}{2}\left(1+2p_1x_4-2p_4x_4-2p_1^2x_1x_4+2p_1p_4x_1x_4-\left(1-\frac{1}{T}\right)p_1^2x_4^2-\left(-1+\frac{1}{T}\right)p_1p_4x_4^2\right)+\frac{1}{2}\left(1-2p_2x_2+2p_7x_2-\left(-1+\frac{1}{T}\right)p_2p_7x_2^2-\left(1-\frac{1}{T}\right)p_7^2x_2^2+2p_2p_7x_2x_7-2p_7^2x_2x_7\right)+\frac{1}{2}\left(1+2p_3x_8-2p_8x_8-2p_3^2x_3x_8+2p_3p_8x_3x_8-\left(1-\frac{1}{T}\right)p_3^2x_8^2-\left(-1+\frac{1}{T}\right)p_3p_8x_8^2\right)-p_9x_9+\frac{1}{2}\left(1-2p_6x_6+2p_9x_6-\left(-1+\frac{1}{T}\right)p_6p_9x_6^2-\left(1-\frac{1}{T}\right)p_9^2x_6^2+2p_6p_9x_6x_9-2p_9^2x_6x_9\right)+p_{10}x_{10}+\frac{1}{2}\left(1+2p_5x_{10}-2p_{10}x_{10}-2p_5^2x_5x_{10}+2p_5p_{10}x_5x_{10}-\left(1-\frac{1}{T}\right)p_5^2x_{10}^2-\left(-1+\frac{1}{T}\right)p_5p_{10}x_{10}^2\right)\right)\Big]$$

In[•]:=
```
Factor@Together[FI@ρ1i[K]]
```

Out[•]=

$$-\frac{(-1+T)^2\,T\,\left(5-4T+5T^2\right)}{\left(2-3T+2T^2\right)^3}$$

In[∘]:=
```
K = Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs];
ν = {lν = 0}; writhe = Total@Cs[[All, 1]];
Do[Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
eqp = ρ1i[K];
eqp1 = T^-writhe CF[eqp /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
    Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1} + T^(s+ν[[i]]) p_1,
       x_i → -T^ν[[i]] p_{i+1} + T^ν[[i]] p_1, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
eqp2 = CF[ρ1i[Flip@K] /. T → T^-1];
FI /@ {eqp, eqp1, eqp2}
```

Out[∘]=

$$\left\{-\frac{(-1+T)^2\,T\,\left(1+T^2\right)}{\left(1-T+T^2\right)^3},\ -\frac{(-1+T)^2\,T\,\left(1+T^2\right)}{\left(1-T+T^2\right)^3},\ -\frac{(-1+T)^2\,T\,\left(1+T^2\right)}{\left(1-T+T^2\right)^3}\right\}$$

In[∘]:=
```
Simplify[eqp1[[1]] - eqp2[[1]]]
```

Out[∘]=

0

In[∘]:=
```
Simplify[eqp1[[2]] - eqp2[[2]]]
```

Out[∘]=

$$\frac{1}{2}\,T\,\Big(-2 - 2\,T\,p_5\,x_1 - 2\,T^2\,p_2\,p_5\,x_1^2 - T\,p_5^2\,x_1^2 + T^2\,p_5^2\,x_1^2 + 2\,p_3\,x_2 + 2\,p_3\,x_3 - 2\,p_6\,x_3 - 2\,T\,p_7\,x_3 -$$
$$p_3\,p_6\,x_3^2 + T\,p_3\,p_6\,x_3^2 + p_6^2\,x_3^2 - T\,p_6^2\,x_3^2 - T\,p_7^2\,x_3^2 + T^2\,p_7^2\,x_3^2 + 2\,T\,p_2\,p_5\,x_1\,x_4 + p_5^2\,x_1\,x_4 - T\,p_5^2\,x_1\,x_4 +$$
$$p_4^2\,x_1\,\left(-\left((-1+T)\,x_1\right) + 2\,x_4\right) - 2\,p_2\,x_5 - 2\,T\,p_3\,x_5 + 2\,p_5\,x_5 + 2\,p_2^2\,x_2\,x_5 + p_3^2\,x_2\,x_5 -$$
$$T\,p_3^2\,x_2\,x_5 - 2\,p_2\,p_5\,x_2\,x_5 + 2\,T\,p_3\,p_6\,x_2\,x_5 + p_2^2\,x_5^2 - T\,p_2^2\,x_5^2 - T\,p_3^2\,x_5^2 + T^2\,p_3^2\,x_5^2 - p_2\,p_5\,x_5^2 +$$
$$T\,p_2\,p_5\,x_5^2 - 2\,T^2\,p_3\,p_6\,x_5^2 - 2\,p_4\,\left(x_1 - x_4 + T\,p_7\,x_3\,\left(T\,x_3 - x_6\right)\right) + 2\,p_7\,x_6 - 2\,p_3\,p_6\,x_3\,x_6 +$$
$$2\,p_6^2\,x_3\,x_6 + p_7^2\,x_3\,x_6 - T\,p_7^2\,x_3\,x_6 - (1+T)\,p_1^2\,\left(T\,x_1^2 + T\,x_3^2 - x_1\,x_4 + x_5\,\left(-x_2 + T\,x_5\right) - x_3\,x_6\right) +$$
$$p_1\,\Big(\left(2\,T^2\,p_2 + (-1+T)\,p_4 + 2\,T\,p_5\right)\,x_1^2 - 2\,x_1\,\left(-1 - T + T\,p_2\,x_4 + p_4\,x_4 + p_5\,x_4\right) + 2\,\Big(T\,\left(T\,p_4 + p_7\right)\,x_3^2 +$$
$$T\,x_5 + T\,p_3\,x_5^2 + T^2\,p_6\,x_5^2 - x_2\,\left(1 + p_3\,x_5 + T\,p_6\,x_5\right) - x_6 + x_3\,\left(T - T\,p_4\,x_6 - p_7\,x_6\right)\Big)\Big)\Big)$$

In[∘]:=
```
FI@EQP[eqp1[[1]], eqp1[[2]] - eqp2[[2]]]
```

Out[∘]=

0

In[∘]:=
```
K = Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs];
ν = {lν = 0};
writhe = Total@Cs[[All, 1]];
Do[Cs /. {{s_, k, j_} :> AppendTo[ν, lν += s], {s_, i_, k} :> AppendTo[ν, lν -= s]}, {k, 2 n}];
eqp = ρ0i[K];
eqp1 = T^-writhe CF[eqp /. Flatten@{{x_{2 n+1} → p_1, p_{2 n+1} → x_{2 n+1}},
        Cs /. {s_Integer, i_, j_} :> {x_j → -T^ν[[i]] p_{j+1} + (1 - T^s) T^ν[[i]] p_{i+1} + T^(s+ν[[i]]) p_1,
           x_i → -T^ν[[i]] p_{i+1} + T^ν[[i]] p_1, p_i → T^-ν[[i]] x_i, p_j → T^(-ν[[i]]-s) x_j}}];
eqp2 = CF[ρ0i[Flip@K] /. T → T^-1];
FI /@ {eqp, eqp1, eqp2}
```

Out[∘]=

$$\left\{-\frac{(-1+T)\ (1+T)}{\left(1-T+T^2\right)^2},\ -\frac{(-1+T)\ (1+T)}{\left(1-T+T^2\right)^2},\ \frac{(-1+T)\ T^2\ (1+T)}{\left(1-T+T^2\right)^2}\right\}$$

In[∘]:=
```
Simplify[eqp1[[1]] - eqp2[[1]]]
```

Out[∘]=

0

In[∘]:=
```
CF[T^2 eqp1[[2]] + eqp2[[2]]]
```

Out[∘]=

$$3 T^2 + T^3 p_2 x_1 - T^3 p_5 x_1 - T p_1 x_2 + T p_5 x_2 + T^2 p_1 x_3 - T^2 p_3 x_3 + T^3 p_4 x_3 - T^3 p_7 x_3 - T p_1 x_4 + T p_7 x_4 + T^2 p_1 x_5 - T^3 p_3 x_5 - T^2 p_5 x_5 + T^3 p_6 x_5 - T p_1 x_6 + T p_3 x_6 + T^2 p_1 x_7 - T^2 p_7 x_7$$

In[∘]:=
```
FI@EQP[eqp1[[1]], T^2 eqp1[[2]] + eqp2[[2]]]
```

Out[∘]=

0