Pensieve header: Palindromicity by flipping and manipulating, in Gaussian integration language.

Initialization and Programs

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];
Once[
  << KnotTheory`;
  << Rot.m
 ];
CF[ℰ_] := Sum[Factor[∂_{x_i,p_j} ℰ] x_i p_j, {i, 0, 2 n + 2}, {j, 0, 2 n + 2}];
```

```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} := {g_{iβ_} ⧴ δ_{iβ} + T^s g_{i+1,β} + (1 - T^s) g_{j+1,β}, g_{jβ_} ⧴ δ_{jβ} + g_{j+1,β},
   g_{α_,i} ⧴ T^-s (g_{α,i+1} - δ_{α,i+1}), g_{α_j} ⧴ g_{α,j+1} - (1 - T^s) g_{αi} - δ_{α,j+1}}
```

pdf

```
In[∘]:= {p*, x*, p̄*, x̄*} = {π, ξ, π̄, ξ̄};  (z_{i__})* := (z*)_i;
Zip_{}[ℰ_] := ℰ;
Zip_{{z_,zs___}}[ℰ_] := (Collect[ℰ // Zip_{{zs}}, z] /. f_. z^d_. ⧴ (D[f, {z*, d}])) /. z* → 0
```

pdf

```
In[∘]:= gPair[ℰ_, w_] := Collect[Zip_{Join@@Table[{p_α,p̄_α,x_α,x̄_α},{α,w}]}[
    ℰ Exp[Sum[g_{α,β} (π_α + π̄_α) (ξ_β + ξ̄_β), {α, w}, {β, w}] - Sum[ξ̄_α π_α, {α, w}]]], g__, Factor]
```

Playing with a single knot

Initialization

```
In[∘]:= K = Mirror@Knot[3, 1]; {Cs, φ} = Rot[K]; n = Length[Cs]; ν = {lν = 0};
Do[
 Cs /. {{s_, k, j_} ⧴ AppendTo[ν, lν += s], {s_, i_, k} ⧴ AppendTo[ν, lν -= s]}, {k, 2 n}];
{Cs, ν}

Out[∘]=
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 1, 0, 1, 0, 1, 0}}
```

The quadratic of K:

```
In[∘]:= Q0 = Echo@CF[Total[
       Cs /. {s_Integer, i_, j_} ⧴ x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + x_{2 n+1} p_{2 n+1}];
```

» $p_1 x_1 - T p_2 x_1 + (-1 + T) p_5 x_1 + p_2 x_2 - p_3 x_2 + p_3 x_3 - T p_4 x_3 + (-1 + T) p_7 x_3 + p_4 x_4 - p_5 x_4 + (-1 + T) p_3 x_5 + p_5 x_5 - T p_6 x_5 + p_6 x_6 - p_7 x_6 + p_7 x_7$

Applying $x_j \to -x_j + (T^{-s} - 1)\, x_i$, $x_i \to -T^{-s}\, x_i$ and splitting off the edge terms:

In[∘]:=
```
Echo@CF[Q0 /. Join @@ (Cs /. {s_Integer, i_, j_} :> {x_j → -x_j + (T^-s - 1) x_i, x_i → -T^-s x_i})] ==
  (Q1 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s p_i x_i - p_j x_i + T^-s p_j x_i - p_j x_j] +
      Sum[x_k p_{k+1}, {k, 1, 2 n}] + x_{2 n+1} p_{2 n+1}])
```

» $-\frac{p_1 x_1}{T}+p_2 x_1-\frac{(-1+T)\, p_4 x_1}{T}-p_2 x_2+p_3 x_2-\frac{p_3 x_3}{T}+p_4 x_3-\frac{(-1+T)\, p_6 x_3}{T}-p_4 x_4+p_5 x_4-\frac{(-1+T)\, p_2 x_5}{T}-\frac{p_5 x_5}{T}+p_6 x_5-p_6 x_6+p_7 x_6+p_7 x_7$

Out[∘]=

True

Transposing, shifting from forward edges to backwards edges:

In[∘]:=
```
(Q1 /. {p → x, x → p}) ==
 Echo@(Q2 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_i - x_j p_i + T^-s x_j p_i - x_j p_j] +
      Sum[p_k x_{k+1}, {k, 1, 2 n}] + p_{2 n+1} x_{2 n+1}]) ==
 CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_i - x_j p_i + T^-s x_j p_i - x_j p_j] +
   Sum[p_{k-1} x_k, {k, 1, 2 n}] - p_0 x_1 + p_{2 n} x_{2 n+1} + p_{2 n+1} x_{2 n+1}]
```

» $-\frac{p_1 x_1}{T}+p_1 x_2-p_2 x_2-\frac{(-1+T)\, p_5 x_2}{T}+p_2 x_3-\frac{p_3 x_3}{T}-\frac{(-1+T)\, p_1 x_4}{T}+p_3 x_4-p_4 x_4+p_4 x_5-\frac{p_5 x_5}{T}-\frac{(-1+T)\, p_3 x_6}{T}+p_5 x_6-p_6 x_6+p_6 x_7+p_7 x_7$

Out[∘]=

True

Permuting the *p* variables and re-absorbing the edge terms into the crossings:

In[∘]:=
```
(Q2 /. {p_{2 n+1} → p_1, p_i_ :> p_{i+1}}) ==
 Echo@(Q3 = CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_{i+1} - x_j p_{i+1} + T^-s x_j p_{i+1} - x_j p_{j+1}] +
      Sum[p_k x_k, {k, 1, 2 n}] - p_1 x_1 + p_{2 n+1} x_{2 n+1} + p_1 x_{2 n+1}]) ==
 CF[Total[Cs /. {s_Integer, i_, j_} :> -T^-s x_i p_{i+1} + (T^-s - 1) x_j p_{i+1} - x_j p_{j+1} + p_i x_i + p_j x_j] -
   p_1 x_1 + p_{2 n+1} x_{2 n+1} + p_1 x_{2 n+1}]
```

» $-\frac{p_2 x_1}{T}+p_2 x_2-p_3 x_2-\frac{(-1+T)\, p_6 x_2}{T}+p_3 x_3-\frac{p_4 x_3}{T}-\frac{(-1+T)\, p_2 x_4}{T}+p_4 x_4-p_5 x_4+p_5 x_5-\frac{p_6 x_5}{T}-\frac{(-1+T)\, p_4 x_6}{T}+p_6 x_6-p_7 x_6+p_1 x_7+p_7 x_7$

Out[∘]=

True

Rescaling by T^v:

In[∘]:=
```
(Q4 = Echo@CF[Q3 /. {p_i_ :> T^ν⟦i⟧ p_i, x_i_ :> T^-ν⟦i⟧ x_i}]) ==
  CF[Total[Cs /. {s_Integer, i_, j_} :> -x_i p_(i+1) + (T^-s - 1) x_j p_(i+1) - T^-s x_j p_(j+1) + p_i x_i + p_j x_j] -
    p_1 x_1 + p_(2 n+1) x_(2 n+1) + p_1 x_(2 n+1)]
```

» $-p_2 x_1+p_2 x_2-\frac{p_3 x_2}{T}-\frac{(-1+T)\ p_6 x_2}{T}+p_3 x_3-p_4 x_3-\frac{(-1+T)\ p_2 x_4}{T}+p_4 x_4-\frac{p_5 x_4}{T}+p_5 x_5-p_6 x_5-\frac{(-1+T)\ p_4 x_6}{T}+p_6 x_6-\frac{p_7 x_6}{T}+p_1 x_7+p_7 x_7$

Out[∘]=

True

Using "col-sum = 0":

In[∘]:=
```
Simplify[Echo@CF[Q4 /. p_k_ /; k > 1 :> p_k - p_1] ==
  (Q5 = CF[Total[Cs /. {s_Integer, i_, j_} :> x_j (p_j - T^-s p_(j+1) + (T^-s - 1) p_(i+1)) + x_i (p_i - p_(i+1))] +
      p_(2 n+1) x_(2 n+1)])]
```

» $p_1 x_1-p_2 x_1+p_2 x_2-\frac{p_3 x_2}{T}-\frac{(-1+T)\ p_6 x_2}{T}+p_3 x_3-p_4 x_3-\frac{(-1+T)\ p_2 x_4}{T}+p_4 x_4-\frac{p_5 x_4}{T}+p_5 x_5-p_6 x_5-\frac{(-1+T)\ p_4 x_6}{T}+p_6 x_6-\frac{p_7 x_6}{T}+p_7 x_7$

Out[∘]=

True

The conjugate quadratic of the flip of *K*:

In[∘]:=
```
Qπ‾ = Echo@CF[Total[
      Cs /. {s_Integer, j_, i_} :> x_i (p_i - T^-s p_(i+1) + (T^-s - 1) p_(j+1)) + x_j (p_j - p_(j+1))] + x_(2 n+1) p_(2 n+1)];
```

» $p_1 x_1-p_2 x_1+p_2 x_2-\frac{p_3 x_2}{T}-\frac{(-1+T)\ p_6 x_2}{T}+p_3 x_3-p_4 x_3-\frac{(-1+T)\ p_2 x_4}{T}+p_4 x_4-\frac{p_5 x_4}{T}+p_5 x_5-p_6 x_5-\frac{(-1+T)\ p_4 x_6}{T}+p_6 x_6-\frac{p_7 x_6}{T}+p_7 x_7$

In[∘]:=
```
Qπ‾ == Q5
```

Out[∘]=

True

Playing with g_{kk}

In[∘]:=
```
P0 = x_i p_i + x_i p_j
```

Out[∘]=

$p_i x_i+p_j x_i$

Applying $x_j\to -x_j+(T^{-s}-1)\,x_i$, $x_i\to -T^{-s}\,x_i$:

In[∘]:=
```
P1 = P0 /. {x_j → -x_j + (T^-s - 1) x_i, x_i → -T^-s x_i}
```

Out[∘]=

$-T^{-s}\ p_i x_i-T^{-s}\ p_j x_i$

Transposing:

In[∘]:=
```
P2 = P1 /. {p → x, x → p}
```
Out[∘]=

$$-T^{-s} p_i x_i - T^{-s} p_i x_j$$

Permuting the *p* variables and re-absorbing the edge terms into the crossings:

In[∘]:=
```
P3 = P2 /. {p2 n+1 → p1, pi_ :→ pi+1}
```
Out[∘]=

$$-T^{-s} p_{1+i} x_i - T^{-s} p_{1+i} x_j$$

Rescaling by T^v:

In[∘]:=
```
P4 = P3 /. {pi_ :→ T^vi pi, xi_ :→ T^-vi xi} /. {vi+1 → vi + s, vj → vi + s, vj+1 → vi}
```
Out[∘]=

$$-p_{1+i} x_i - T^{-s} p_{1+i} x_j$$

In[∘]:=
```
P5 = P4 /. pk_ :→ pk - p1
```
Out[∘]=

$$-((-p_1 + p_{1+i}) x_i) - T^{-s} (-p_1 + p_{1+i}) x_j$$

The conjugate quadratic of the flip of *K*:

In[∘]:=
```
P0 /. {i → j, j → i}
```
Out[∘]=

$$p_i x_j + p_j x_j$$

Playing with R_1

r_1 is taken from Talks/Oaxaca-2210/Rho.nb

In[∘]:=
```
P0 = s (-1 + 2 pi xi - 2 pj xi + (-1 + T^s) pi pj xi^2 + (1 - T^s) pj^2 xi^2 - 2 pi pj xi xj + 2 pj^2 xi xj) / 2
```
Out[∘]=

$$\frac{1}{2} s \left(-1 + 2 p_i x_i - 2 p_j x_i + (-1 + T^s) p_i p_j x_i^2 + (1 - T^s) p_j^2 x_i^2 - 2 p_i p_j x_i x_j + 2 p_j^2 x_i x_j\right)$$

Applying $x_j \to -x_j + (T^{-s} - 1) x_i$, $x_i \to -T^{-s} x_i$:

In[∘]:=
```
P1 = P0 /. {xj → -xj + (T^-s - 1) xi, xi → -T^-s xi}
```
Out[∘]=

$$\frac{1}{2} s \left(-1 - 2 T^{-s} p_i x_i + 2 T^{-s} p_j x_i + T^{-2 s} (-1 + T^s) p_i p_j x_i^2 + T^{-2 s} (1 - T^s) p_j^2 x_i^2 + 2 T^{-s} p_i p_j x_i ((-1 + T^{-s}) x_i - x_j) - 2 T^{-s} p_j^2 x_i ((-1 + T^{-s}) x_i - x_j)\right)$$

Transposing:

In[∘]:= **P2 = P1 /. {p → x, x → p}**

Out[∘]=

$$\frac{1}{2} s \left(-1 - 2 T^{-s} p_i x_i + 2 T^{-s} p_i x_j + T^{-2 s} (-1 + T^s) p_i^2 x_i x_j + 2 T^{-s} p_i ((-1 + T^{-s}) p_i - p_j) x_i x_j + T^{-2 s} (1 - T^s) p_i^2 x_j^2 - 2 T^{-s} p_i ((-1 + T^{-s}) p_i - p_j) x_j^2\right)$$

Permuting the *p* variables and re-absorbing the edge terms into the crossings:

In[∘]:= **P3 = P2 /. {p_{2 n+1} → p_1, p_{i_} :→ p_{i+1}}**

Out[∘]=

$$\frac{1}{2} s \left(-1 - 2 T^{-s} p_{1+i} x_i + 2 T^{-s} p_{1+i} x_j + T^{-2 s} (-1 + T^s) p_{1+i}^2 x_i x_j + 2 T^{-s} p_{1+i} ((-1 + T^{-s}) p_{1+i} - p_{1+j}) x_i x_j + T^{-2 s} (1 - T^s) p_{1+i}^2 x_j^2 - 2 T^{-s} p_{1+i} ((-1 + T^{-s}) p_{1+i} - p_{1+j}) x_j^2\right)$$

Rescaling by T^v:

In[∘]:= **P4 = P3 /. {p_{i_} :→ T^{v_i} p_i, x_{i_} :→ T^{-v_i} x_i}**

Out[∘]=

$$\frac{1}{2} s \left(-1 - 2 T^{-s-v_i+v_{1+i}} p_{1+i} x_i + 2 T^{-s+v_{1+i}-v_j} p_{1+i} x_j + T^{-2 s-v_i+2 v_{1+i}-v_j} (-1 + T^s) p_{1+i}^2 x_i x_j + 2 T^{-s-v_i+v_{1+i}-v_j} p_{1+i} (T^{v_{1+i}} (-1 + T^{-s}) p_{1+i} - T^{v_{1+j}} p_{1+j}) x_i x_j + T^{-2 s+2 v_{1+i}-2 v_j} (1 - T^s) p_{1+i}^2 x_j^2 - 2 T^{-s+v_{1+i}-2 v_j} p_{1+i} (T^{v_{1+i}} (-1 + T^{-s}) p_{1+i} - T^{v_{1+j}} p_{1+j}) x_j^2\right)$$

In[∘]:= **P4 = Simplify[P3 /. {p_{i_} :→ T^{v_i} p_i, x_{i_} :→ T^{-v_i} x_i} /. {v_{i+1} → v_i + s, v_j → v_i + s, v_{j+1} → v_i}]**

Out[∘]=

$$-\frac{1}{2} s T^{-2 s} \left(T^{2 s} + (-1 + T^s) p_{1+i}^2 (T^s x_i - x_j) x_j + 2 p_{1+i} (T^s x_i - x_j) (T^s + p_{1+j} x_j)\right)$$

In[∘]:= **P5 = Simplify[P4 /. p_{k_} :→ p_k - p_1]**

Out[∘]=

$$-\frac{1}{2} s T^{-2 s} \left(T^{2 s} + (-1 + T^s) (p_1 - p_{1+i})^2 (T^s x_i - x_j) x_j + 2 (-p_1 + p_{1+i}) (T^s x_i - x_j) (T^s + (-p_1 + p_{1+j}) x_j)\right)$$

The conjugate perturbation of the flip of *K*:

In[∘]:= **P0 /. {T → T^{-1}, i → j, j → i}**

Out[∘]=

$$\frac{1}{2} s \left(-1 - 2 p_i x_j + 2 p_j x_j + 2 p_i^2 x_i x_j - 2 p_i p_j x_i x_j + \left(1 - \left(\frac{1}{T}\right)^s\right) p_i^2 x_j^2 + \left(-1 + \left(\frac{1}{T}\right)^s\right) p_i p_j x_j^2\right)$$

In[∘]:= **Simplify@PowerExpand[P5 - (P0 /. {T → T^{-1}, i → j, j → i})]**

Out[∘]=

$$\frac{1}{2} s \left(2 p_i x_j - 2 p_j x_j - 2 p_i^2 x_i x_j + 2 p_i p_j x_i x_j - T^{-2 s} (-1 + T^s) (p_1 - p_{1+i})^2 (T^s x_i - x_j) x_j - (1 - T^{-s}) p_i^2 x_j^2 - (-1 + T^{-s}) p_i p_j x_j^2 - 2 T^{-2 s} (-p_1 + p_{1+i}) (T^s x_i - x_j) (T^s + (-p_1 + p_{1+j}) x_j)\right)$$

In[∘]:=
```
gPair[Simplify@PowerExpand[P5 - (P0 /. {T → T^-1, i → j, j → i})], {i, j}]
```

Out[∘]=

$$-s\,T^{-s}\,(-1+T^{s})\,g_{i,j}^{2}-s\,g_{j,j}+s\,g_{i,i}\,g_{j,j}+g_{i,j}\,(s-2\,s\,g_{i,i}+s\,g_{j,i}+s\,T^{-s}\,(-1+T^{s})\,g_{j,j})$$

In[∘]:=
```
Simplify[gPair[Simplify@PowerExpand[P5 - (P0 /. {T → T^-1, i → j, j → i})] /. {p → x, x → p},
  {i, j}] //. gRules_{s,i,j}]
```

Out[∘]=

$$s\,\left(\left(-2+2\,T^{-3\,s}-7\,T^{-2\,s}+7\,T^{-s}\right)g_{1+j,1+i}^{2}+(-1+g_{1+i,1+i})\,g_{1+j,1+j}+g_{1+j,1+i}\left(-1+2\,T^{-s}+(2-4\,T^{-s})\,g_{1+i,1+i}+g_{1+i,1+j}-2\,g_{1+j,1+j}-T^{-2\,s}\,g_{1+j,1+j}+3\,T^{-s}\,g_{1+j,1+j}\right)\right)$$