Pensieve header: Palindromicity by flipping and manipulating, in Gaussian integration language.

Programs

In[◦]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APAI"];
Once[
  << KnotTheory`;
  << Rot.m
 ];
CF[ℰ_] := Sum[Factor[∂_{x_i,p_j} ℰ] x_i p_j, {i, 0, 2 n + 2}, {j, 0, 2 n + 2}]
```

In[◦]:=
```
K = Mirror@Knot[3, 1];
{Cs, ρ} = Rot[K]
 n = Length[Cs];
ν = {lν = 0};
For[k = 1, k ≤ 2 n, ++k,
 Cs /. {{s_, k, j_} ⧴ AppendTo[ν, lν += s], {s_, i_, k} ⧴ AppendTo[ν, lν -= s]}];
ν
Q0 = CF@Total[Cs /. {s_Integer, j_, i_} ⧴ x_i (p_i - T^s p_{i+1} + (T^s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] -
   x_{2 n+1} p_{2 n+1}
CF[Q0 /. Join @@ (Cs /. {s_Integer, j_, i_} ⧴ {x_i → T^-s x_i, x_j → x_j + (1 - T^-s) x_i})] ==
 (Q1 = CF[Total[Cs /. {s_Integer, j_, i_} ⧴ T^-s p_i x_i + p_j x_i - T^-s p_j x_i + p_j x_j] -
      Sum[x_k p_{k+1}, {k, 1, 2 n}] - x_{2 n+1} p_{2 n+1}])
(Q1 /. {p → x, x → p}) ==
 (Q2 = CF[Total[Cs /. {s_Integer, j_, i_} ⧴ T^-s x_i p_i + x_j p_i - T^-s x_j p_i + x_j p_j] -
      Sum[p_k x_{k+1}, {k, 1, 2 n}] - p_{2 n+1} x_{2 n+1}]) ==
 CF[Total[Cs /. {s_Integer, j_, i_} ⧴ T^-s x_i p_i + x_j p_i - T^-s x_j p_i + x_j p_j] -
    Sum[p_{k-1} x_k, {k, 1, 2 n}] + p_0 x_1 - p_{2 n} x_{2 n+1} - p_{2 n+1} x_{2 n+1}]
(Q2 /. {p_{2 n+1} → p_1, p_i_ ⧴ p_{i+1}}) ==
 (Q3 = CF[Total[Cs /. {s_Integer, j_, i_} ⧴ T^-s x_i p_{i+1} + x_j p_{i+1} - T^-s x_j p_{i+1} + x_j p_{j+1}] -
      Sum[p_k x_k, {k, 1, 2 n}] + p_1 x_1 - p_{2 n+1} x_{2 n+1} - p_1 x_{2 n+1}]) ==
 CF[Total[Cs /. {s_Integer, j_, i_} ⧴ T^-s x_i p_{i+1} + (1 - T^-s ) x_j p_{i+1} + x_j p_{j+1} - x_i p_i - x_j p_j] +
    p_1 x_1 - p_{2 n+1} x_{2 n+1} - p_1 x_{2 n+1}]
CF[Q4 = Q3 /. {p_i_ ⧴ T^-ν[[i]] p_i, x_i_ ⧴ -T^ν[[i]] x_i}] ==
 CF[Total[Cs /. {s_Integer, i_, j_} ⧴ x_i (p_i - T^-s p_{i+1} + (T^-s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] -
    p_1 x_1 + p_{2 n+1} x_{2 n+1} + p_1 x_{2 n+1}]
CF[Q4 /. p_k_ /; k > 1 ⧴ p_k - p_1] == CF[
  Total[Cs /. {s_Integer, i_, j_} ⧴ x_i (p_i - T^-s p_{i+1} + (T^-s - 1) p_{j+1}) + x_j (p_j - p_{j+1})] + p_{2 n+1} x_{2 n+1}]
```

Out[◦]=
```
{{{1, 1, 4}, {1, 3, 6}, {1, 5, 2}}, {0, 0, 0, -1, 0, 0}}
```

Out[◦]=
```
{0, 1, 0, 1, 0, 1, 0}
```

Out[ ]=

$$p_1 x_1 - p_2 x_1 + p_2 x_2 - T p_3 x_2 + (-1 + T) p_6 x_2 + p_3 x_3 - p_4 x_3 + (-1 + T) p_2 x_4 + p_4 x_4 - T p_5 x_4 + p_5 x_5 - p_6 x_5 + (-1 + T) p_4 x_6 + p_6 x_6 - T p_7 x_6 - p_7 x_7$$

Out[ ]=

True

Out[ ]=

True

Out[ ]=

True

Out[ ]=

True

Out[ ]=

True

In[ ]:=

```
% // Simplify
```

Out[ ]=

$$p_7 x_7 == 0$$

In[•]:=

```
AlexanderMatrices[K_] := Module[{},
  {Cs, ρ} = Rot[K]; n = Length[Cs];
  ν = {lν = 0};
  For[k = 1, k ≤ 2 n, ++k,
   Cs /. {{s_, k, j_} :→ AppendTo[ν, lν += s], {s_, i_, k} :→ AppendTo[ν, lν -= s]}];
  ν = DiagonalMatrix[T^ν]; νi = Inverse@ν;
  A0 = Table[0, 2 n + 1, 2 n + 1]; A0[[2 n + 1, 2 n + 1]] = 1;
  C2 = C1 = Aπ = A1 = A2 = A3 = A4 = A0;
  P1 = RotateLeft[IdentityMatrix[2 n + 1]];
  (*P1=Table[0,2n+1,2n+1]; Do[P1[[i,i+1]]=1,{i,2n}];*)
  P2 = IdentityMatrix[2 n + 1]; Do[P2[[i, 1]] = -1, {i, 2, 2 n + 1}];
  E_{i_,j_} := ReplacePart[Table[0, 2 n + 1, 2 n + 1], {i, j} → 1];
  Cases[Cs, {s_Integer, i_, j_} :→ (Aπ[[{i, j}, {i, j, i + 1, j + 1}]] = (1 0 -1 0; 0 1 T^s - 1 -T^s);
     A0[[{i, j}, {i, j, i + 1, j + 1}]] = (1 0 -T^s T^s - 1; 0 1 0 -1);
     C1[[{i, j}, {i, j}]] = (-T^-s T^-s - 1; 0 -1); C2[[{i, j}, {i, j}]] = (-T^-s 0; T^-s - 1 -1);
     A1[[{i, j}, {i, j, i + 1, j + 1}]] = (-T^-s T^-s - 1 1 0; 0 -1 0 1);
     A2[[{i, j, i + 1, j + 1}, {i, j}]] = (-T^-s 0; T^-s - 1 -1; 1 0; 0 1);
     A3[[{i, j}, {i, j, i + 1, j + 1}]] = (1 0 -T^-s 0; 0 1 T^-s - 1 -1);
     A4[[{i, j}, {i, j, i + 1, j + 1}]] = (1 0 -1 0; 0 1 T^-s - 1 -T^-s);)]; ]

AlexanderMatrices[Knot[10, 165]]
Det /@ {Aπ, A0, A1, A2, A3, A4}
Simplify[{C1.A0 == A1, (C1.A0)^T == A2, A0^T.C2.P1 + E_{1,1} == A3,
  νi.A0^T.C2.P1.ν + E_{1,1} == A4, νi.A0^T.C2.P1.ν.P2 == (Aπ /. T → T^-1)}]
```

Out[•]=

$$\left\{\frac{-2T^3+10T^4-15T^5+10T^6-2T^7}{T}, \frac{-2T^2+10T^3-15T^4+10T^5-2T^6}{T^2}, -\frac{2T^2-10T^3+15T^4-10T^5+2T^6}{T^8}, -\frac{2T^9-10T^{10}+15T^{11}-10T^{12}+2T^{13}}{T^{15}}, \frac{-2T^9+10T^{10}-15T^{11}+10T^{12}-2T^{13}}{T^{15}}, \frac{-2T^2+10T^3-15T^4+10T^5-2T^6}{T^8}\right\}$$

Out[•]=

```
{True, True, False, False, True}
```

In[•]:=

```
Total@Table[AlexanderMatrices[K];
  Simplify[νi.A0ᵀ.C2.P1.ν.P2 == (Aπ /. T → T^-1) ∧ (C2 /. T → T^-1) == Inverse[C2]],
  {K, AllKnots[{3, 10}]}]
```

Out[•]=

```
249 True
```

In[•]:=

```
AlexanderMatrices[Knot[3, 1]];
{lhs, rhs} = Simplify@{νi.A0ᵀ.C2.P1.ν.P2, Aπ /. T → T^-1};
MatrixForm /@ {C2, C2.P1, P1.ν.P2, Inverse[P1.ν.P2], ν, lhs, rhs}
Det /@ {lhs, rhs}
Simplify[lhs == rhs]
```

Out[•]=

$$\left\{\begin{pmatrix} -1 & 0 & 0 & -1+T & 0 & 0 & 0 \\ 0 & -T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1+T & 0 \\ 0 & 0 & 0 & -T & 0 & 0 & 0 \\ 0 & -1+T & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 & 0 & -1+T & 0 & 0 \\ 0 & 0 & -T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1+T \\ 0 & 0 & 0 & 0 & -T & 0 & 0 \\ 0 & 0 & -1+T & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -T \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\right.$$

$$\begin{pmatrix} -T & T & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -T & 0 & 0 & T & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ -T & 0 & 0 & 0 & 0 & T & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{T} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{T} & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\left.\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\}$$

Out[•]=

$$\{1 - T + T^2, 1 - T + T^2\}$$

Out[•]=

```
True
```

In[∘]:=
```
AlexanderMatrices[Knot[5, 2]];
{lhs, rhs} = Simplify@{νi.A0ᵀ.C2.P1.ν.P2, Aπ /. T → T^-1};
MatrixForm /@ {C2, Inverse[C2], ν, P1, P2, A0, Aπ}
Simplify[(C2 /. T → T^-1) == Inverse[C2]]
lhs == rhs
```

Out[∘]=

$$\left\{\begin{pmatrix}
-1 & 0 & 0 & -1+T & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1+T & 0 & 0 & 0\\
0 & 0 & 0 & -T & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1+T & 0\\
0 & 0 & 0 & 0 & 0 & -T & 0 & 0 & 0 & 0 & 0\\
0 & -1+T & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -T & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1+T & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},\right.$$

$$\begin{pmatrix}
-1 & 0 & 0 & \frac{1-T}{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1-T}{T} & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & \frac{1-T}{T} & 0\\
0 & 0 & 0 & 0 & 0 & -\frac{1}{T} & 0 & 0 & 0 & 0 & 0\\
0 & \frac{1-T}{T} & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{T} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \frac{1-T}{T} & 0 & 0 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{T} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & T & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & T & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & T & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & T & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & -1+\frac{1}{T} & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1+\frac{1}{T} & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & -1+\frac{1}{T} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1+\frac{1}{T} & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1+\frac{1}{T} & 0 & 0 & 0 & 1 & -\frac{1}{T} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},$$

$$\begin{pmatrix}
1 & -\frac{1}{T} & 0 & 0 & -1+\frac{1}{T} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & -1+\frac{1}{T} & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 & 0 & -1+\frac{1}{T} \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1+\frac{1}{T} & 0 & 0 & 0 & 1 & -\frac{1}{T} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1+\frac{1}{T} & 0 & 1 & -\frac{1}{T} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}\Big\}$$

Out[*]=

True

Out[*]=

True