Pensieve header: Mathematica notebook for A Perturbed Alexander Invariant.

In[ ]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\APerturbedAlexanderInvariant"];
```

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Two of the main reasons we like ρ_1 is that it is very easy to implement and even an unsophisticated implementation runs very fast. To highlight these points we include a full implementation here, a step-by-step run-through, and a demo run. We write in Mathematica~\cite{Wolfram:Mathematica}, and you can find the notebook displayed here at~\cite[APAI.nb]{Self}.

We start by loading the library \verb$KnotTheory`$~\cite{Bar-NatanMorrison:KnotTheory} (it is used here only for the list of knots that it contains, and to compute other invariants for comparisons). We also load minor conversion routine~\cite[RVK.nb / RVK.m]{Self} whose internal workings are simple and yet irrelevant here.

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In[ ]:=
```
Once[<< KnotTheory`; << RVK.m];
```

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\subsection{The Program} This done, here is the full ρ_1 program:

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In[ ]:=
```
ρ[K_] := ρ[K] = Module[{Cs, r, n, B, A, c, s, i, j, Δ, G, g, ρ1},
    {Cs, r} = List @@ RVK[K]; n = Length[Cs]; B = Table[0, 2 n, 2 n + 1];
    Do[{s, i, j} = c;
     B[[{i, j}, {i, j, i + 1, j + 1}]] = ({{1 0  -1   0}, {0 1 T^s - 1 -T^s}}), {c, Cs}];
    A = B[[All, 2 ;;]];
    Δ = T^((Total[r]-Total[First/@Cs])/2 Det[A];
    G = Prepend[Table[0, 2 n]][Inverse[A]]; g_α_,β_ := G[[α, β]];
    ρ1 = Δ^2 Sum[{s, i, j} = c;
       s ((1 - T^s) g_ij (g_ij - g_jj) + 2 g_ii g_ij - g_ij g_ji - g_ii g_jj - g_ij + g_jj - 1 / 2), {c, Cs}];
    ρ1 += Δ^2 Sum[r[[k]] (g_kk - 1 / 2), {k, 2 n}] ;
    Factor@{Δ, ρ1}];
```

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The program uses mostly the same symbols as the text, so even without any knowledge of Mathematica, the reader should be able to recognize at least formulas~\eqref{eq:B}, \eqref{eq:Delta}, and~\eqref{eq:rho1} within it. As a further hint we add that the variables \verb$Cs$ ends up storing the list of crossing in a knot K, where each crossing is stored as a triple $\{s,i,j\}$, where s, i, and j have the same meaning as in~\eqref{eq:B}. The conversion routine \verb$RVK$ automatically produces \verb$Cs$, as well as a list \verb$r$ of rotation numbers, given any other knot presentation known to the package \verb$KnotTheory`$.

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Note that the program outputs the ordered pair (Δ,ρ_1). The Alexander polynomial Δ is anyway computed internally, and we consider the aggregate (Δ,ρ_1) as more interesting than any of its pieces by itself.

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\subsection{A Step-by-Step Run-Through} We start by setting K to be the knot diagram on page~1 using the \verb$PD$ notation of \verb$KnotTheory`$. We then print \verb$RVK[K]$, which is a list of crossings followed by a list of rotation numbers:

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```
In[•]:= K = PD[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]];
        RVK[K]
```

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```
Out[•]= RVK[{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}]
```

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Next we set \verb$Cs$ and \verb$r$ to be the list of crossings, and the list of rotation numbers, respectively.

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```
In[•]:= {Cs, r} = List @@ RVK[K]
```

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```
Out[•]= {{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}
```

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We set \verb$n$ to be the number of crossings, \verb$B$ to be the zero matrix of dimensions $2n\times(2n+1)$, and then we iterate over \verb$c$ in \verb$Cs$, adding a block as in~\eqref{eq:B} for each crossings.

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```
In[•]:= n = Length[Cs]; B = Table[0, 2 n, 2 n + 1];
        Do[{s, i, j} = c;
          B[[{i, j}, {i, j, i + 1, j + 1}]] = {{1, 0, -1, 0}, {0, 1, T^s - 1, -T^s}}, {c, Cs}];
```

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Here's what \verb$B$ comes out to be:

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```
In[•]:= B // MatrixForm
```

Out[•]//MatrixForm=
pdf

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -T & 0 & 0 & -1+T & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & -1+T & 0 & 1 & -T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1+T & 0 & 1 & -T \end{pmatrix}$$

Here's the same, in TeXForm:

In[•]:=
```
B // MatrixForm // TeXForm
```

Out[•]//TeXForm=
```
\left(
\begin{array}{ccccccc}
 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & -T & 0 & 0 & T-1 & 0 \\
 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
 0 & T-1 & 0 & 1 & -T & 0 & 0 \\
 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
 0 & 0 & 0 & T-1 & 0 & 1 & -T \\
\end{array}
\right)
```

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Next we set \verb$A$ to be the matrix whose rows are \verb$All$ the rows of \verb$B$, and whose columns are the columns of \verb$B$ starting from column 2. We set Δ to be the determinant of \verb$A$, with a correction as in~\eqref{eq:Delta}. So Δ is the Alexander polynomial of K.

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In[•]:=
```
A = B[[All, 2 ;;]];
Δ = T^((Total[r]-Total[First/@Cs])/2 Det[A]
```

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Out[•]=
$$\frac{T-T^2+T^3}{T^2}$$

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\verb$G$ is now the \verb$Inverse$ of \verb$A$ with a row of 0's added at the start. We set ${\mathtt g}_{\alpha\beta}$ to be the matrix entries of \verb$G$, and print \verb$G$:

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In[•]:=
```
G = Prepend[Table[0, 2 n]][Inverse[A]]; g_α_,β_ := G[[α, β]];
G // MatrixForm
```

Out[•]//MatrixForm=
pdf

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & 0 & 0 & 0 & 0 & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & \frac{T-T^2}{T-T^2+T^3} & \frac{T-T^2}{T-T^2+T^3} & \frac{T^2-T^3}{T-T^2+T^3} & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & -\frac{T^3}{T-T^2+T^3} & \frac{T-T^2}{T-T^2+T^3} & \frac{T^2-T^3}{T-T^2+T^3} & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & -\frac{T}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & \frac{T-T^2}{T-T^2+T^3} & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & -\frac{T}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & -\frac{T^2}{T-T^2+T^3} & -\frac{T^3}{T-T^2+T^3} & 0 \\
\frac{-T+T^2-T^3}{T-T^2+T^3} & \frac{-1+T-T^2}{T-T^2+T^3} & \frac{-T+T^2-T^3}{T-T^2+T^3} & \frac{-1+T-T^2}{T-T^2+T^3} & \frac{-T+T^2-T^3}{T-T^2+T^3} & \frac{-1+T-T^2}{T-T^2+T^3}
\end{pmatrix}$$

Here's the same, in TeXForm:

In[•]:=
```
G // MatrixForm // TeXForm
```

Out[•]//TeXForm=
```
\left(
\begin{array}{cccccc}
 0 & 0 & 0 & 0 & 0 & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & 0 & 0 & 0 & 0 & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & -\frac{T^2}{T^3-T^2+T} & \frac{T-T^2}{T^3-T^2+T} &
   \frac{T-T^2}{T^3-T^2+T} & \frac{T^2-T^3}{T^3-T^2+T} & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & -\frac{T^2}{T^3-T^2+T} & -\frac{T^3}{T^3-T^2+T} &
   \frac{T-T^2}{T^3-T^2+T} & \frac{T^2-T^3}{T^3-T^2+T} & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & -\frac{T}{T^3-T^2+T} & -\frac{T^2}{T^3-T^2+T} &
   -\frac{T^2}{T^3-T^2+T} & \frac{T-T^2}{T^3-T^2+T} & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & -\frac{T}{T^3-T^2+T} & -\frac{T^2}{T^3-T^2+T} &
   -\frac{T^2}{T^3-T^2+T} & -\frac{T^3}{T^3-T^2+T} & 0 \\
 \frac{-T^3+T^2-T}{T^3-T^2+T} & \frac{-T^2+T-1}{T^3-T^2+T} & \frac{-T^3+T^2-T}{T^3-T^2+T}
   & \frac{-T^2+T-1}{T^3-T^2+T} & \frac{-T^3+T^2-T}{T^3-T^2+T} &
   \frac{-T^2+T-1}{T^3-T^2+T} \\
\end{array}
\right)
```

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It remains to blindly follow the two parts of Equation~\eqref{eq:rho1}:

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In[•]:=
```
ρ1 = Δ^2 Sum[{s, i, j} = c;
    s ((1 - T^s) g_ij (g_ij - g_jj) + 2 g_ii g_ij - g_ij g_ji - g_ii g_jj - g_ij + g_jj - 1 / 2), {c, Cs}]
```

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Out[•]=
$$\frac{\left(T-T^2+T^3\right)^2 \left(-\frac{3}{2}+\frac{(1-T)\,T^2}{\left(T-T^2+T^3\right)^2}-\frac{2\,T\left(T-T^2\right)}{\left(T-T^2+T^3\right)^2}+\frac{T}{T-T^2+T^3}+\frac{T-T^2}{T-T^2+T^3}\right)}{T^4}$$

pdf

In[•]:=
```
ρ1 += Δ^2 Sum[r[[k]] (g_kk - 1 / 2), {k, 2 n}]
```

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Out[•]=
$$\frac{\left(T-T^2+T^3\right)^2 \left(\frac{1}{2}-\frac{T-T^2}{T-T^2+T^3}\right)}{T^4}+\frac{\left(T-T^2+T^3\right)^2 \left(-\frac{3}{2}+\frac{(1-T)\,T^2}{\left(T-T^2+T^3\right)^2}-\frac{2\,T\left(T-T^2\right)}{\left(T-T^2+T^3\right)^2}+\frac{T}{T-T^2+T^3}+\frac{T-T^2}{T-T^2+T^3}\right)}{T^4}$$

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And to output both Δ and ρ_1. We factor them just to put them in a nicer form:

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In[•]:=
```
Factor@{Δ, ρ1}
```

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Out[•]=
$$\left\{\frac{1-T+T^2}{T},\ -\frac{(-1+T)^2\left(1+T^2\right)}{T^2}\right\}$$

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\subsection{A Demo Run} Here are Δ and ρ_1 of all the knots with up to 6 crossings:

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In[•]:=
```
Do[Echo[K → ρ[K]], {K, AllKnots[{3, 6}]}]
```

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KnotTheory: Loading precomputed data in PD4Knots`.

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» $\text{Knot}[3,1] \to \left\{\frac{1-T+T^2}{T}, \frac{(-1+T)^2(1+T^2)}{T^2}\right\}$

pdf

» $\text{Knot}[4,1] \to \left\{-\frac{1-3T+T^2}{T}, 0\right\}$

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» $\text{Knot}[5,1] \to \left\{\frac{1-T+T^2-T^3+T^4}{T^2}, \frac{(-1+T)^2(1+T^2)(2+T^2+2T^4)}{T^4}\right\}$

pdf

» $\text{Knot}[5,2] \to \left\{\frac{2-3T+2T^2}{T}, \frac{(-1+T)^2(5-4T+5T^2)}{T^2}\right\}$

pdf

» $\text{Knot}[6,1] \to \left\{-\frac{(-2+T)(-1+2T)}{T}, \frac{(-1+T)^2(1-4T+T^2)}{T^2}\right\}$

pdf

» $\text{Knot}[6,2] \to \left\{-\frac{1-3T+3T^2-3T^3+T^4}{T^2}, \frac{(-1+T)^2(1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}\right\}$

pdf

» $\text{Knot}[6,3] \to \left\{\frac{1-3T+5T^2-3T^3+T^4}{T^2}, 0\right\}$

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\begin{figure}
\[\resizebox{6in}{!}{\input{GST48-Marked.pdf_t}} \]
\caption{A 48-crossing knot from~\cite{GompfScharlemannThompson:Counterexample}.}
\label{fig:GST48}
\end{figure}

Next is one of our favourites, a knot from~\cite{GompfScharlemannThompson:Counterexample} (see Figure~\ref{fig:GST48}), which is a potential counterexample to the ribbon$=$slice conjecture. It takes about one minute to compute ρ_1 for this 48 crossing knot (note that Mathematica prints \verb$Timing$ information is seconds):

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In[•]:=
```
Timing@ρ[EPD[X[14,1], X[2,29], X[3,40], X[43,4], X[26,5], X[6,95], X[96,7], X[13,8], X[9,28], X[10,41], X[42,11], X[27,12],
  X[30,15], X[16,61], X[17,72], X[18,83], X[19,34], X[89,20], X[21,92], X[79,22], X[68,23], X[57,24], X[25,56], X[62,31],
  X[73,32], X[84,33], X[50,35], X[36,81], X[37,70], X[38,59], X[39,54], X[44,55], X[58,45], X[69,46], X[80,47], X[48,91],
  X[90,49], X[51,82], X[52,71], X[53,60], X[63,74], X[64,85], X[76,65], X[87,66], X[67,94], X[75,86], X[88,77], X[78,93]]]
```

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Out[•]=
$$\left\{77.1563, \left\{-\frac{(-1+2T-T^2-T^3+2T^4-T^5+T^8)(-1+T^3-2T^4+T^5+T^6-2T^7+T^8)}{T^8},\right.\right.$$
$$\frac{1}{T^{16}}(-1+T)^2(5-18T+33T^2-32T^3+2T^4+42T^5-62T^6-8T^7+166T^8-242T^9+108T^{10}+$$
$$\left.\left.132T^{11}-226T^{12}+148T^{13}-11T^{14}-36T^{15}-11T^{16}+148T^{17}-226T^{18}+132T^{19}+108T^{20}-242T^{21}+166T^{22}-8T^{23}-62T^{24}+42T^{25}+2T^{26}-32T^{27}+33T^{28}-18T^{29}+5T^{30})\right\}\right\}$$

```
In[•]:= Timing@
          Jones[EPD[X_{14,1}, X̄_{2,29}, X_{3,40}, X_{43,4}, X̄_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, X̄_{9,28}, X_{10,41}, X_{42,11}, X̄_{27,12}, X_{30,15},
              X̄_{16,61}, X̄_{17,72}, X̄_{18,83}, X_{19,34}, X̄_{89,20}, X̄_{21,92}, X̄_{79,22}, X̄_{68,23}, X̄_{57,24}, X̄_{25,56}, X_{62,31}, X_{73,32}, X_{84,33},
              X̄_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, X̄_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91}, X_{90,49}, X_{51,82}, X_{52,71},
              X_{53,60}, X̄_{63,74}, X̄_{64,85}, X̄_{76,65}, X̄_{87,66}, X̄_{67,94}, X̄_{75,86}, X̄_{88,77}, X̄_{78,93}]][q] /. P[1, 97] → 1 + q
```

$$\left\{58.2344,\ \frac{1}{q^{13/2}}-\frac{1}{q^{11/2}}+\frac{1}{q^{5/2}}-\frac{3}{q^{3/2}}+\frac{3}{\sqrt{q}}-5\sqrt{q}+5q^{3/2}-5q^{5/2}+3q^{7/2}+q^{9/2}-3q^{11/2}+\right.$$
$$\left.4q^{13/2}-2q^{15/2}+q^{17/2}-q^{19/2}+q^{21/2}-q^{23/2}+q^{25/2}-3q^{27/2}+4q^{29/2}-3q^{31/2}+q^{33/2}\right\}$$

The knot below was taken from the Dunfield list; its EPD presentation was computed at GST48.nb:

```
In[•]:= Timing@ρ[EPD[X_{72,11}, X_{2,83}, X_{33,20}, X̄_{14,7}, X_{30,27}, X_{74,57}, X_{21,34}, X_{6,19}, X_{60,29}, X̄_{89,92}, X_{87,50}, X̄_{18,13},
              X_{12,73}, X_{5,94}, X̄_{8,17}, X_{55,48}, X_{46,75}, X̄_{91,52}, X_{71,68}, X_{69,66}, X_{58,61}, X_{49,54}, X_{76,47}, X̄_{81,40},
              X_{95,4}, X_{63,24}, X_{28,59}, X_{39,42}, X̄_{65,62}, X_{43,84}, X_{67,70}, X̄_{16,9}, X̄_{77,56}, X_{82,1}, X_{23,64}, X̄_{41,80},
              X_{93,78}, X̄_{51,90}, X_{26,31}, X_{35,32}, X_{36,3}, X_{45,86}, X_{79,38}, X̄_{37,96}, X_{85,44}, X̄_{10,15}, X̄_{25,22}, X_{53,88}]]
```

$$\left\{85.8281,\right.$$
$$\left\{\frac{1}{T^{12}}\left(1-T+T^2\right)\left(16-254T+1793T^2-7557T^3+21126T^4-40030T^5+46606T^6-9674T^7-93093T^8+\right.\right.$$
$$245384T^9-386265T^{10}+443897T^{11}-386265T^{12}+245384T^{13}-93093T^{14}-$$
$$9674T^{15}+46606T^{16}-40030T^{17}+21126T^{18}-7557T^{19}+1793T^{20}-254T^{21}+16T^{22}\Big),$$
$$-\frac{1}{T^{24}}(-1+T)^2\left(1720-54728T+821057T^2-7773930T^3+52327950T^4-266751514T^5+\right.$$
$$1067934146T^6-3422377568T^7+8807134930T^8-17830580990T^9+26179241761T^{10}-$$
$$18120014484T^{11}-37650843764T^{12}+179456394542T^{13}-421154821337T^{14}+700721793322$$
$$T^{15}-836694883356T^{16}+539303589844T^{17}+488825640017T^{18}-2380033057788T^{19}+$$
$$4941886324068T^{20}-7623935664316T^{21}+9675216710671T^{22}-10444802180416T^{23}+$$
$$9675216710671T^{24}-7623935664316T^{25}+4941886324068T^{26}-2380033057788T^{27}+$$
$$488825640017T^{28}+539303589844T^{29}-836694883356T^{30}+700721793322T^{31}-$$
$$421154821337T^{32}+179456394542T^{33}-37650843764T^{34}-18120014484T^{35}+$$
$$26179241761T^{36}-17830580990T^{37}+8807134930T^{38}-3422377568T^{39}+1067934146T^{40}-$$
$$\left.\left.266751514T^{41}+52327950T^{42}-7773930T^{43}+821057T^{44}-54728T^{45}+1720T^{46}\right)\right\}\right\}$$

```
In[•]:= Timing@Jones[EPD[X72,11, X2,83, X33,20, X̄14,7, X30,27, X74,57, X21,34, X6,19, X60,29, X̄89,92, X87,50, X̄18,13,
          X12,73, X5,94, X̄8,17, X55,48, X46,75, X̄91,52, X71,68, X69,66, X58,61, X49,54, X76,47, X̄81,40, X95,4,
          X63,24, X28,59, X39,42, X̄65,62, X43,84, X67,70, X̄16,9, X̄77,56, X82,1, X23,64, X̄41,80, X93,78, X̄51,90,
          X26,31, X35,32, X36,3, X45,86, X79,38, X̄37,96, X85,44, X̄10,15, X̄25,22, X53,88]][q] /. P[1, 97] → 1 + q
```

$$\left\{29.7188,\ -\frac{1}{q^{7/2}}+\frac{11}{q^{5/2}}-\frac{70}{q^{3/2}}+\frac{320}{\sqrt{q}}-1158\sqrt{q}+3514\,q^{3/2}-9227\,q^{5/2}+21\,415\,q^{7/2}-\right.$$
$$44\,564\,q^{9/2}+83\,962\,q^{11/2}-144\,164\,q^{13/2}+226\,434\,q^{15/2}-325\,748\,q^{17/2}+428\,430\,q^{19/2}-$$
$$511\,994\,q^{21/2}+548\,301\,q^{23/2}-510\,177\,q^{25/2}+380\,093\,q^{27/2}-158\,051\,q^{29/2}-134\,490\,q^{31/2}+$$
$$456\,934\,q^{33/2}-758\,231\,q^{35/2}+989\,514\,q^{37/2}-1\,116\,571\,q^{39/2}+1\,127\,964\,q^{41/2}-1\,036\,182\,q^{43/2}+$$
$$871\,892\,q^{45/2}-673\,838\,q^{47/2}+478\,190\,q^{49/2}-310\,789\,q^{51/2}+184\,108\,q^{53/2}-98\,705\,q^{55/2}+$$
$$\left.47\,423\,q^{57/2}-20\,146\,q^{59/2}+7429\,q^{61/2}-2317\,q^{63/2}+588\,q^{65/2}-114\,q^{67/2}+15\,q^{69/2}-q^{71/2}\right\}$$

```
In[•]:= Length[AllKnots[{3, 13}]]
Out[•]= 12 965

In[•]:= Timing[Length@Union[ρ /@ AllKnots[{3, 10}]]]
Out[•]= {128.125, 249}

In[•]:= Timing[Length@Union[{Kh[PD@#][q, t], HOMFLYPT[PD@#][a, z]} & /@ AllKnots[{3, 10}]]]
KnotTheory: The Khovanov homology program JavaKh–v2 is an update of Jeremy Green's program JavaKh–v1, written by Scott Morrison in 2008 at Microsoft Station Q.
KnotTheory: The HOMFLYPT program was written by Scott Morrison.
Out[•]= {6., 248}

In[•]:= Monitor[Timing[Tallyρ13 = Tally[Last /@ Tally@Table[ρ[K], {K, AllKnots[{3, 13}]}]]], K]
KnotTheory: Loading precomputed data in DTCode4KnotsTo11`.
KnotTheory: The GaussCode to PD conversion was written by Siddarth Sankaran at the University of Toronto in the summer of 2005.
KnotTheory: Loading precomputed data in KnotTheory/12A.dts.
KnotTheory: Loading precomputed data in KnotTheory/12N.dts.
General: Further output of KnotTheory::loading will be suppressed during this calculation.
Out[•]= {20 270.5, {{1, 11 140}, {2, 809}, {4, 33}, {3, 23}, {6, 1}}}

In[•]:= Total[Times @@@ Rest[Tallyρ13]]
Out[•]= 1825

In[•]:= Monitor[Timing[TallyHKh13 = Tally[Last /@
          Tally@Table[{Kh[PD@K][q, t], HOMFLYPT[PD@K][a, z]}, {K, AllKnots[{3, 13}]}]]], K]
Out[•]= {950., {{1, 9714}, {2, 1269}, {3, 150}, {4, 47}, {5, 10}, {6, 3}, {7, 1}}}

In[•]:= Total[Times @@@ Rest[TallyHKh13]]
Out[•]= 3251
```

```
In[•]:= {NumberOfKnots[14, Alternating], NumberOfKnots[14, NonAlternating]}
Out[•]= {19 536, 27 436}

In[•]:= Monitor[Timing[Tallyρ14 = Tally[Last /@ Tally@Table[ρ[K], {K, AllKnots[{3, 14}]}]]], K]
KnotTheory: Loading precomputed data in KnotTheory/14A.dts.
KnotTheory: Loading precomputed data in KnotTheory/14N.dts.
Out[•]= {207 320., {{1, 48 336}, {2, 4814}, {3, 217}, {4, 291}, {6, 19}, {5, 4}, {8, 3}}}

In[•]:= Monitor[Timing[TallyHKh14 = Tally[Last /@
          Tally@Table[{Kh[PD@K][q, t], HOMFLYPT[PD@K][a, z]}, {K, AllKnots[{3, 14}]}]]], K]
Out[•]= {6727.34, {{1, 40 661}, {2, 6969}, {3, 965},
  {5, 85}, {4, 411}, {6, 43}, {8, 6}, {10, 1}, {9, 1}, {7, 7}}}

In[•]:= {Total[Times @@@ Rest[Tallyρ14]], Total[Times @@@ Rest[TallyHKh14]]}
Out[•]= {11 601, 19 276}

In[•]:= K = EPD[X_{14,1}, X̄_{2,29}, X_{3,40}, X_{43,4}, X̄_{26,5}, X_{6,95}, X_{96,7}, X_{13,8}, X̄_{9,28}, X_{10,41}, X_{42,11}, X̄_{27,12},
      X_{30,15}, X̄_{16,61}, X̄_{17,72}, X̄_{18,83}, X_{19,34}, X̄_{89,20}, X̄_{21,92}, X̄_{79,22}, X̄_{68,23}, X̄_{57,24}, X̄_{25,56}, X_{62,31},
      X_{73,32}, X_{84,33}, X̄_{50,35}, X_{36,81}, X_{37,70}, X_{38,59}, X̄_{39,54}, X_{44,55}, X_{58,45}, X_{69,46}, X_{80,47}, X_{48,91},
      X_{90,49}, X_{51,82}, X_{52,71}, X_{53,60}, X̄_{63,74}, X̄_{64,85}, X̄_{76,65}, X̄_{87,66}, X̄_{67,94}, X̄_{75,86}, X̄_{88,77}, X̄_{78,93}];
    Echo@ρ[K];
    Do[Echo[i → ρ[ReplacePart[K, i → (K[[i]] /. {X_{α_,β_} :> X̄_{β,α}, X̄_{α_,β_} :> X_{β,α}})]]], {i, Length[K]}]
» {-(-1 + 2 T - T^2 - T^3 + 2 T^4 - T^5 + T^8) (-1 + T^3 - 2 T^4 + T^5 + T^6 - 2 T^7 + T^8) / T^8,
   1/T^16 (-1 + T)^2 (5 - 18 T + 33 T^2 - 32 T^3 + 2 T^4 + 42 T^5 - 62 T^6 - 8 T^7 + 166 T^8 - 242 T^9 +
     108 T^10 + 132 T^11 - 226 T^12 + 148 T^13 - 11 T^14 - 36 T^15 - 11 T^16 + 148 T^17 - 226 T^18 + 132 T^19 +
     108 T^20 - 242 T^21 + 166 T^22 - 8 T^23 - 62 T^24 + 42 T^25 + 2 T^26 - 32 T^27 + 33 T^28 - 18 T^29 + 5 T^30)}

» 1 → {-(1 - 2 T + 2 T^2 - 4 T^3 + 7 T^4 - 4 T^5 - 5 T^6 + 9 T^7 - 5 T^8 - 4 T^9 + 7 T^10 - 4 T^11 + 2 T^12 - 2 T^13 + T^14) / T^7,
   1/T^14 (-1 + T)^2 (6 - 10 T + 3 T^2 + 7 T^4 - 16 T^5 + 46 T^6 - 110 T^7 + 88 T^8 + 66 T^9 - 136 T^10 + 124 T^11 - 164 T^12 + 236 T^13 -
     164 T^14 + 124 T^15 - 136 T^16 + 66 T^17 + 88 T^18 - 110 T^19 + 46 T^20 - 16 T^21 + 7 T^22 + 3 T^24 - 10 T^25 + 6 T^26)}

» 2 → {-1/T^9 (1 - 3 T + 5 T^2 - 5 T^3 - 3 T^4 + 22 T^5 - 39 T^6 +
     33 T^7 - 8 T^8 - 7 T^9 - 8 T^10 + 33 T^11 - 39 T^12 + 22 T^13 - 3 T^14 - 5 T^15 + 5 T^16 - 3 T^17 + T^18),
   1/T^18 (-1 + T)^2 (4 - 16 T + 36 T^2 - 62 T^3 + 80 T^4 - 40 T^5 - 122 T^6 + 398 T^7 - 620 T^8 + 478 T^9 +
     248 T^10 - 1256 T^11 + 1975 T^12 - 1994 T^13 + 911 T^14 + 1758 T^15 - 5124 T^16 + 6744 T^17 -
     5124 T^18 + 1758 T^19 + 911 T^20 - 1994 T^21 + 1975 T^22 - 1256 T^23 + 248 T^24 + 478 T^25 -
     620 T^26 + 398 T^27 - 122 T^28 - 40 T^29 + 80 T^30 - 62 T^31 + 36 T^32 - 16 T^33 + 4 T^34)}
```

» $$3 \to \left\{ \frac{1}{T^9} \left(1 - 5\,T + 10\,T^2 - 10\,T^3 + 17\,T^5 - 28\,T^6 + 23\,T^7 - 7\,T^8 - T^9 - 7\,T^{10} + 23\,T^{11} - 28\,T^{12} + 17\,T^{13} - 10\,T^{15} + 10\,T^{16} - 5\,T^{17} + T^{18}\right), \frac{1}{T^{18}} (-1+T)^2 \left(5 - 40\,T + 142\,T^2 - 302\,T^3 + 413\,T^4 - 292\,T^5 - 135\,T^6 + 664\,T^7 - 913\,T^8 + 638\,T^9 + 133\,T^{10} - 1304\,T^{11} + 2575\,T^{12} - 3208\,T^{13} + 2268\,T^{14} + 444\,T^{15} - 3561\,T^{16} + 4968\,T^{17} - 3561\,T^{18} + 444\,T^{19} + 2268\,T^{20} - 3208\,T^{21} + 2575\,T^{22} - 1304\,T^{23} + 133\,T^{24} + 638\,T^{25} - 913\,T^{26} + 664\,T^{27} - 135\,T^{28} - 292\,T^{29} + 413\,T^{30} - 302\,T^{31} + 142\,T^{32} - 40\,T^{33} + 5\,T^{34}\right) \right\}$$

» $$4 \to \left\{ \frac{1}{T^9} \left(1 - 6\,T + 14\,T^2 - 16\,T^3 + 2\,T^4 + 23\,T^5 - 35\,T^6 + 19\,T^7 + 11\,T^8 - 25\,T^9 + 11\,T^{10} + 19\,T^{11} - 35\,T^{12} + 23\,T^{13} + 2\,T^{14} - 16\,T^{15} + 14\,T^{16} - 6\,T^{17} + T^{18}\right), \frac{1}{T^{18}} (-1+T)^2 \left(5 - 50\,T + 216\,T^2 - 538\,T^3 + 833\,T^4 - 700\,T^5 - 62\,T^6 + 1028\,T^7 - 1450\,T^8 + 1088\,T^9 - 362\,T^{10} - 614\,T^{11} + 2293\,T^{12} - 4232\,T^{13} + 4206\,T^{14} - 418\,T^{15} - 5209\,T^{16} + 7956\,T^{17} - 5209\,T^{18} - 418\,T^{19} + 4206\,T^{20} - 4232\,T^{21} + 2293\,T^{22} - 614\,T^{23} - 362\,T^{24} + 1088\,T^{25} - 1450\,T^{26} + 1028\,T^{27} - 62\,T^{28} - 700\,T^{29} + 833\,T^{30} - 538\,T^{31} + 216\,T^{32} - 50\,T^{33} + 5\,T^{34}\right) \right\}$$

» $$5 \to \left\{ -\frac{1}{T^9} \left(2 - 9\,T + 20\,T^2 - 23\,T^3 + 2\,T^4 + 37\,T^5 - 57\,T^6 + 27\,T^7 + 31\,T^8 - 61\,T^9 + 31\,T^{10} + 27\,T^{11} - 57\,T^{12} + 37\,T^{13} + 2\,T^{14} - 23\,T^{15} + 20\,T^{16} - 9\,T^{17} + 2\,T^{18}\right), \frac{1}{T^{18}} (-1+T)^2 \left(15 - 106\,T + 381\,T^2 - 874\,T^3 + 1344\,T^4 - 1272\,T^5 + 294\,T^6 + 1294\,T^7 - 2739\,T^8 + 3546\,T^9 - 3304\,T^{10} + 772\,T^{11} + 5198\,T^{12} - 12\,004\,T^{13} + 12\,564\,T^{14} - 2090\,T^{15} - 13\,610\,T^{16} + 21\,236\,T^{17} - 13\,610\,T^{18} - 2090\,T^{19} + 12\,564\,T^{20} - 12\,004\,T^{21} + 5198\,T^{22} + 772\,T^{23} - 3304\,T^{24} + 3546\,T^{25} - 2739\,T^{26} + 1294\,T^{27} + 294\,T^{28} - 1272\,T^{29} + 1344\,T^{30} - 874\,T^{31} + 381\,T^{32} - 106\,T^{33} + 15\,T^{34}\right) \right\}$$

» $$6 \to \left\{ -\frac{1 - T - T^2 - T^3 + 6\,T^4 - 4\,T^5 - 5\,T^6 + 9\,T^7 - 5\,T^8 - 4\,T^9 + 6\,T^{10} - T^{11} - T^{12} - T^{13} + T^{14}}{T^7}, \frac{1}{T^{14}} (-1+T)^2 \left(6 - 8\,T + T^2 - 2\,T^3 - 13\,T^4 + 40\,T^5 + 18\,T^6 - 150\,T^7 + 102\,T^8 + 150\,T^9 - 178\,T^{10} - 12\,T^{11} + 57\,T^{12} + 20\,T^{13} + 57\,T^{14} - 12\,T^{15} - 178\,T^{16} + 150\,T^{17} + 102\,T^{18} - 150\,T^{19} + 18\,T^{20} + 40\,T^{21} - 13\,T^{22} - 2\,T^{23} + T^{24} - 8\,T^{25} + 6\,T^{26}\right) \right\}$$

» $$7 \to \left\{ -\frac{1 - T - T^2 - T^3 + 6\,T^4 - 4\,T^5 - 5\,T^6 + 9\,T^7 - 5\,T^8 - 4\,T^9 + 6\,T^{10} - T^{11} - T^{12} - T^{13} + T^{14}}{T^7}, \frac{1}{T^{14}} (-1+T)^2 \left(6 - 8\,T + T^2 - 2\,T^3 - 13\,T^4 + 40\,T^5 + 18\,T^6 - 150\,T^7 + 102\,T^8 + 150\,T^9 - 178\,T^{10} - 12\,T^{11} + 57\,T^{12} + 20\,T^{13} + 57\,T^{14} - 12\,T^{15} - 178\,T^{16} + 150\,T^{17} + 102\,T^{18} - 150\,T^{19} + 18\,T^{20} + 40\,T^{21} - 13\,T^{22} - 2\,T^{23} + T^{24} - 8\,T^{25} + 6\,T^{26}\right) \right\}$$

» $$8 \to \left\{ -\frac{1 - 2\,T + 5\,T^3 - 9\,T^4 + 8\,T^5 - 4\,T^6 + T^7 - T^8 + T^9 - 4\,T^{10} + 8\,T^{11} - 9\,T^{12} + 5\,T^{13} - 2\,T^{15} + T^{16}}{T^8}, \frac{1}{T^{16}} (-1+T)^2 \left(5 - 16\,T + 20\,T^2 + 6\,T^3 - 65\,T^4 + 124\,T^5 - 155\,T^6 + 132\,T^7 - 20\,T^8 - 180\,T^9 + 415\,T^{10} - 524\,T^{11} + 334\,T^{12} + 244\,T^{13} - 916\,T^{14} + 1232\,T^{15} - 916\,T^{16} + 244\,T^{17} + 334\,T^{18} - 524\,T^{19} + 415\,T^{20} - 180\,T^{21} - 20\,T^{22} + 132\,T^{23} - 155\,T^{24} + 124\,T^{25} - 65\,T^{26} + 6\,T^{27} + 20\,T^{28} - 16\,T^{29} + 5\,T^{30}\right) \right\}$$

» $9 \to \left\{ -\frac{1}{T^9}\left(1 - 3\,T + 5\,T^2 - 5\,T^3 - 2\,T^4 + 18\,T^5 - 31\,T^6 + 24\,T^7 - 3\,T^8 - 9\,T^9 - 3\,T^{10} + 24\,T^{11} - 31\,T^{12} + 18\,T^{13} - 2\,T^{14} - 5\,T^{15} + 5\,T^{16} - 3\,T^{17} + T^{18}\right), \frac{1}{T^{18}}(-1+T)^2\left(4 - 16\,T + 36\,T^2 - 62\,T^3 + 81\,T^4 - 50\,T^5 - 78\,T^6 + 280\,T^7 - 427\,T^8 + 316\,T^9 + 183\,T^{10} - 864\,T^{11} + 1374\,T^{12} - 1474\,T^{13} + 924\,T^{14} + 674\,T^{15} - 2797\,T^{16} + 3848\,T^{17} - 2797\,T^{18} + 674\,T^{19} + 924\,T^{20} - 1474\,T^{21} + 1374\,T^{22} - 864\,T^{23} + 183\,T^{24} + 316\,T^{25} - 427\,T^{26} + 280\,T^{27} - 78\,T^{28} - 50\,T^{29} + 81\,T^{30} - 62\,T^{31} + 36\,T^{32} - 16\,T^{33} + 4\,T^{34}\right)\right\}$

» $10 \to \left\{ \frac{1 - 5\,T + 10\,T^2 - 10\,T^3 + 17\,T^5 - 26\,T^6 + 17\,T^7 + T^8 - 9\,T^9 + T^{10} + 17\,T^{11} - 26\,T^{12} + 17\,T^{13} - 10\,T^{15} + 10\,T^{16} - 5\,T^{17} + T^{18}}{T^9}, \frac{1}{T^{18}}(-1+T)^2\left(5 - 40\,T + 142\,T^2 - 302\,T^3 + 413\,T^4 - 292\,T^5 - 127\,T^6 + 618\,T^7 - 807\,T^8 + 540\,T^9 + 35\,T^{10} - 824\,T^{11} + 1798\,T^{12} - 2516\,T^{13} + 2056\,T^{14} + 92\,T^{15} - 2863\,T^{16} + 4168\,T^{17} - 2863\,T^{18} + 92\,T^{19} + 2056\,T^{20} - 2516\,T^{21} + 1798\,T^{22} - 824\,T^{23} + 35\,T^{24} + 540\,T^{25} - 807\,T^{26} + 618\,T^{27} - 127\,T^{28} - 292\,T^{29} + 413\,T^{30} - 302\,T^{31} + 142\,T^{32} - 40\,T^{33} + 5\,T^{34}\right)\right\}$

» $11 \to \left\{ \frac{1 - 5\,T + 10\,T^2 - 10\,T^3 + 17\,T^5 - 26\,T^6 + 17\,T^7 + T^8 - 9\,T^9 + T^{10} + 17\,T^{11} - 26\,T^{12} + 17\,T^{13} - 10\,T^{15} + 10\,T^{16} - 5\,T^{17} + T^{18}}{T^9}, \frac{1}{T^{18}}(-1+T)^2\left(5 - 40\,T + 142\,T^2 - 302\,T^3 + 413\,T^4 - 292\,T^5 - 127\,T^6 + 618\,T^7 - 807\,T^8 + 540\,T^9 + 35\,T^{10} - 824\,T^{11} + 1798\,T^{12} - 2516\,T^{13} + 2056\,T^{14} + 92\,T^{15} - 2863\,T^{16} + 4168\,T^{17} - 2863\,T^{18} + 92\,T^{19} + 2056\,T^{20} - 2516\,T^{21} + 1798\,T^{22} - 824\,T^{23} + 35\,T^{24} + 540\,T^{25} - 807\,T^{26} + 618\,T^{27} - 127\,T^{28} - 292\,T^{29} + 413\,T^{30} - 302\,T^{31} + 142\,T^{32} - 40\,T^{33} + 5\,T^{34}\right)\right\}$

» $12 \to \left\{ -\frac{1}{T^9}\left(2 - 8\,T + 16\,T^2 - 17\,T^3 + T^4 + 27\,T^5 - 42\,T^6 + 22\,T^7 + 18\,T^8 - 39\,T^9 + 18\,T^{10} + 22\,T^{11} - 42\,T^{12} + 27\,T^{13} + T^{14} - 17\,T^{15} + 16\,T^{16} - 8\,T^{17} + 2\,T^{18}\right), \frac{1}{T^{18}}(-1+T)^2\left(15 - 92\,T + 291\,T^2 - 604\,T^3 + 865\,T^4 - 786\,T^5 + 175\,T^6 + 812\,T^7 - 1736\,T^8 + 2156\,T^9 - 1620\,T^{10} - 380\,T^{11} + 3822\,T^{12} - 6882\,T^{13} + 6276\,T^{14} - 308\,T^{15} - 7770\,T^{16} + 11\,588\,T^{17} - 7770\,T^{18} - 308\,T^{19} + 6276\,T^{20} - 6882\,T^{21} + 3822\,T^{22} - 380\,T^{23} - 1620\,T^{24} + 2156\,T^{25} - 1736\,T^{26} + 812\,T^{27} + 175\,T^{28} - 786\,T^{29} + 865\,T^{30} - 604\,T^{31} + 291\,T^{32} - 92\,T^{33} + 15\,T^{34}\right)\right\}$

» $13 \to \left\{ -\frac{1 - 3\,T + 3\,T^2 + T^3 - 8\,T^4 + 13\,T^5 - 12\,T^6 + 7\,T^7 - 5\,T^8 + 7\,T^9 - 12\,T^{10} + 13\,T^{11} - 8\,T^{12} + T^{13} + 3\,T^{14} - 3\,T^{15} + T^{16}}{T^8}, \frac{1}{T^{16}}(-1+T)^2\left(5 - 20\,T + 34\,T^2 - 14\,T^3 - 59\,T^4 + 144\,T^5 - 164\,T^6 + 78\,T^7 + 117\,T^8 - 384\,T^9 + 668\,T^{10} - 780\,T^{11} + 519\,T^{12} + 184\,T^{13} - 962\,T^{14} + 1316\,T^{15} - 962\,T^{16} + 184\,T^{17} + 519\,T^{18} - 780\,T^{19} + 668\,T^{20} - 384\,T^{21} + 117\,T^{22} + 78\,T^{23} - 164\,T^{24} + 144\,T^{25} - 59\,T^{26} - 14\,T^{27} + 34\,T^{28} - 20\,T^{29} + 5\,T^{30}\right)\right\}$

» $14 \to \left\{ -\frac{2 - 5\,T + 16\,T^3 - 27\,T^4 + 12\,T^5 + 22\,T^6 - 41\,T^7 + 22\,T^8 + 12\,T^9 - 27\,T^{10} + 16\,T^{11} - 5\,T^{13} + 2\,T^{14}}{T^7}, \right.$

$\frac{1}{T^{14}}\,(-1+T)^2\,\left(1 - 2\,T + 11\,T^2 - 40\,T^3 + 41\,T^4 + 122\,T^5 - 511\,T^6 + 726\,T^7 + 116\,T^8 - \right.$

$2032\,T^9 + 3104\,T^{10} - 1160\,T^{11} - 2801\,T^{12} + 4892\,T^{13} - 2801\,T^{14} - 1160\,T^{15} + 3104\,T^{16} -$

$\left.\left. 2032\,T^{17} + 116\,T^{18} + 726\,T^{19} - 511\,T^{20} + 122\,T^{21} + 41\,T^{22} - 40\,T^{23} + 11\,T^{24} - 2\,T^{25} + T^{26}\right)\right\}$

» $15 \to \left\{ \frac{1}{T^8}\left(1 - 5\,T + 5\,T^2 + 11\,T^3 - 35\,T^4 + 39\,T^5 - \right.\right.$

$\left. 9\,T^6 - 34\,T^7 + 55\,T^8 - 34\,T^9 - 9\,T^{10} + 39\,T^{11} - 35\,T^{12} + 11\,T^{13} + 5\,T^{14} - 5\,T^{15} + T^{16}\right),$

$-\frac{1}{T^{15}}\,(-1+T)^2\,\left(2 - 13\,T + 32\,T^2 - 21\,T^3 - 106\,T^4 + 350\,T^5 - 536\,T^6 + 499\,T^7 + 212\,T^8 - 2504\,T^9 + \right.$

$5662\,T^{10} - 5947\,T^{11} + 276\,T^{12} + 8356\,T^{13} - 12\,568\,T^{14} + 8356\,T^{15} + 276\,T^{16} - 5947\,T^{17} + 5662\,T^{18} -$

$\left.\left. 2504\,T^{19} + 212\,T^{20} + 499\,T^{21} - 536\,T^{22} + 350\,T^{23} - 106\,T^{24} - 21\,T^{25} + 32\,T^{26} - 13\,T^{27} + 2\,T^{28}\right)\right\}$

» $16 \to \left\{ \frac{\left(1 - T + T^2\right)^2\,\left(1 - 3\,T - 2\,T^2 + 11\,T^3 - 8\,T^4 - 5\,T^5 + 13\,T^6 - 5\,T^7 - 8\,T^8 + 11\,T^9 - 2\,T^{10} - 3\,T^{11} + T^{12}\right)}{T^8}, \right.$

$-\frac{1}{T^{15}}\,(-1+T)^2\,\left(1 - T + T^2\right)\,\left(2 - 11\,T + 21\,T^2 - 31\,T^3 + 16\,T^4 + 177\,T^5 - 633\,T^6 + 910\,T^7 - 61\,T^8 - \right.$

$2054\,T^9 + 3287\,T^{10} - 1174\,T^{11} - 3273\,T^{12} + 5614\,T^{13} - 3273\,T^{14} - 1174\,T^{15} + 3287\,T^{16} -$

$\left.\left. 2054\,T^{17} - 61\,T^{18} + 910\,T^{19} - 633\,T^{20} + 177\,T^{21} + 16\,T^{22} - 31\,T^{23} + 21\,T^{24} - 11\,T^{25} + 2\,T^{26}\right)\right\}$

» $17 \to \left\{ -\frac{\left(1 - T + T^2\right)\,\left(1 - T^2 + T^4\right)\,\left(3 - 8\,T + 8\,T^2 - 13\,T^4 + 19\,T^5 - 13\,T^6 + 8\,T^8 - 8\,T^9 + 3\,T^{10}\right)}{T^8}, \right.$

$-\frac{1}{T^{18}}\,(-1+T)^2\,\left(2 - 10\,T + 6\,T^2 + 52\,T^3 - 154\,T^4 + 134\,T^5 + 232\,T^6 - 906\,T^7 + 1489\,T^8 - \right.$

$1578\,T^9 + 1001\,T^{10} + 460\,T^{11} - 2959\,T^{12} + 5398\,T^{13} - 5176\,T^{14} + 660\,T^{15} + 5722\,T^{16} -$

$8780\,T^{17} + 5722\,T^{18} + 660\,T^{19} - 5176\,T^{20} + 5398\,T^{21} - 2959\,T^{22} + 460\,T^{23} + 1001\,T^{24} -$

$\left.\left. 1578\,T^{25} + 1489\,T^{26} - 906\,T^{27} + 232\,T^{28} + 134\,T^{29} - 154\,T^{30} + 52\,T^{31} + 6\,T^{32} - 10\,T^{33} + 2\,T^{34}\right)\right\}$

» $18 \to \left\{ -\frac{1 - T - 3\,T^2 + 8\,T^3 - 8\,T^4 + T^5 + 7\,T^6 - 11\,T^7 + 7\,T^8 + T^9 - 8\,T^{10} + 8\,T^{11} - 3\,T^{12} - T^{13} + T^{14}}{T^7}, \right.$ $\frac{1}{T^{14}}\,(-1+T)^2$

$\left(2 - 4\,T - 5\,T^2 + 28\,T^3 - 27\,T^4 - 38\,T^5 + 73\,T^6 + 36\,T^7 - 78\,T^8 - 156\,T^9 + 417\,T^{10} - 206\,T^{11} - 384\,T^{12} + 720\,T^{13} - \right.$

$\left.\left. 384\,T^{14} - 206\,T^{15} + 417\,T^{16} - 156\,T^{17} - 78\,T^{18} + 36\,T^{19} + 73\,T^{20} - 38\,T^{21} - 27\,T^{22} + 28\,T^{23} - 5\,T^{24} - 4\,T^{25} + 2\,T^{26}\right)\right\}$

» $19 \to \left\{ \frac{1}{T^8}\left(2 - 10\,T + 15\,T^2 + 2\,T^3 - 37\,T^4 + 53\,T^5 - 19\,T^6 - \right.\right.$

$\left. 43\,T^7 + 75\,T^8 - 43\,T^9 - 19\,T^{10} + 53\,T^{11} - 37\,T^{12} + 2\,T^{13} + 15\,T^{14} - 10\,T^{15} + 2\,T^{16}\right),$

$\frac{1}{T^{17}}\,(-1+T)^2\,\left(2 - 9\,T + 14\,T^2 + 2\,T^3 - 20\,T^4 - 47\,T^5 + 236\,T^6 - 425\,T^7 + 662\,T^8 - 1363\,T^9 + 1824\,T^{10} + 628\,T^{11} - \right.$

$6418\,T^{12} + 9673\,T^{13} - 3684\,T^{14} - 8502\,T^{15} + 14\,892\,T^{16} - 8502\,T^{17} - 3684\,T^{18} + 9673\,T^{19} - 6418\,T^{20} +$

$\left.\left. 628\,T^{21} + 1824\,T^{22} - 1363\,T^{23} + 662\,T^{24} - 425\,T^{25} + 236\,T^{26} - 47\,T^{27} - 20\,T^{28} + 2\,T^{29} + 14\,T^{30} - 9\,T^{31} + 2\,T^{32}\right)\right\}$

» $20 \to \left\{ -\frac{1 - T - 4\,T^2 + 11\,T^3 - 11\,T^4 + 14\,T^6 - 21\,T^7 + 14\,T^8 - 11\,T^{10} + 11\,T^{11} - 4\,T^{12} - T^{13} + T^{14}}{T^7}, \right.$ $\frac{1}{T^{14}}\,(-1+T)^2$

$\left(2 - 4\,T - 4\,T^2 + 24\,T^3 - 18\,T^4 - 48\,T^5 + 66\,T^6 + 50\,T^7 + 25\,T^8 - 528\,T^9 + 913\,T^{10} - 320\,T^{11} - 969\,T^{12} + 1664\,T^{13} - \right.$

$\left.\left. 969\,T^{14} - 320\,T^{15} + 913\,T^{16} - 528\,T^{17} + 25\,T^{18} + 50\,T^{19} + 66\,T^{20} - 48\,T^{21} - 18\,T^{22} + 24\,T^{23} - 4\,T^{24} - 4\,T^{25} + 2\,T^{26}\right)\right\}$

» 21 →

$$\left\{-\frac{2-5\,T+16\,T^3-27\,T^4+12\,T^5+22\,T^6-41\,T^7+22\,T^8+12\,T^9-27\,T^{10}+16\,T^{11}-5\,T^{13}+2\,T^{14}}{T^7},\right.$$

$$\frac{1}{T^{14}}\,(-1+T)^2\left(1-2\,T+11\,T^2-40\,T^3+41\,T^4+122\,T^5-511\,T^6+726\,T^7+116\,T^8-\right.$$

$$\left.\left.2032\,T^9+3104\,T^{10}-1160\,T^{11}-2801\,T^{12}+4892\,T^{13}-2801\,T^{14}-1160\,T^{15}+3104\,T^{16}-\right.\right.$$

$$\left.\left.2032\,T^{17}+116\,T^{18}+726\,T^{19}-511\,T^{20}+122\,T^{21}+41\,T^{22}-40\,T^{23}+11\,T^{24}-2\,T^{25}+T^{26}\right)\right\}$$

» 22 →

$$\left\{-\frac{1}{T^{10}}\left(1-4\,T+4\,T^2+7\,T^3-21\,T^4+13\,T^5+22\,T^6-48\,T^7+25\,T^8+32\,T^9-63\,T^{10}+32\,T^{11}+25\,T^{12}-48\,T^{13}+22\,T^{14}+\right.\right.$$

$$\left.13\,T^{15}-21\,T^{16}+7\,T^{17}+4\,T^{18}-4\,T^{19}+T^{20}\right),$$

$$\frac{1}{T^{20}}\,(-1+T)^2\left(3-18\,T+37\,T^2-144\,T^4+272\,T^5-109\,T^6-374\,T^7+709\,T^8-526\,T^9+180\,T^{10}+\right.$$

$$196\,T^{11}-1549\,T^{12}+3534\,T^{13}-2476\,T^{14}-3704\,T^{15}+9193\,T^{16}-5458\,T^{17}-5843\,T^{18}+$$

$$12\,160\,T^{19}-5843\,T^{20}-5458\,T^{21}+9193\,T^{22}-3704\,T^{23}-2476\,T^{24}+3534\,T^{25}-1549\,T^{26}+196\,T^{27}+$$

$$\left.\left.180\,T^{28}-526\,T^{29}+709\,T^{30}-374\,T^{31}-109\,T^{32}+272\,T^{33}-144\,T^{34}+37\,T^{36}-18\,T^{37}+3\,T^{38}\right)\right\}$$

» 23 →

$$\left\{-\frac{1}{T^{10}}\left(1-4\,T+4\,T^2+7\,T^3-21\,T^4+13\,T^5+22\,T^6-48\,T^7+25\,T^8+32\,T^9-63\,T^{10}+32\,T^{11}+25\,T^{12}-48\,T^{13}+22\,T^{14}+\right.\right.$$

$$\left.13\,T^{15}-21\,T^{16}+7\,T^{17}+4\,T^{18}-4\,T^{19}+T^{20}\right),$$

$$\frac{1}{T^{20}}\,(-1+T)^2\left(3-18\,T+37\,T^2-144\,T^4+272\,T^5-109\,T^6-374\,T^7+709\,T^8-526\,T^9+180\,T^{10}+\right.$$

$$196\,T^{11}-1549\,T^{12}+3534\,T^{13}-2476\,T^{14}-3704\,T^{15}+9193\,T^{16}-5458\,T^{17}-5843\,T^{18}+$$

$$12\,160\,T^{19}-5843\,T^{20}-5458\,T^{21}+9193\,T^{22}-3704\,T^{23}-2476\,T^{24}+3534\,T^{25}-1549\,T^{26}+196\,T^{27}+$$

$$\left.\left.180\,T^{28}-526\,T^{29}+709\,T^{30}-374\,T^{31}-109\,T^{32}+272\,T^{33}-144\,T^{34}+37\,T^{36}-18\,T^{37}+3\,T^{38}\right)\right\}$$

» 24 →

$$\left\{-\frac{1-3\,T+4\,T^2-9\,T^4+15\,T^5-11\,T^6+7\,T^8-6\,T^9+3\,T^{10}-6\,T^{11}+7\,T^{12}-11\,T^{14}+15\,T^{15}-9\,T^{16}+4\,T^{18}-3\,T^{19}+T^{20}}{T^{10}},\right.$$

$$-\frac{1}{T^{20}}\,(-1+T)^2\left(1-T+T^2\right)\left(1-3\,T-17\,T^2+70\,T^3-103\,T^4+T^5+262\,T^6-501\,T^7+\right.$$

$$446\,T^8+29\,T^9-733\,T^{10}+1266\,T^{11}-1144\,T^{12}+148\,T^{13}+1214\,T^{14}-1828\,T^{15}+926\,T^{16}+$$

$$786\,T^{17}-1668\,T^{18}+786\,T^{19}+926\,T^{20}-1828\,T^{21}+1214\,T^{22}+148\,T^{23}-1144\,T^{24}+1266\,T^{25}-$$

$$\left.\left.733\,T^{26}+29\,T^{27}+446\,T^{28}-501\,T^{29}+262\,T^{30}+T^{31}-103\,T^{32}+70\,T^{33}-17\,T^{34}-3\,T^{35}+T^{36}\right)\right\}$$

» 25 →

$$\left\{-\frac{\left(1-T+T^2\right)\left(1-5\,T^2+5\,T^3+4\,T^4-7\,T^5-6\,T^6+15\,T^7-6\,T^8-7\,T^9+4\,T^{10}+5\,T^{11}-5\,T^{12}+T^{14}\right)}{T^8},\right.$$

$$-\frac{1}{T^{17}}\,(-1+T)^2$$

$$\left(2-13\,T+24\,T^2+12\,T^3-150\,T^4+328\,T^5-274\,T^6-304\,T^7+1314\,T^8-2070\,T^9+1624\,T^{10}+476\,T^{11}-3292\,T^{12}+\right.$$

$$4423\,T^{13}-2072\,T^{14}-2293\,T^{15}+4512\,T^{16}-2293\,T^{17}-2072\,T^{18}+4423\,T^{19}-3292\,T^{20}+476\,T^{21}+$$

$$\left.\left.1624\,T^{22}-2070\,T^{23}+1314\,T^{24}-304\,T^{25}-274\,T^{26}+328\,T^{27}-150\,T^{28}+12\,T^{29}+24\,T^{30}-13\,T^{31}+2\,T^{32}\right)\right\}$$

» $$26 \to \left\{ -\frac{1 - 2\,T + 3\,T^3 - T^4 - 5\,T^5 + 3\,T^6 + 10\,T^7 - 19\,T^8 + 10\,T^9 + 3\,T^{10} - 5\,T^{11} - T^{12} + 3\,T^{13} - 2\,T^{15} + T^{16}}{T^8}, \right.$$

$$\frac{1}{T^{16}}\,(-1+T)^2\,(1+T)^2\,\left(1 - T + T^2\right)\,\left(5 - 23\,T + 47\,T^2 - 44\,T^3 - 29\,T^4 + 173\,T^5 - 299\,T^6 + \right.$$

$$\left. \left. 248\,T^7 + 59\,T^8 - 435\,T^9 + 510\,T^{10} - 81\,T^{11} - 571\,T^{12} + 888\,T^{13} - 571\,T^{14} - 81\,T^{15} + 510\,T^{16} - \right.\right.$$

$$\left. \left. 435\,T^{17} + 59\,T^{18} + 248\,T^{19} - 299\,T^{20} + 173\,T^{21} - 29\,T^{22} - 44\,T^{23} + 47\,T^{24} - 23\,T^{25} + 5\,T^{26}\right)\right\}$$

» $$27 \to \left\{ -\frac{1}{T^9}\left(1 - 4\,T + 8\,T^2 - 9\,T^3 + T^4 + 14\,T^5 - 24\,T^6 + \right.\right.$$

$$\left. \left. 19\,T^7 - 4\,T^8 - 5\,T^9 - 4\,T^{10} + 19\,T^{11} - 24\,T^{12} + 14\,T^{13} + T^{14} - 9\,T^{15} + 8\,T^{16} - 4\,T^{17} + T^{18}\right), \right.$$

$$\frac{1}{T^{19}}\,(-1+T)^2\,\left(2 - 4\,T - 4\,T^2 + 32\,T^3 - 64\,T^4 + 51\,T^5 + 44\,T^6 - 203\,T^7 + 344\,T^8 - 405\,T^9 + 356\,T^{10} - \right.$$

$$83\,T^{11} - 582\,T^{12} + 1610\,T^{13} - 2368\,T^{14} + 1800\,T^{15} + 522\,T^{16} - 3397\,T^{17} + 4716\,T^{18} -$$

$$3397\,T^{19} + 522\,T^{20} + 1800\,T^{21} - 2368\,T^{22} + 1610\,T^{23} - 582\,T^{24} - 83\,T^{25} + 356\,T^{26} -$$

$$\left. \left. 405\,T^{27} + 344\,T^{28} - 203\,T^{29} + 44\,T^{30} + 51\,T^{31} - 64\,T^{32} + 32\,T^{33} - 4\,T^{34} - 4\,T^{35} + 2\,T^{36}\right)\right\}$$

» $$28 \to \left\{ -\frac{1 - 2\,T + 3\,T^3 - T^4 - 5\,T^5 + 3\,T^6 + 10\,T^7 - 19\,T^8 + 10\,T^9 + 3\,T^{10} - 5\,T^{11} - T^{12} + 3\,T^{13} - 2\,T^{15} + T^{16}}{T^8}, \right.$$

$$\frac{1}{T^{16}}\,(-1+T)^2\,(1+T)^2\,\left(1 - T + T^2\right)\,\left(5 - 23\,T + 47\,T^2 - 44\,T^3 - 29\,T^4 + 173\,T^5 - 299\,T^6 + \right.$$

$$\left. \left. 248\,T^7 + 59\,T^8 - 435\,T^9 + 510\,T^{10} - 81\,T^{11} - 571\,T^{12} + 888\,T^{13} - 571\,T^{14} - 81\,T^{15} + 510\,T^{16} - \right.\right.$$

$$\left. \left. 435\,T^{17} + 59\,T^{18} + 248\,T^{19} - 299\,T^{20} + 173\,T^{21} - 29\,T^{22} - 44\,T^{23} + 47\,T^{24} - 23\,T^{25} + 5\,T^{26}\right)\right\}$$

» $$29 \to \left\{ -\frac{1 - 2\,T + T^2 + 3\,T^4 - 8\,T^5 + 4\,T^6 + 10\,T^7 - 19\,T^8 + 10\,T^9 + 4\,T^{10} - 8\,T^{11} + 3\,T^{12} + T^{14} - 2\,T^{15} + T^{16}}{T^8}, \right.$$

$$\frac{1}{T^{16}}\,(-1+T)^2\,\left(3 - 10\,T + 13\,T^2 + 4\,T^3 - 51\,T^4 + 106\,T^5 - 92\,T^6 - 64\,T^7 + 261\,T^8 - 208\,T^9 - \right.$$

$$250\,T^{10} + 800\,T^{11} - 824\,T^{12} + 88\,T^{13} + 943\,T^{14} - 1416\,T^{15} + 943\,T^{16} + 88\,T^{17} - 824\,T^{18} + 800\,T^{19} -$$

$$\left. \left. 250\,T^{20} - 208\,T^{21} + 261\,T^{22} - 64\,T^{23} - 92\,T^{24} + 106\,T^{25} - 51\,T^{26} + 4\,T^{27} + 13\,T^{28} - 10\,T^{29} + 3\,T^{30}\right)\right\}$$

» $$30 \to \left\{ -\frac{\left(-2 + 3\,T - 2\,T^2 - T^3 + 2\,T^4 - T^5 - T^7 + T^8\right)\,\left(-1 + T + T^3 - 2\,T^4 + T^5 + 2\,T^6 - 3\,T^7 + 2\,T^8\right)}{T^8}, \right.$$

$$-\frac{1}{T^{18}}\,(-1+T)^2\,\left(1 + T^2\right)\,\left(2 - 8\,T + T^2 + 40\,T^3 - 85\,T^4 + 42\,T^5 + 117\,T^6 - 264\,T^7 + 238\,T^8 + 8\,T^9 - 324\,T^{10} + \right.$$

$$372\,T^{11} + 67\,T^{12} - 596\,T^{13} + 437\,T^{14} + 402\,T^{15} - 910\,T^{16} + 402\,T^{17} + 437\,T^{18} - 596\,T^{19} + 67\,T^{20} +$$

$$\left. \left. 372\,T^{21} - 324\,T^{22} + 8\,T^{23} + 238\,T^{24} - 264\,T^{25} + 117\,T^{26} + 42\,T^{27} - 85\,T^{28} + 40\,T^{29} + T^{30} - 8\,T^{31} + 2\,T^{32}\right)\right\}$$

» $$31 \to \left\{ -\frac{1}{T^9}\left(1 - 4\,T + 8\,T^2 - 9\,T^3 + T^4 + 14\,T^5 - 24\,T^6 + \right.\right.$$

$$\left. \left. 19\,T^7 - 4\,T^8 - 5\,T^9 - 4\,T^{10} + 19\,T^{11} - 24\,T^{12} + 14\,T^{13} + T^{14} - 9\,T^{15} + 8\,T^{16} - 4\,T^{17} + T^{18}\right), \right.$$

$$\frac{1}{T^{19}}\,(-1+T)^2\,\left(2 - 4\,T - 4\,T^2 + 32\,T^3 - 64\,T^4 + 51\,T^5 + 44\,T^6 - 203\,T^7 + 344\,T^8 - 405\,T^9 + 356\,T^{10} - \right.$$

$$83\,T^{11} - 582\,T^{12} + 1610\,T^{13} - 2368\,T^{14} + 1800\,T^{15} + 522\,T^{16} - 3397\,T^{17} + 4716\,T^{18} -$$

$$3397\,T^{19} + 522\,T^{20} + 1800\,T^{21} - 2368\,T^{22} + 1610\,T^{23} - 582\,T^{24} - 83\,T^{25} + 356\,T^{26} -$$

$$\left. \left. 405\,T^{27} + 344\,T^{28} - 203\,T^{29} + 44\,T^{30} + 51\,T^{31} - 64\,T^{32} + 32\,T^{33} - 4\,T^{34} - 4\,T^{35} + 2\,T^{36}\right)\right\}$$

» 32 →
$$\left\{\frac{1}{T^{10}}\left(1-4T+4T^2+4T^3-13T^4+8T^5+12T^6-25T^7+13T^8+14T^9-27T^{10}+14T^{11}+13T^{12}-25T^{13}+12T^{14}+8T^{15}-13T^{16}+4T^{17}+4T^{18}-4T^{19}+T^{20}\right),\right.$$
$$\left.\frac{1}{T^{20}}(-1+T)^2(1+T)^2\left(4-32T+108T^2-190T^3+131T^4+180T^5-568T^6+620T^7-171T^8-408T^9+703T^{10}-758T^{11}+573T^{12}+350T^{13}-2008T^{14}+2878T^{15}-1323T^{16}-1802T^{17}+3432T^{18}-1802T^{19}-1323T^{20}+2878T^{21}-2008T^{22}+350T^{23}+573T^{24}-758T^{25}+703T^{26}-408T^{27}-171T^{28}+620T^{29}-568T^{30}+180T^{31}+131T^{32}-190T^{33}+108T^{34}-32T^{35}+4T^{36}\right)\right\}$$

» 33 →
$$\left\{\frac{1}{T^{10}}\left(1-T+T^2\right)^2\left(1-2T-5T^2+11T^3+5T^4-22T^5+9T^6+13T^7-19T^8+13T^9+9T^{10}-22T^{11}+5T^{12}+11T^{13}-5T^{14}-2T^{15}+T^{16}\right),\right.$$
$$\left.\frac{1}{T^{20}}(-1+T)^2\left(4-24T+36T^2+70T^3-312T^4+324T^5+372T^6-1374T^7+1250T^8+554T^9-2239T^{10}+2072T^{11}-1413T^{12}+1676T^{13}+95T^{14}-6484T^{15}+11\,425T^{16}-5396T^{17}-9230T^{18}+17\,200T^{19}-9230T^{20}-5396T^{21}+11\,425T^{22}-6484T^{23}+95T^{24}+1676T^{25}-1413T^{26}+2072T^{27}-2239T^{28}+554T^{29}+1250T^{30}-1374T^{31}+372T^{32}+324T^{33}-312T^{34}+70T^{35}+36T^{36}-24T^{37}+4T^{38}\right)\right\}$$

» 34 →
$$\left\{-\frac{1}{T^8}\left(1-T+T^2\right)\left(2-5T+T^2+13T^3-23T^4+10T^5+20T^6-37T^7+20T^8+10T^9-23T^{10}+13T^{11}+T^{12}-5T^{13}+2T^{14}\right),\right.$$
$$\left.\frac{1}{T^{16}}(-1+T)^2\left(9-36T+51T^2+26T^3-212T^4+360T^5-371T^6+444T^7-653T^8+126T^9+2323T^{10}-5726T^{11}+6225T^{12}-910T^{13}-7334T^{14}+11\,380T^{15}-7334T^{16}-910T^{17}+6225T^{18}-5726T^{19}+2323T^{20}+126T^{21}-653T^{22}+444T^{23}-371T^{24}+360T^{25}-212T^{26}+26T^{27}+51T^{28}-36T^{29}+9T^{30}\right)\right\}$$

» 35 →
$$\left\{-\frac{\left(1-T+T^2\right)\left(2-4T-2T^2+14T^3-16T^4+T^5+15T^6-21T^7+15T^8+T^9-16T^{10}+14T^{11}-2T^{12}-4T^{13}+2T^{14}\right)}{T^8},\right.$$
$$\left.\frac{1}{T^{16}}(-1+T)^2\left(9-36T+48T^2+36T^3-226T^4+330T^5-130T^6-156T^7+16T^8+184T^9+939T^{10}-3394T^{11}+4291T^{12}-868T^{13}-5012T^{14}+7972T^{15}-5012T^{16}-868T^{17}+4291T^{18}-3394T^{19}+939T^{20}+184T^{21}+16T^{22}-156T^{23}-130T^{24}+330T^{25}-226T^{26}+36T^{27}+48T^{28}-36T^{29}+9T^{30}\right)\right\}$$

» 36 →
$$\left\{-\frac{\left(1-T+T^2\right)\left(3-6T-T^2+12T^3-12T^4+T^5+10T^6-15T^7+10T^8+T^9-12T^{10}+12T^{11}-T^{12}-6T^{13}+3T^{14}\right)}{T^8},\right.$$
$$\left.-\frac{1}{T^{18}}(-1+T)^2\left(2-8T-4T^2+62T^3-117T^4+16T^5+285T^6-484T^7+244T^8+164T^9-70T^{10}-336T^{11}-130T^{12}+1698T^{13}-2566T^{14}+892T^{15}+2340T^{16}-4008T^{17}+2340T^{18}+892T^{19}-2566T^{20}+1698T^{21}-130T^{22}-336T^{23}-70T^{24}+164T^{25}+244T^{26}-484T^{27}+285T^{28}+16T^{29}-117T^{30}+62T^{31}-4T^{32}-8T^{33}+2T^{34}\right)\right\}$$

» 37 →

$$\left\{-\frac{\left(1-T+T^2\right)\left(3-6T-T^2+12T^3-12T^4+T^5+10T^6-15T^7+10T^8+T^9-12T^{10}+12T^{11}-T^{12}-6T^{13}+3T^{14}\right)}{T^8},\right.$$

$$-\frac{1}{T^{18}}(-1+T)^2\left(2-8T-4T^2+62T^3-117T^4+16T^5+285T^6-484T^7+244T^8+164T^9-70T^{10}-336T^{11}-130T^{12}+\right.$$
$$\left.\left.1698T^{13}-2566T^{14}+892T^{15}+2340T^{16}-4008T^{17}+2340T^{18}+892T^{19}-2566T^{20}+1698T^{21}-130T^{22}-\right.\right.$$
$$\left.\left.336T^{23}-70T^{24}+164T^{25}+244T^{26}-484T^{27}+285T^{28}+16T^{29}-117T^{30}+62T^{31}-4T^{32}-8T^{33}+2T^{34}\right)\right\}$$

» 38 → $\left\{-\frac{1}{T^8}\left(2-5T+3T^2+7T^3-21T^4+28T^5-\right.\right.$

$$\left.24T^6+16T^7-13T^8+16T^9-24T^{10}+28T^{11}-21T^{12}+7T^{13}+3T^{14}-5T^{15}+2T^{16}\right),$$

$$\frac{1}{T^{16}}(-1+T)^2\left(7-20T+6T^2+76T^3-173T^4+142T^5+83T^6-400T^7+641T^8-860T^9+1208T^{10}-\right.$$
$$1428T^{11}+817T^{12}+1056T^{13}-3319T^{14}+4368T^{15}-3319T^{16}+1056T^{17}+817T^{18}-1428T^{19}+$$
$$\left.\left.1208T^{20}-860T^{21}+641T^{22}-400T^{23}+83T^{24}+142T^{25}-173T^{26}+76T^{27}+6T^{28}-20T^{29}+7T^{30}\right)\right\}$$

» 39 → $\left\{-\frac{\left(1-T+T^2\right)\left(1-T+T^2-T^3+T^4\right)\left(2-T-7T^2+8T^3+3T^4-11T^5+3T^6+8T^7-7T^8-T^9+2T^{10}\right)}{T^8},\right.$

$$\frac{1}{T^{16}}(-1+T)^2\left(9-26T+10T^2+88T^3-222T^4+204T^5+108T^6-542T^7+751T^8-810T^9+1145T^{10}-\right.$$
$$1528T^{11}+973T^{12}+992T^{13}-3307T^{14}+4348T^{15}-3307T^{16}+992T^{17}+973T^{18}-1528T^{19}+$$
$$\left.\left.1145T^{20}-810T^{21}+751T^{22}-542T^{23}+108T^{24}+204T^{25}-222T^{26}+88T^{27}+10T^{28}-26T^{29}+9T^{30}\right)\right\}$$

» 40 → $\left\{-\frac{1-2T+5T^3-9T^4+8T^5-4T^6+T^7-T^8+T^9-4T^{10}+8T^{11}-9T^{12}+5T^{13}-2T^{15}+T^{16}}{T^8},\right.$

$$\frac{1}{T^{16}}(-1+T)^2\left(5-14T+10T^2+30T^3-91T^4+102T^5-15T^6-144T^7+282T^8-310T^9+243T^{10}-\right.$$
$$148T^{11}+44T^{12}+164T^{13}-422T^{14}+556T^{15}-422T^{16}+164T^{17}+44T^{18}-148T^{19}+243T^{20}-$$
$$\left.\left.310T^{21}+282T^{22}-144T^{23}-15T^{24}+102T^{25}-91T^{26}+30T^{27}+10T^{28}-14T^{29}+5T^{30}\right)\right\}$$

» 41 → $\left\{-\frac{1-T-2T^2+2T^3+4T^4-5T^5-6T^6+13T^7-6T^8-5T^9+4T^{10}+2T^{11}-2T^{12}-T^{13}+T^{14}}{T^7},\right.$

$$\frac{1}{T^{14}}(-1+T)^2\left(4-10T+2T^2+42T^3-79T^4-20T^5+189T^6-156T^7-\right.$$
$$46T^8+210T^9-273T^{10}+170T^{11}+201T^{12}-432T^{13}+201T^{14}+170T^{15}-273T^{16}+$$
$$\left.\left.210T^{17}-46T^{18}-156T^{19}+189T^{20}-20T^{21}-79T^{22}+42T^{23}+2T^{24}-10T^{25}+4T^{26}\right)\right\}$$

» 42 → $\left\{-\frac{1-2T+T^2+T^3+T^4-10T^5+15T^6-10T^7+T^8+T^9+T^{10}-2T^{11}+T^{12}}{T^6},\right.$

$$\frac{1}{T^{14}}(-1+T)^2\left(4-14T+27T^2-52T^3+93T^4-120T^5+29T^6+278T^7-679T^8+\right.$$
$$902T^9-662T^{10}+8T^{11}+724T^{12}-1040T^{13}+724T^{14}+8T^{15}-662T^{16}+902T^{17}-$$
$$\left.\left.679T^{18}+278T^{19}+29T^{20}-120T^{21}+93T^{22}-52T^{23}+27T^{24}-14T^{25}+4T^{26}\right)\right\}$$

» $$43 \to \left\{ -\frac{1 - T - 2T^2 + 2T^3 + 4T^4 - 5T^5 - 6T^6 + 13T^7 - 6T^8 - 5T^9 + 4T^{10} + 2T^{11} - 2T^{12} - T^{13} + T^{14}}{T^7}, \right.$$

$$\left. \frac{1}{T^{14}} (-1+T)^2 \left(4 - 10T + 2T^2 + 42T^3 - 79T^4 - 20T^5 + 189T^6 - 156T^7 - 46T^8 + 210T^9 - 273T^{10} + 170T^{11} + 201T^{12} - 432T^{13} + 201T^{14} + 170T^{15} - 273T^{16} + 210T^{17} - 46T^{18} - 156T^{19} + 189T^{20} - 20T^{21} - 79T^{22} + 42T^{23} + 2T^{24} - 10T^{25} + 4T^{26}\right) \right\}$$

» $$44 \to \left\{ -\frac{1 - 2T + T^2 + T^3 + T^4 - 10T^5 + 15T^6 - 10T^7 + T^8 + T^9 + T^{10} - 2T^{11} + T^{12}}{T^6}, \right.$$

$$\left. \frac{1}{T^{14}} (-1+T)^2 \left(4 - 14T + 27T^2 - 52T^3 + 93T^4 - 120T^5 + 29T^6 + 278T^7 - 679T^8 + 902T^9 - 662T^{10} + 8T^{11} + 724T^{12} - 1040T^{13} + 724T^{14} + 8T^{15} - 662T^{16} + 902T^{17} - 679T^{18} + 278T^{19} + 29T^{20} - 120T^{21} + 93T^{22} - 52T^{23} + 27T^{24} - 14T^{25} + 4T^{26}\right) \right\}$$

» $$45 \to \left\{ \frac{1 - 4T + 4T^2 + 2T^3 - 6T^4 + 3T^5 - 2T^6 + 11T^7 - 17T^8 + 11T^9 - 2T^{10} + 3T^{11} - 6T^{12} + 2T^{13} + 4T^{14} - 4T^{15} + T^{16}}{T^8}, \right.$$

$$\left. \frac{1}{T^{17}} (-1+T)^2 \left(2 - 6T + 32T^3 - 64T^4 + 14T^5 + 122T^6 - 161T^7 - 38T^8 + 231T^9 - 134T^{10} - 21T^{11} - 28T^{12} + 125T^{13} - 32T^{14} - 110T^{15} + 176T^{16} - 110T^{17} - 32T^{18} + 125T^{19} - 28T^{20} - 21T^{21} - 134T^{22} + 231T^{23} - 38T^{24} - 161T^{25} + 122T^{26} + 14T^{27} - 64T^{28} + 32T^{29} - 6T^{31} + 2T^{32}\right) \right\}$$

» $$46 \to \left\{ -\frac{1 - T - 2T^2 + 2T^3 + 4T^4 - 5T^5 - 6T^6 + 13T^7 - 6T^8 - 5T^9 + 4T^{10} + 2T^{11} - 2T^{12} - T^{13} + T^{14}}{T^7}, \right.$$

$$\left. \frac{1}{T^{14}} (-1+T)^2 \left(4 - 10T + 2T^2 + 42T^3 - 79T^4 - 20T^5 + 189T^6 - 156T^7 - 46T^8 + 210T^9 - 273T^{10} + 170T^{11} + 201T^{12} - 432T^{13} + 201T^{14} + 170T^{15} - 273T^{16} + 210T^{17} - 46T^{18} - 156T^{19} + 189T^{20} - 20T^{21} - 79T^{22} + 42T^{23} + 2T^{24} - 10T^{25} + 4T^{26}\right) \right\}$$

» $$47 \to \left\{ -\frac{1 - T - 2T^2 + 2T^3 + 4T^4 - 5T^5 - 6T^6 + 13T^7 - 6T^8 - 5T^9 + 4T^{10} + 2T^{11} - 2T^{12} - T^{13} + T^{14}}{T^7}, \right.$$

$$\left. \frac{1}{T^{14}} (-1+T)^2 \left(4 - 10T + 2T^2 + 42T^3 - 79T^4 - 20T^5 + 189T^6 - 156T^7 - 46T^8 + 210T^9 - 273T^{10} + 170T^{11} + 201T^{12} - 432T^{13} + 201T^{14} + 170T^{15} - 273T^{16} + 210T^{17} - 46T^{18} - 156T^{19} + 189T^{20} - 20T^{21} - 79T^{22} + 42T^{23} + 2T^{24} - 10T^{25} + 4T^{26}\right) \right\}$$

» $$48 \to \left\{ \frac{1 - 5T + 7T^2 - T^3 - 5T^4 + 2T^5 + T^6 + 7T^7 - 13T^8 + 7T^9 + T^{10} + 2T^{11} - 5T^{12} - T^{13} + 7T^{14} - 5T^{15} + T^{16}}{T^8}, \right.$$

$$\left. \frac{1}{T^{17}} (-1+T)^2 \left(2 - 8T + 6T^2 + 31T^3 - 82T^4 + 18T^5 + 254T^6 - 478T^7 + 246T^8 + 202T^9 - 232T^{10} + 42T^{11} - 202T^{12} + 458T^{13} - 220T^{14} - 289T^{15} + 540T^{16} - 289T^{17} - 220T^{18} + 458T^{19} - 202T^{20} + 42T^{21} - 232T^{22} + 202T^{23} + 246T^{24} - 478T^{25} + 254T^{26} + 18T^{27} - 82T^{28} + 31T^{29} + 6T^{30} - 8T^{31} + 2T^{32}\right) \right\}$$