

# A PERTURBED-ALEXANDER INVARIANT

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*Dedicated to the memory of V. F. R. Jones, 1952–2020, a friend and a mentor.*

ABSTRACT. In this note we give concise formulas, which lead to a simple and fast computer program that computes a powerful knot invariant. This invariant  $\rho_1$  is not new, yet our formulas are by far the simplest and fastest: given a knot we write one of the standard matrices  $A$  whose determinant is its Alexander polynomial, yet instead of computing the determinant we consider a certain quadratic expression in the entries of  $A^{-1}$ . The proximity of our formulas to the Alexander polynomial suggest that they should have a topological explanation. This we don't have yet. Yet we do know that  $\rho_1$  yields a lower bound on the genus of a knot, and that it is sometimes better than the lower bound coming from the Alexander polynomial.

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## 1. THE FORMULAS

One of the selling points for this article is that the formulas in it are concise. Thus we start by running through these formulas for a knot invariant  $\rho_1$  as quickly as we can. In Section 2 we turn the formulas into a short yet very fast computer program, in Section 3

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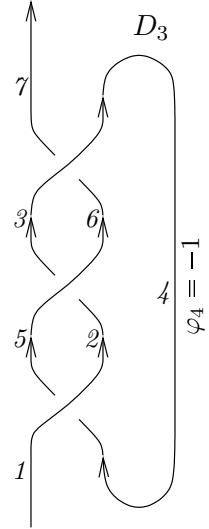
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we give a partial interpretation of the formulas in terms of car traffic on a knot diagram and use it to prove the invariance of  $\rho_1$ , and in Section 4 we quickly sketch the context: Alexander, Burau, Jones, Melvin, Morton, Rozansky, Overbay, and our own prior work. This article accompanies a talk, [BN9] (video and handout available), and a future talk, [BN10] (registration link and handout available).

Given an oriented  $n$ -crossing knot  $K$ , we draw it in the plane as a long knot diagram  $D$  in such a way that the two strands intersecting at each crossing are pointed up (that's always possible because we can always rotate crossings as needed), and so that at its beginning and at its end the knot is oriented upward. We call such a diagram an *upright knot diagram*. An example of an upright knot diagram  $D_3$  is shown on the right.

We then label each edge of the diagram with two integer labels: a running index  $k$  which runs from 1 to  $2n + 1$ , and a “rotation number”  $\varphi_k$ , the geometric rotation number of that edge (the signed number of times the tangent to the edge is horizontal and heading right, with cups counted with +1 signs and caps with  $-1$ ; this number is well defined because at their ends, all edges are headed up). On the right the running index runs from 1 to 7, and the rotation numbers for all edges are 0 (and hence are omitted) except for  $\varphi_4$ , which is  $-1$ .



A *Technicality*. Some Reidemeister moves create or lose an edge and to avoid the need for renumbering it is beneficial to also allow labelling the edges with non-consecutive labels. Hence we allow that, and write  $i^+$  for the successor of the label  $i$  along the knot, and  $i^{++}$  for the successor of  $i^+$  (these are  $i + 1$  and  $i + 2$  if the labelling is by consecutive integers). Also, “1” will always refer to the label of the first edge, and “ $2n + 1$ ” will always refer to the label of the last.

We let  $A$  be the  $(2n + 1) \times (2n + 1)$  matrix of Laurent polynomials in the formal variable  $T$  defined by

$$A := I - \sum_c (T^s E_{i,i^+} + (1 - T^s) E_{i,j^+} + E_{j,j^+}),$$

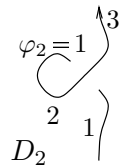
where  $I$  is the identity matrix,  $E_{\alpha\beta}$  denotes the elementary matrix with 1 in row  $\alpha$  and column  $\beta$  and zeros elsewhere. The summation is over the crossings  $c$  of the diagram  $D$ , and once  $c$  is chosen,  $s$  denotes its sign and  $i$  and  $j$  denote the labels below the crossing where the label  $i$  belongs to the over-strand and  $j$  to the under-strand.

Alternatively,  $A = I + \sum_c A_c$ , where  $A_c$  is a matrix of zeros except for the blocks as follows:

$$\begin{array}{cc} \begin{array}{c} j^+ \uparrow \quad i^+ \uparrow \\ \diagdown \quad \diagup \\ i \quad j \\ s = +1 \end{array} & \begin{array}{c} i^+ \uparrow \quad j^+ \uparrow \\ \diagdown \quad \diagup \\ j \quad i \\ s = -1 \end{array} \end{array} \longrightarrow \begin{array}{c|cc} A_c & \text{column } i^+ & \text{column } j^+ \\ \hline \text{row } i & -T^s & T^s - 1 \\ \text{row } j & 0 & -1 \end{array} \quad (1)$$

For example, if  $D = D_1 = \uparrow$  is the diagram with no crossings, the resulting matrix  $A$  is the  $1 \times 1$  identity matrix (1). If  $D = D_2$  is the diagram on the right (here  $s = +1$ ,  $(i, j) = (2, 1)$ , and  $(i^+, j^+) = (3, 2)$ ), then

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 0 & T - 1 & -T \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & T & -T \\ 0 & 0 & 1 \end{pmatrix},$$



and for  $D_3$  as on the first page, we have

$$A = \begin{pmatrix} 1 & -T & 0 & 0 & T-1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & T-1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & T-1 & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We note without supplying details that the matrix  $A$  comes in a straightforward way from Fox calculus as it is applied to the Wirtinger presentation of the fundamental group of the complement of  $K$  (using the diagram  $D$ ). Hence the determinant of  $A$  is equal up to a unit to the normalized Alexander polynomial  $\Delta$  of  $K$  (which satisfies  $\Delta(T) = \Delta(T^{-1})$  and  $\Delta(1) = 1$ ). In fact, we have that

$$\Delta = T^{(-\varphi(D)-w(D))/2} \det(A), \quad (2)$$

where  $\varphi(D) := \sum_k \varphi_k$  is the total rotation number of  $D$  and where  $w(D) = \sum_c s_c$  is the writhe of  $D$ , namely the sum of the signs  $s_c$  of all the crossings  $c$  in  $D$ .

For our example  $D_2$ ,  $\det(A) = T$ ,  $\varphi(D) = 1$ , and  $w(D) = 1$ , so  $\Delta = T^{(-1-1)/2} \cdot T = 1$ , as expected for a diagram of the unknot. For  $D_3$ ,  $\det(A) = 1 - T + T^2$ ,  $\varphi(D) = -1$ , and  $w(D) = 3$ , so  $\Delta = T^{(1-3)/2} (1 - T + T^2) = T - 1 + T^{-1}$ , as expected for the trefoil knot.

We set<sup>1</sup>  $G = (g_{\alpha\beta}) = A^{-1}$ , and taking our inspiration from physics, we name  $g_{\alpha\beta}$  the *Green function* for the diagram  $D$ . For our three examples  $D_1$ ,  $D_2$ , and  $D_3$ , the Green function  $G$  is respectively

$$(1), \quad \begin{pmatrix} 1 & T^{-1} & 1 \\ 0 & T^{-1} & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 & \frac{T^3-T^2+T}{T^2-T+1} & 1 \\ 0 & 1 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T}{T^2-T+1} & \frac{T^2}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & \frac{1-T}{T^2-T+1} & \frac{T-T^2}{T^2-T+1} & \frac{1}{T^2-T+1} & \frac{T}{T^2-T+1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3)$$

We can now define our invariant  $\rho_1$ . It is the sum of two sums. The first is a sum of a term  $R_1(c)$  over all crossings  $c$  in  $D$ , where for such a crossing we let  $s$  denote its sign and we let  $i$  and  $j$  denote the edge labels of the incoming over- and under-strands, respectively and where

$$R_1(c) := s (g_{ji} (g_{j+,j} + g_{j,j+} - g_{ij}) - g_{ii} (g_{j,j+} - 1) - 1/2). \quad (4)$$

The second sum is a sum over the edges  $k$  of  $D$  of a correction term dependent on the rotation number  $\varphi_k$ . We multiply the result by  $\Delta^2$  to “clear the denominators”<sup>2</sup>:

$$\rho_1 := \Delta^2 \left( \sum_c R_1(c) - \sum_k \varphi_k (g_{kk} - 1/2) \right), \quad (5)$$

Direct calculations show that  $\rho_1(D_1) = 0$  (as the sums are empty),  $\rho_1(D_2) = 0$ , and  $\rho_1(D_3) = -T^2 + 2T - 2 + 2T^{-1} - T^{-2}$ .

<sup>1</sup>At  $T = 1$  the matrix  $A$  has 1’s on the main diagonal,  $(-1)$ ’s on the diagonal above it, and 0’s everywhere else. Hence  $A$  is invertible at  $T = 1$  and hence over the field of rational functions.

<sup>2</sup> $R_1(s)$  is quadratic in the entries of  $G$  and hence it has denominators proportional to  $\Delta^2$ .

**Theorem 1** (“Invariance”, proofs in Section 3). *The quantity  $\rho_1$  is a knot invariant.*

As we shall see in the next section,  $\rho_1$  has more separation power than the Jones polynomial, yet it is closer to the more topologically-meaningful Alexander polynomial  $\Delta$ : it is cooked up from the same matrix  $A$  and in terms of computational complexity, computing  $\rho_1$  is not very different from computing  $\Delta$ . In order to compute  $\Delta$  we need to compute the determinant of  $A$ , while to compute  $\rho_1$  we need to invert  $A$  and then compute a sum of  $O(n)$  terms that are quadratic in the entries of  $A^{-1}$ .<sup>3</sup> We have computed  $\rho_1$  for knots with over 200 crossings using the unsophisticated implementation presented in Section 2.

Topologists should be intrigued!  $\rho_1$  is derived from the same matrix as the Alexander polynomial  $\Delta$ , yet we have no topological interpretation for  $\rho_1$ . Even without an interpretation, we have an application:

**Theorem 2** (“Genus”). *For any knot  $K$ , the breadth of the Laurent polynomial  $\rho_1(K)$ , namely the difference between the highest power of  $T$  in it and the lowest, is at least four times the genus of  $K$ .*

A well-known theorem (e.g., [Li, Proposition 6.13]) asserts that the breadth of the Alexander polynomial  $\Delta(K)$  is at least twice the genus of  $K$ . In Section ?? we show examples for which the genus bound coming from  $\rho_1$  is better than the genus bound coming from  $\Delta$ .


*Proof of Theorem 2.*


## 2. IMPLEMENTATION AND POWER

Two of the main reasons we like  $\rho_1$  is that it is very easy to implement and even an unsophisticated implementation runs very fast. To highlight these points we include a full implementation here, a step-by-step run-through, and a demo run. We write in Mathematica [Wo], and you can find the notebook displayed here at [BV4, APAI.nb].

We start by loading the library `KnotTheory`` [BM] (it is used here only for the list of knots that it contains, and to compute other invariants for comparisons). We also load minor conversion routine [BV4, Rot.nb / Rot.m] whose internal workings are irrelevant here.

 `Once[<< KnotTheory` ; << Rot.m];`

 Loading `KnotTheory`` version of February 2, 2020, 10:53:45.2097.  
Read more at <http://katlas.org/wiki/KnotTheory>.

 Loading `Rot.m` from <http://drorbn.net/APAI> to compute rotation numbers.

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<sup>3</sup>We prefer not to be more specific about the complexity of computing  $\rho_1$ . It is the same as the complexity of inverting  $A$ , and matrix inversion is poly-time, with a rather small exponent, even for matrices with entries in a ring of polynomials (e.g. [St]). We have not explored how much one can further gain by exploiting the fact that  $A$  is very sparse.

2.1. **The Program.** This done, here is the full  $\rho_1$  program:

```

☹️  $R_1[s_-, i_-, j_-] := s (g_{ji} (g_{j^+, j} + g_{j, j^+} - g_{ij}) - g_{ii} (g_{j, j^+} - 1) - 1/2);$ 
❤️  $\rho[K_-] := \text{Module}[\{\text{Cs}, \varphi, n, A, s, i, j, k, \Delta, G, \rho_1\},$ 
   $\{\text{Cs}, \varphi\} = \text{Rot}[K]; n = \text{Length}[\text{Cs}];$ 
   $A = \text{IdentityMatrix}[2n + 1];$ 
   $\text{Cases}[\text{Cs}, \{s_-, i_-, j_-\} \Rightarrow \left( A[\{i, j\}, \{i+1, j+1\}] += \begin{pmatrix} -T^s & T^s - 1 \\ \theta & -1 \end{pmatrix} \right)];$ 
   $\Delta = T^{(-\text{Total}[\varphi] - \text{Total}[\text{Cs}[\text{All}, 1]])/2} \text{Det}[A];$ 
   $G = \text{Inverse}[A];$ 
   $\rho_1 = \sum_{k=1}^n R_1 @@ \text{Cs}[\{k\}] - \sum_{k=1}^{2n} \varphi[\{k\}] (g_{kk} - 1/2);$ 
   $\text{Factor}@\{\Delta, \Delta^2 \rho_1 / . \alpha_-^+ \Rightarrow \alpha + 1 / . g_{\alpha, \beta_-} \Rightarrow G[\{\alpha, \beta\}]\};$ 

```

The program uses mostly the same symbols as the text, so even without any knowledge of Mathematica, the reader should be able to recognize at least formulas (1), (2), and (5) within it. As a further hint we add that the variables `Cs` ends up storing the list of crossings in a knot  $K$ , where each crossing is stored as a triple  $(s, i, j)$ , where  $s, i$ , and  $j$  have the same meaning as in (1). The conversion routine `Rot` automatically produces `Cs`, as well as a list  $\varphi$  of rotation numbers, given any other knot presentation known to the package `KnotTheory'`.

Note that the program outputs the ordered pair  $(\Delta, \rho_1)$ . The Alexander polynomial  $\Delta$  is anyway computed internally, and we consider the aggregate  $(\Delta, \rho_1)$  as more interesting than any of its pieces by itself.

2.2. **A Step-by-Step Run-Through.** We start by setting  $K$  to be the knot diagram on page 1 using the PD notation of `KnotTheory'` [BM]. We then print `Rot[K]`, which is a list of crossings followed by a list of rotation numbers:

```

☹️  $K = \text{PD}[X[4, 2, 5, 1], X[2, 6, 3, 5], X[6, 4, 7, 3]];$ 
   $\text{Rot}[K]$ 
  {{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}

```

Next we set `Cs` and  $\varphi$  to be the list of crossings, and the list of rotation numbers, respectively.

```

☹️  $\{\text{Cs}, \varphi\} = \text{Rot}[K]$ 
  {{{{1, 1, 4}, {1, 5, 2}, {1, 3, 6}}, {0, 0, 0, -1, 0, 0}}

```

We set  $n$  to be the number of crossings,  $A$  to be the  $(2n + 1)$ -dimensional identity matrix, and then we iterate over  $c$  in `Cs`, adding a block as in (1) for each crossing.


```

☹️  $n = \text{Length}[\text{Cs}];$ 
  ❤️  $A = \text{IdentityMatrix}[2n + 1];$ 
   $\text{Cases}[\text{Cs}, \{s_-, i_-, j_-\} \Rightarrow \left( A[\{i, j\}, \{i+1, j+1\}] += \begin{pmatrix} -T^s & T^s - 1 \\ \theta & -1 \end{pmatrix} \right)];$ 


```


Here's what A comes out to be:

 **A // MatrixForm**

 
$$\begin{pmatrix} 1 & -T & 0 & 0 & -1+T & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -T & 0 & 0 & -1+T \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1+T & 0 & 1 & -T & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We set  $\Delta$  to be the determinant of A, with a correction as in (2). So  $\Delta$  is the Alexander polynomial of  $K$ .


  $\Delta = T^{(-\text{Total}[\varphi] - \text{Total}[\text{Cs}[\text{All}, 1]])/2} \text{Det}[A]$

 
$$\frac{1 - T + T^2}{T}$$

G is now the Inverse of A:

 **G = Inverse[A];**

**G // MatrixForm**

 
$$\begin{pmatrix} 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 & \frac{T-T^2+T^3}{1-T+T^2} & 1 \\ 0 & 1 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T}{1-T+T^2} & \frac{T^2}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & \frac{1-T}{1-T+T^2} & \frac{T-T^2}{1-T+T^2} & \frac{1}{1-T+T^2} & \frac{T}{1-T+T^2} & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

It remains to blindly follow the two parts of Equation (5):

$$\rho_1 = \sum_{k=1}^n R_1 @ @ C_s [[k]] - \sum_{k=1}^{12n} \varphi [[k]] (g_{kk} - 1 / 2)$$

$$-2 + g_{4,4} - g_{1,1} (-1 + g_{4,4^+}) - (-1 + g_{2,2^+}) g_{5,5} - g_{3,3} (-1 + g_{6,6^+}) + g_{2,5} (g_{2,2^+} - g_{5,2} + g_{2^+,2}) + g_{4,1} (-g_{1,4} + g_{4,4^+} + g_{4^+,4}) + g_{6,3} (-g_{3,6} + g_{6,6^+} + g_{6^+,6})$$

We replace each  $g_{\alpha\beta}$  with the appropriate entry of G:

$$\Delta^2 \rho_1 / . \alpha_-^+ \Rightarrow \alpha + 1 / . g_{\alpha, \beta} \Rightarrow G[[\alpha, \beta]]$$

$$\frac{(1 - T + T^2)^2 \left( -1 + \frac{T}{(1-T+T^2)^2} - \frac{-1 + \frac{1}{1-T+T^2}}{1-T+T^2} \right)}{T^2}$$

Finally, we output both  $\Delta$  and  $\rho_1$ . We factor them just to put them in a nicer form:


$$\text{Factor}@\{\Delta, \Delta^2 \rho_1 / . \alpha_-^+ \Rightarrow \alpha + 1 / . g_{\alpha, \beta} \Rightarrow G[[\alpha, \beta]]\}$$

$$\left\{ \frac{1 - T + T^2}{T}, -\frac{(-1 + T)^2 (1 + T^2)}{T^2} \right\}$$

2.3. **A Demo Run.** Here are  $\Delta$  and  $\rho_1$  of all the knots with up to 6 crossings (a table up to 10 crossings is printed at [BV3]):

$$\text{TableForm}[\text{Table}[\text{Join}[\{K\}, \rho[K]], \{K, \text{AllKnots}[\{3, 6\}]\}], \text{TableAlignments} \rightarrow \text{Center}]$$

 **KnotTheory:** Loading precomputed data in PD4Knots`.

 Knot [3, 1]	$\frac{1-T+T^2}{T}$	$\frac{(-1+T)^2 (1+T^2)}{T^2}$
Knot [4, 1]	$-\frac{1-3T+T^2}{T}$	0
Knot [5, 1]	$\frac{1-T+T^2-T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1+T^2) (2+T^2+2T^4)}{T^4}$
Knot [5, 2]	$\frac{2-3T+2T^2}{T}$	$\frac{(-1+T)^2 (5-4T+5T^2)}{T^2}$
Knot [6, 1]	$-\frac{(-2+T) (-1+2T)}{T}$	$\frac{(-1+T)^2 (1-4T+T^2)}{T^2}$
Knot [6, 2]	$-\frac{1-3T+3T^2-3T^3+T^4}{T^2}$	$\frac{(-1+T)^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{T^4}$
Knot [6, 3]	$\frac{1-3T+5T^2-3T^3+T^4}{T^2}$	0

Some comments are in order:

- $\rho_1$  flips its sign when switching to the mirror of a knot. Indeed in (5) both  $R_1(c)$  and  $\varphi_k$  flip sign under reflection in a plane perpendicular to the plane of the knot diagram. Hence  $\rho_1$  vanishes on amphicheiral knots, such as  $4_1$  and  $6_3$  above.
- $\rho_1$  seems to always be divisible by  $(T - 1)^2$  and seems to always be palindromic ( $\rho_1(T) = \rho_1(T^{-1})$ ). We are not sure why this is so.

Next is one of our favourites, a knot from [GST] (see Figure 2.1), which is a potential counterexample to the ribbon=slice conjecture [Fo]. It takes about two minutes to compute

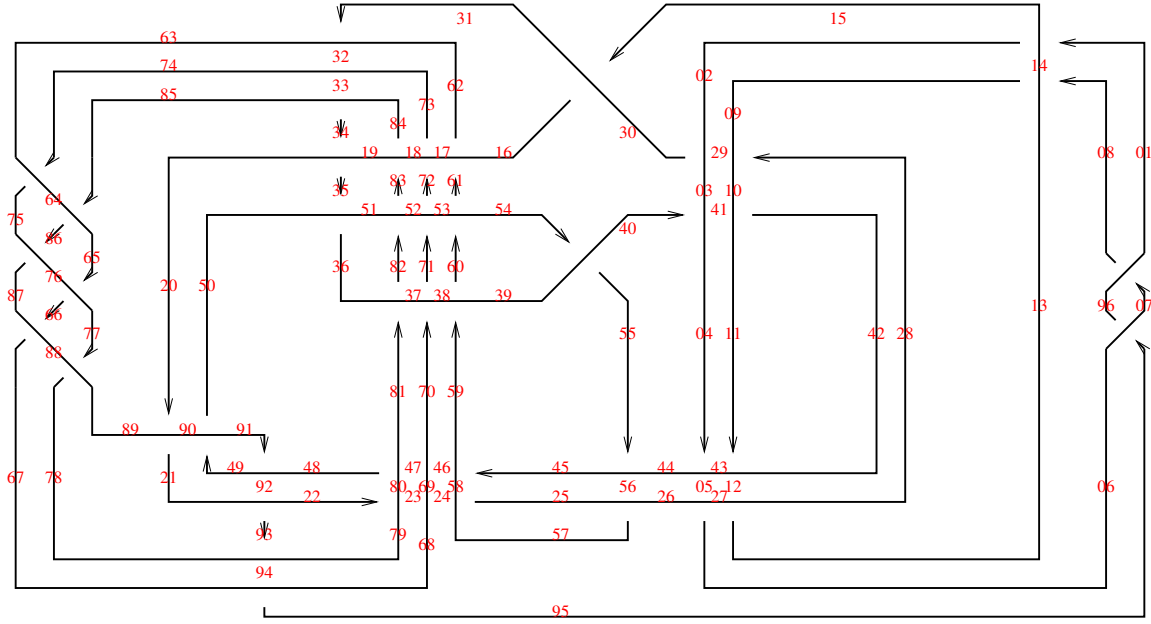








FIGURE 2.1. A 48-crossing knot from [GST].

$\rho_1$  for this 48 crossing knot (note that Mathematica prints Timing information in seconds, and that this information is highly dependent on the CPU used, how loaded it is, and even on its temperature at the time of the computation):

 Timing@ $\rho$  [EPD [X<sub>14,1</sub>, X̄<sub>2,29</sub>, X<sub>3,40</sub>, X<sub>43,4</sub>, X̄<sub>26,5</sub>, X<sub>6,95</sub>, X<sub>96,7</sub>, X<sub>13,8</sub>, X̄<sub>9,28</sub>, X<sub>10,41</sub>, X<sub>42,11</sub>,  
 X̄<sub>27,12</sub>, X<sub>30,15</sub>, X̄<sub>16,61</sub>, X̄<sub>17,72</sub>, X̄<sub>18,83</sub>, X<sub>19,34</sub>, X̄<sub>89,20</sub>, X̄<sub>21,92</sub>, X̄<sub>79,22</sub>, X̄<sub>68,23</sub>, X̄<sub>57,24</sub>,  
 X̄<sub>25,56</sub>, X<sub>62,31</sub>, X<sub>73,32</sub>, X<sub>84,33</sub>, X̄<sub>50,35</sub>, X<sub>36,81</sub>, X<sub>37,70</sub>, X<sub>38,59</sub>, X̄<sub>39,54</sub>, X<sub>44,55</sub>, X<sub>58,45</sub>,  
 X<sub>69,46</sub>, X<sub>80,47</sub>, X<sub>48,91</sub>, X<sub>90,49</sub>, X<sub>51,82</sub>, X<sub>52,71</sub>, X<sub>53,60</sub>, X̄<sub>63,74</sub>, X̄<sub>64,85</sub>, X̄<sub>76,65</sub>, X̄<sub>87,66</sub>,  
 X̄<sub>67,94</sub>, X̄<sub>75,86</sub>, X̄<sub>88,77</sub>, X̄<sub>78,93</sub> ] ]

 { 158.625, { -  $\frac{(-1 + 2T - T^2 - T^3 + 2T^4 - T^5 + T^8)(-1 + T^3 - 2T^4 + T^5 + T^6 - 2T^7 + T^8)}{T^8}$ ,  
 $\frac{1}{T^{16}} (-1 + T)^2 (5 - 18T + 33T^2 - 32T^3 + 2T^4 + 42T^5 - 62T^6 - 8T^7 + 166T^8 - 242T^9 + 108T^{10} +$   
 $132T^{11} - 226T^{12} + 148T^{13} - 11T^{14} - 36T^{15} - 11T^{16} + 148T^{17} - 226T^{18} + 132T^{19} + 108T^{20} -$   
 $242T^{21} + 166T^{22} - 8T^{23} - 62T^{24} + 42T^{25} + 2T^{26} - 32T^{27} + 33T^{28} - 18T^{29} + 5T^{30})$  } }

2.4. The Separation Power of  $\rho_1$ . Let us check how powerful  $\rho_1$  is on knots with up to 12 crossings:

 {NumberOfKnots[{3, 12}],  
 Length@Union@Table[ $\rho$ [K], {K, AllKnots[{3, 12}]},  
 Length@Union@Table[{HOMFLYPT[K], Kh[K]}, {K, AllKnots[{3, 12}]},  
 {2977, 2882, 2785}



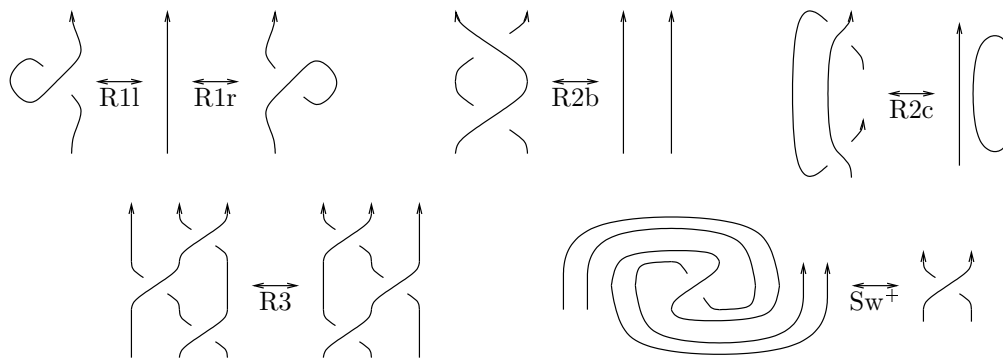


FIGURE 3.1. The upright Reidemeister moves: Reidemeister 1 left and right, Reidemeister 2 braid-like and cyclic, Reidemeister 3, and (the +) Swirl.

So the pair  $(\Delta, \rho_1)$  attains 2,882 distinct values on the 2,977 prime knots with up to 12 crossings (a deficit of 95), whereas the pair  $(H, Kh) = (\text{HOMFLYPT polynomial, Khovanov Homology})$  attains only 2,785 distinct values on the same knots (a deficit of 192).

In our spare time we computed all of these invariants on all the prime knots with up to 14 crossings. On these 59,937 knots the pair  $(\Delta, \rho_1)$  attains 53,684 distinct values (a deficit of 6,253) whereas the pair  $(H, Kh)$  attains only 49,149 distinct values on the same knots (a deficit of 10,788).

Hence the pair  $(\Delta, \rho_1)$ , computable in polynomial time by simple programs, seems stronger than the pair  $(H, Kh)$ , which is more difficult to program and (for all we know) cannot be computed in polynomial time. We are not aware of another poly-time invariant as strong as the pair  $(\Delta, \rho_1)$ .

### 3. PROOFS OF THEOREM 1, THE INVARIANCE THEOREM

We tell the proof of the Invariance Theorem (Theorem 1) in two ways: an elegant and intuitive though slightly lacking telling in Section 3.2, and a complete though slightly dull telling in Section 3.3. But first, a few common elements.

**3.1. Common Elements.** Two upright knot diagrams are considered the same (as diagrams) if their underlying knot diagrams are the same, and if the respective rotation numbers of their edges are all the same. It is clear that  $\rho_1$  is well defined on upright knot diagrams. To prove Theorem 1 we need to know what to prove. Namely, when do two upright knot diagrams represent the same knot? This is answered in the spirit of the classical Reidemeister theorem by the following:

**Theorem 3** (“Upright Reidemeister”). *Two upright knot diagrams represent the same knot if and only if they differ by a sequence of  $R1l$ ,  $R2r$ ,  $R2b$ ,  $R2c$ ,  $R3$ , and  $Sw^+$  moves as in Figure 3.1.*

*Sketch of the proof.* In the case of round knots (i.e., not “long”), knot diagrams can be turned upright by rotating individual crossings. The only ambiguity here is by powers of the full rotation, the *swirls*  $Sw^+$  and  $Sw^-$  (where  $Sw^-$  is the same as  $Sw^+$  except with a negative crossing, and we don’t need to impose it separately as it follows from  $Sw^+$  and R2). Hence we have a well-defined map  $\{\text{knot diagrams}\} \rightarrow \{\text{upright knot diagrams}\}/Sw^\pm$ . It remains to write the usual Reidemeister moves between knot diagrams as moves between upright knot

diagrams. The result are the moves R1l, R2r, R2b, R2c, and R3. Note that unoriented knot theory is presented with just three Reidemeister moves, but these split into several versions in the oriented case. The sufficiency of the versions we picked can be found in [Po]. In the case of long knots a minor further complication arises, regarding the rotation numbers of the initial and final edges. We leave the details of the problem and its resolution to the reader.  $\square$

Our key formulas, (4) and (5) involve the Green function  $g_{\alpha\beta}$ . We need to know that it is subject to some relations, the  $g$ -rules of Lemma 4 below, whose proof is so easy that it comes first:

*Proof of Lemma 4.* The first set of  $g$ -rules reads out column  $\beta$  of the equality  $AG = I$ , and the second set of  $g$ -rules reads out row  $\alpha$  of the equality  $GA = I$ .  $\square$

**Lemma 4** (“ $g$ -rules”). *Given a fixed upright knot diagram  $D$ , its corresponding matrix  $A$ , and its inverse  $G = (g_{\alpha\beta})$ , and given a crossing  $c = (s, i, j)$  in  $D$  (with  $s, i$ , and  $j$  as before), the following two sets of relations (the  $g$ -rules) hold (with  $\delta$  denoting the Kronecker delta):*

$$g_{i\beta} = \delta_{i\beta} + T^s g_{i^+, \beta} + (1 - T^s) g_{j^+, \beta}, \quad g_{j\beta} = \delta_{j\beta} + g_{j^+, \beta}, \quad g_{2n+1, \beta} = \delta_{2n+1, \beta} \quad (6)$$

and

$$g_{\alpha i} = T^{-s} (g_{\alpha, i^+} - \delta_{\alpha, i^+}), \quad g_{\alpha j} = g_{\alpha, j^+} - (1 - T^s) g_{\alpha i} - \delta_{\alpha, j^+}, \quad g_{\alpha, 1} = \delta_{\alpha, 1}. \quad (7)$$

Furthermore, for each fixed  $\beta$  there are  $2n + 1$   $g$ -rules of type (6) (the first two depend on a choice of one of  $n$  crossings, and the third is fixed, to a total of  $2n + 1$  rules). These fully determine the  $2n + 1$  scalars  $g_{\alpha\beta}$  corresponding to varying  $\alpha$ . Similarly, for each fixed  $\alpha$  there are  $2n + 1$   $g$ -rules of type (7). These fully determine the  $2n + 1$  scalars  $g_{\alpha\beta}$  corresponding to varying  $\beta$ .

For use later, we teach our computer about  $g$ -rules:

```

(◉)  $\delta_{i,j} := \text{If}[i === j, 1, 0];$ 
(♥)  $gRules_{s,i,j} := \{g_{i\beta} \mapsto \delta_{i\beta} + T^s g_{i^+, \beta} + (1 - T^s) g_{j^+, \beta}, g_{j\beta} \mapsto \delta_{j\beta} + g_{j^+, \beta},$ 
 $g_{\alpha,i} \mapsto T^{-s} (g_{\alpha, i^+} - \delta_{\alpha, i^+}), g_{\alpha,j} \mapsto g_{\alpha, j^+} - (1 - T^s) g_{\alpha i} - \delta_{\alpha, j^+}\}$ 
 $(\alpha_+)^+ := \alpha^{++}; \quad (* \text{ this is for cosmetic reasons only } *)$ 

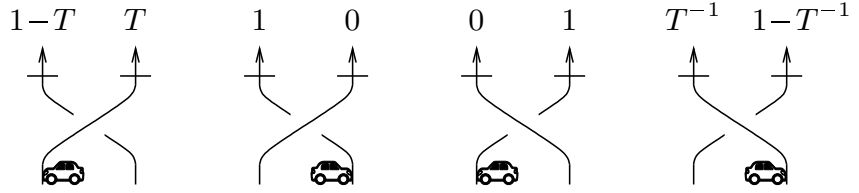
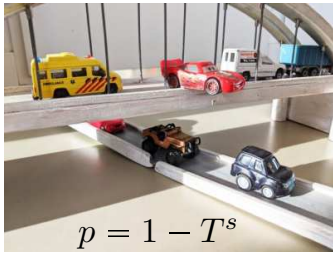
```

**3.2. Cars, Traffic Counters, and Interchanges.** Our first proof of Theorem 1 is slightly informal as it uses the language and intuition of probability theory even though our “probabilities” are merely algebraic formulae and not numbers between 0 and 1. Seasoned mathematicians should see that there is no real problem here. Yet just to be safe, we also include a fully formal proof in Section 3.3.

Cars (🚗, 🚘) travel on knot diagrams subject to the following three rules, inspired by Jones’ “bowling balls” [Jo] and by Lin, Tian, and Wang’s “random walks” [LTW]):

- On plain roads (edges) they travel following the orientation of the edge.
- When reaching an underpass (the lower strand of a crossing), cars just pass through.
- When reaching an overpass cars pass through with probability  $T^s$  (where  $s = \pm 1$  is the sign of the crossing), yet drop over to the lower strand with the complementary probability of  $1 - T^s$ .

These rules can be summarized by the following pictures:

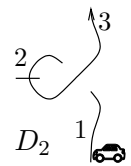


In these pictures the horizontal struts represent “traffic counters” which measure the amount of traffic that passes through their respective roads, and the output reading of these counters is printed above them. Thus for example, the last interchange picture indicates that if a unit stream of cars is injected into the diagram on the bottom right and two traffic counters are placed at the top, then the first of these will read a car intensity of  $T^{-1}$  and the second  $(1 - T^{-1})$ .

Note that our probabilities aren’t really probabilities, if only because  $T$  and  $T^{-1}$  cannot both be between 0 and 1 simultaneously. Yet we will manipulate them algebraically as if they are probabilities, restricting ourselves to equalities and avoiding inequalities. With this restriction, we can use intuition from probability theory. We will pretend that  $T^s \sim 1$ , or equivalently, that  $1 - T^s \sim 0$ . This has an algebraic meaning that does not refer to inequalities. Namely, certain series can be deemed summable. For example,

$$\sum_{r \geq 0} (1 - T^s)^r = \frac{1}{1 - (1 - T^s)} = T^{-s}.$$

**Example 5.** Cars are injected on edge #1 of the diagram  $D_2$  of Section 1 as indicated on the right. What does the indicated traffic counter on edge #2 measure?



*Solution.* Every car coming through the interchange from #1 passes through the underpass and comes to #2, so the counter reads “1” just for this traffic. But then these cars continue and pass on the overpass, and  $(1 - T)$  of them fall down and continue through edge #2 and get counted again. But then these fallen cars continue and pass on the overpass once again, and  $(1 - T)$  of them, meaning  $(1 - T)^2$  of the original traffic, fall once more and contribute a further reading of  $(1 - T)^2$ . This process continues and the overall counter reading is

$$1 + (1 - T) + (1 - T)^2 + (1 - T)^3 + \dots = \frac{1}{1 - (1 - T)} = T^{-1}.$$

Note that this is exactly the row 1 column 2 entry of the matrix  $G$  computed for this tangle in (3).

We claim that this is general:

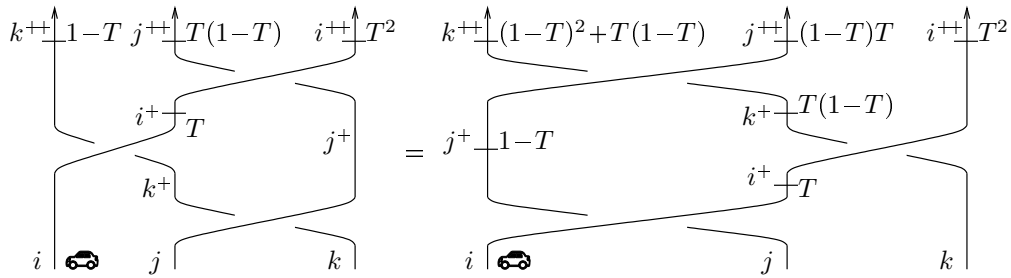
**Proposition 6.** *For a general knot diagram  $D$ , the entry  $g_{\alpha\beta}$  of its Green function is equal to the reading of a traffic counter placed at  $\beta$  given that traffic is injected into  $D$  at  $\alpha$ . (In the case where  $\alpha = \beta$ , the counter is placed after where the traffic is injected, not before).*

*Proof.* Consider the  $g$ -rules of type (6). The third,  $g_{2n+1,\beta} = \delta_{2n+1,\beta}$  is the statement that if traffic is injected on the outgoing edge of  $D$ , it can only be measured on the outgoing

edge of  $D$  (so traffic never flows backwards). The second,  $g_{j\beta} = \delta_{j\beta} + g_{j^+,\beta}$ , is the statement that traffic goes through underpasses undisturbed, so  $g_{j\beta} = g_{j^+,\beta}$  unless the traffic counter  $\beta$  is placed between  $j$  and  $j^+$ , in which case it measures one unit more if the cars are injected before it, at  $j$ , rather than after it, at  $j^+$ . Similarly the first of these  $g$ -rules,  $g_{i\beta} = \delta_{i\beta} + T^s g_{i^+,\beta} + (1 - T^s) g_{j^+,\beta}$ , is the statement of the behaviour of traffic at overpasses. Thus the rules in (6) are obeyed by cars and traffic counters, and as the rules in (6) determine  $g_{\alpha\beta}$ , the proposition follows.  $\square$

**Proposition 7.** *The quantity  $\rho_1$  is invariant under R3.*

*Proof.* We first show that cars entering a multiple interchange styled as the left hand side of the R3 move, exit it with the exact same distribution as cars entering the multiple interchange styled as the right hand side. The hardest part of that computation is when cars enter at the bottom left (at  $i$ ) and it boils down to the equality  $1 - T = (1 - T)^2 + T(1 - T)$ :



If cars enter in the middle or at the bottom right (at  $j$  or at  $k$ ), the computation is even easier.<sup>4</sup>

The conclusion is that performing the R3 move does not affect traffic patterns outside the area of the move itself; namely, the Green function  $g_{\alpha\beta}$  is unchanged if both  $\alpha$  and  $\beta$  are outside the area of the move.

Thus the only contribution to  $\rho_1$  that may change (see (5)) is the contribution coming from the three  $R_1$  terms corresponding to the crossings that move, and we need to know if the following equality holds:

$$R_1(+1, j, k) + R_1(+1, i, k^+) + R_1(+1, i^+, j^+) \stackrel{?}{=} R_1(+1, i, j) + R_1(+1, i^+, k) + R_1(+1, j^+, k^+)$$

Both sides here are messy quadratics involving the  $g_{\alpha\beta}$ 's of both sides, evaluated at  $\alpha, \beta \in \{i, j, k, i^+, j^+, k^+, i^{++}, j^{++}, k^{++}\}$ . But we can use the traffic rules (aka the  $g$ -rules) to rewrite these quadratics in terms of the  $g_{\alpha\beta}$ 's with  $\alpha, \beta \in \{i^{++}, j^{++}, k^{++}\}$ , and these are unchanged between the sides. So we simply need to know whether the above equality holds *after* the relevant  $g$ -rules have been applied to both sides. We could do that by hand, but it's simpler to appeal to a higher wisdom:

```

(☹) lhs = R1[1, j, k] + R1[1, i, k+] + R1[1, i+, j+] // . gRules1,j,k ∪ gRules1,i,k+ ∪ gRules1,i+,j+;
(♥) rhs = R1[1, i, j] + R1[1, i+, k] + R1[1, j+, k+] // . gRules1,i,j ∪ gRules1,i+,k ∪ gRules1,j+,k+;
Simplify[lhs == rhs]

```

True

$\square$

<sup>4</sup>Note that this computation is exactly the one that proves that the Bureau representation [Bu] respects R3.

*First Proof of Theorem 1, “Invariance”.* We’ve shown invariance under R3. Invariance under the other moves is shown in a similar way: first one shows that overall traffic patterns are unchanged by each of the moves, and then one verifies that the local contributions to  $\rho_1$  coming from the area changed by each move are equal once the  $g$ -rules are used to rewrite them in terms of  $g_{\alpha\beta}$ ’s that are unaffected by the moves. This is shown in greater detail in the following section.  $\square$

**3.3. A More Formal Version of the Proof.** Again we start with the hardest, R3:

**Proposition 8.** *The quantity  $\rho_1$  is invariant under R3.*

*Proof.* We need to know how the Green function  $g_{\alpha\beta}$  changes under R3. Here are the two sides of the move, along with the  $g$ -rules of type (6) corresponding to the crossings within, written with the assumption that  $\beta$  isn’t in  $\{i^+, j^+, k^+\}$ , so several of the Kronecker deltas can be ignored. We use  $g$  for the Green function of the left side of R3, and  $g'$  for the right side:

	$g_{i^+,\beta} = Tg_{i^{++},\beta} + (1-T)g_{j^{++},\beta}$ $g_{j^+,\beta} = g_{j^{++},\beta}$ $g_{i,\beta} = \delta_{i\beta} + Tg_{i^+,\beta} + (1-T)g_{k^{++},\beta}$ $g_{k^+,\beta} = g_{k^{++},\beta}$ $g_{j,\beta} = \delta_{j\beta} + Tg_{j^+,\beta} + (1-T)g_{k^+,\beta}$ $g_{k,\beta} = \delta_{k\beta} + g_{k^+,\beta}$		$g'_{j^+,\beta} = Tg'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta}$ $g'_{k^+,\beta} = g'_{k^{++},\beta}$ $g'_{i^+,\beta} = Tg'_{i^{++},\beta} + (1-T)g'_{k^+,\beta}$ $g'_{k,\beta} = \delta_{k\beta} + g'_{k^+,\beta}$ $g'_{i,\beta} = \delta_{i\beta} + Tg'_{i^+,\beta} + (1-T)g'_{j^+,\beta}$ $g'_{j,\beta} = \delta_{j\beta} + g'_{j^+,\beta}$
--	----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------	--	----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------

A routine computation (eliminating  $g_{i^+,\beta}$ ,  $g_{j^+,\beta}$ , and  $g_{k^+,\beta}$ ) shows that the first system of 6 equations is equivalent to the following 3 equations:

$$g_{i,\beta} = \delta_{i\beta} + T^2g_{i^{++},\beta} + T(1-T)g_{j^{++},\beta} + (1-T)g_{k^{++},\beta},$$

$$g_{j,\beta} = \delta_{j\beta} + Tg_{j^{++},\beta} + (1-T)g_{k^{++},\beta}, \quad \text{and} \quad g_{k,\beta} = \delta_{k\beta} + g_{k^{++},\beta}.$$

Similarly eliminating  $g'_{i^+,\beta}$ ,  $g'_{j^+,\beta}$ , and  $g'_{k^+,\beta}$  from the second set of equations, we find that it is equivalent to

$$g'_{i,\beta} = \delta_{i\beta} + T^2g'_{i^{++},\beta} + T(1-T)g'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta},$$

$$g'_{j,\beta} = \delta_{j\beta} + Tg'_{j^{++},\beta} + (1-T)g'_{k^{++},\beta}, \quad \text{and} \quad g'_{k,\beta} = \delta_{k\beta} + g'_{k^{++},\beta}.$$

But these two sets of equations are the same, and as stated in  $g$ -rules lemma (Lemma 4), along with the  $g$ -rules corresponding to the other crossings in  $D$  (which are also the same between  $g$  and  $g'$ ), these equations determine  $g_{\alpha\beta}$  and  $g'_{\alpha\beta}$ , for  $\alpha, \beta \notin \{i^+, j^+, k^+\}$ . So with this exclusion on  $\alpha$  and  $\beta$ , we have that  $g_{\alpha\beta} = g'_{\alpha\beta}$ . But this means that the summations (5) in the definitions of  $\rho_1$  are equal for the two sides of R3, except perhaps for the three summands on each side that come from the crossings that touch  $\{i^+, j^+, k^+\}$ .

What remains is completely mechanical. We just need to compute the sum of those three summands for both sides of R3, and apply to it the  $g$ -rules of types (6) and (7) that eliminate the indices  $\{i^+, j^+, k^+\}$ . The computation is easy enough to be done by hand, yet why bother? Here’s the machine version (it takes less typing to apply all relevant  $g$ -rules and also eliminate the indices  $\{i, j, k\}$ ):

☹️ lhs = Simplify[

❤️  $R_1[1, j, k] + R_1[1, i, k^+] + R_1[1, i^+, j^+] // . \text{gRules}_{1,j,k} \cup \text{gRules}_{1,i,k^+} \cup \text{gRules}_{1,i^+,j^+}$

💻 
$$-\frac{1}{2T^2} \left( -2(-1+T)Tg_{j^{++},i^{++}}^2 + 2g_{j^{++},i^{++}} \right. \\ \left. (T^2 + T^2g_{i^{++},j^{++}} - 2T^2g_{j^{++},j^{++}} + g_{k^{++},i^{++}} - 2Tg_{k^{++},i^{++}} + T^2g_{k^{++},i^{++}} - Tg_{k^{++},j^{++}} + T^2g_{k^{++},j^{++}}) + \right. \\ \left. 2g_{i^{++},i^{++}} \left( -2T^2 + (-1+T)Tg_{j^{++},i^{++}} + T^2g_{j^{++},j^{++}} - g_{k^{++},i^{++}} + Tg_{k^{++},i^{++}} + T^2g_{k^{++},k^{++}} \right) + \right. \\ \left. T \left( 3T - 2(-1+T)g_{k^{++},i^{++}}^2 + 2Tg_{k^{++},j^{++}} + 2Tg_{j^{++},k^{++}}g_{k^{++},j^{++}} + 2g_{k^{++},j^{++}}^2 - \right. \right. \\ \left. \left. 2Tg_{k^{++},j^{++}}^2 + 2g_{j^{++},j^{++}} \left( (-1+T)g_{k^{++},i^{++}} + (-1+T)g_{k^{++},j^{++}} + T(-1+g_{k^{++},k^{++}}) \right) \right) - \right. \\ \left. \left. 4Tg_{k^{++},j^{++}}g_{k^{++},k^{++}} + 2g_{k^{++},i^{++}} \left( T + Tg_{i^{++},k^{++}} - 2(-1+T)g_{k^{++},j^{++}} - 2Tg_{k^{++},k^{++}} \right) \right) \right)$$

☹️ rhs = Simplify[

❤️  $R_1[1, i, j] + R_1[1, i^+, k] + R_1[1, j^+, k^+] // . \text{gRules}_{1,i,j} \cup \text{gRules}_{1,i^+,k} \cup \text{gRules}_{1,j^+,k^+}$

💻 
$$-\frac{1}{2T^2} \left( -2(-1+T)Tg_{j^{++},i^{++}}^2 + 2g_{j^{++},i^{++}} \right. \\ \left. (T^2 + T^2g_{i^{++},j^{++}} - 2T^2g_{j^{++},j^{++}} + g_{k^{++},i^{++}} - 2Tg_{k^{++},i^{++}} + T^2g_{k^{++},i^{++}} - Tg_{k^{++},j^{++}} + T^2g_{k^{++},j^{++}}) + \right. \\ \left. 2g_{i^{++},i^{++}} \left( -2T^2 + (-1+T)Tg_{j^{++},i^{++}} + T^2g_{j^{++},j^{++}} - g_{k^{++},i^{++}} + Tg_{k^{++},i^{++}} + T^2g_{k^{++},k^{++}} \right) + \right. \\ \left. T \left( 3T - 2(-1+T)g_{k^{++},i^{++}}^2 + 2Tg_{k^{++},j^{++}} + 2Tg_{j^{++},k^{++}}g_{k^{++},j^{++}} + 2g_{k^{++},j^{++}}^2 - \right. \right. \\ \left. \left. 2Tg_{k^{++},j^{++}}^2 + 2g_{j^{++},j^{++}} \left( (-1+T)g_{k^{++},i^{++}} + (-1+T)g_{k^{++},j^{++}} + T(-1+g_{k^{++},k^{++}}) \right) \right) - \right. \\ \left. \left. 4Tg_{k^{++},j^{++}}g_{k^{++},k^{++}} + 2g_{k^{++},i^{++}} \left( T + Tg_{i^{++},k^{++}} - 2(-1+T)g_{k^{++},j^{++}} - 2Tg_{k^{++},k^{++}} \right) \right) \right)$$

☹️ lhs == rhs

💻 True

□

**Proposition 9.** *The quantity  $\rho_1$  is invariant under R2c.*

*Proof.* We follow the exact same steps as in the case of R3. First, we write the  $g$ -rules, assuming  $\beta$  isn't in  $\{i, j, i^+, j^+\}$ :

$$\begin{array}{c}
 j^+ \\
 \parallel \\
 \mathcal{G}_{j^+} \\
 \parallel \\
 i^+
 \end{array}
 \left(
 \begin{array}{l}
 \uparrow i^{++} \\
 \backslash j \\
 i^+ \\
 \nearrow j^{++} \\
 i
 \end{array}
 \right)
 \begin{array}{l}
 g_{i^+,\beta} = Tg_{i^{++},\beta} + (1-T)g_{j^+,\beta} \\
 g_{j,\beta} = g_{j^+,\beta} \\
 g_{i,\beta} = T^{-1}g_{i^+,\beta} + (1-T^{-1})g_{j^{++},\beta} \\
 g_{j^+,\beta} = g_{j^{++},\beta}
 \end{array}
 \begin{array}{c}
 \uparrow \\
 \parallel \\
 \mathcal{G}_{j^+} \\
 \parallel \\
 i^{++}
 \end{array}
 \left(
 \begin{array}{l}
 j^{++} \\
 \circlearrowright \\
 i^{++}
 \end{array}
 \right)
 \emptyset$$

Note that for the right hand side we allowed ourselves to label the edges  $i^{++}$  and  $j^{++}$  as the computation is independent of the labelling and the labelling need not be by contiguous integers (outside of the move area, we assume that the left hand side and the right hand side are labelled in the same way). Note also that for the right hand side, there are no relevant  $g$ -rules. Now as in the case of R3, for the left hand side we eliminate  $g_{i^+,\beta}$  and  $g_{j^+,\beta}$  and we are left with the relations

$$g_{i,\beta} = g_{i^{++},\beta} \quad \text{and} \quad g_{j,\beta} = g_{j^{++},\beta}.$$

Otherwise the  $g$ -rules for the left and for the right are the same, and so their Green functions are the same except if the indices are in  $\{i, j, i^+, j^+\}$  (these indices do not even appear in the

right hand side). Thus the contribution to  $\rho_1$  from outside the area of the move is the same for both sides.

Next we write the contribution to  $\rho_1$  coming from the two crossings and one rotation that appear on the left, and use the  $g$ -rules to push all the indices in  $\{i, j, i^+, j^+\}$  up to  $i^{++}$  and  $j^{++}$ . This can be done by hand, but seeing that we have tools, we use them:

$$\begin{aligned} & \text{Simplify}[R_1[-1, i, j^+] + R_1[1, i^+, j] - (g_{j^+,j^+} - 1/2)] \\ \text{lhs} = & \text{Simplify}[R_1[-1, i, j^+] + R_1[1, i^+, j] - (g_{j^+,j^+} - 1/2) // . gRules_{-1,i,j^+} \cup gRules_{1,i^+,j}] \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} - (-1 + g_{j,j^+}) g_{i^+,i^+} + g_{j,i^+} (g_{j,j^+} - g_{i^+,j} + g_{j^+,j}) + \\ & g_{i,i} (-1 + g_{j^+,j^{++}}) - g_{j^+,i} (-g_{i,j^+} + g_{j^{++},j^+} + g_{j^+,j^{++}}) - g_{j^+,j^+} \end{aligned}$$

$$\frac{1}{2} - g_{j^{++},j^{++}}$$

This result is clearly equal to the single rotation contribution to  $\rho_1$  that comes from the right hand side.  $\square$

**Proposition 10.** *The quantity  $\rho_1$  is invariant under R1l.*

*Proof.* We start with the relevant  $g$ -rules:

$$\begin{array}{ccc} \begin{array}{c} i^{++} \\ \nearrow \\ i^+ \text{---} \text{C} \text{---} \\ \searrow \\ i \end{array} & \begin{array}{l} g_{i^+,\beta} = \delta_{i^+,\beta} + T g_{i^{++},\beta} + (1-T) g_{i,\beta} \\ g_{i,\beta} = \delta_{i,\beta} + g_{i^+,\beta} \end{array} & \begin{array}{c} \uparrow \\ \emptyset \\ i^{++} \end{array} \end{array}$$

The first of these rules is equivalent to  $g_{i^+,\beta} = T^{-1}\delta_{i^+,\beta} + g_{i^{++},\beta}$ . For  $\beta \neq i, i^+$  we find as before that  $g_{i,\beta} = g_{i^+,\beta}$  and we can ignore the contributions to  $\rho_1$  coming from outside the area of the move. The contribution to  $\rho_1$  coming from the single crossing and single rotation on the left hand side is computed below, and is equal to the empty contribution coming from the right hand side:

$$\begin{aligned} \text{lhs1} = & R_1[1, i^+, i] - (g_{i^+,i^+} - 1/2) \\ \text{lhs2} = & \text{lhs1} // . \{g_{i^+,\beta_-} \Rightarrow T^{-1} \delta_{i^+,\beta} + g_{i^{++},\beta}, g_{i,\beta_-} \Rightarrow \delta_{i,\beta} + g_{i^+,\beta}\} \\ & \text{Simplify}[\text{lhs2}] \end{aligned}$$



$$g_{i^+,i^+}^2 - g_{i^+,i^+} - (-1 + g_{i,i^+}) g_{i^+,i^+}$$

$$-\frac{1}{T} - g_{i^{++},i^+} - \left(-1 + \frac{1}{T} + g_{i^{++},i^+}\right) \left(\frac{1}{T} + g_{i^{++},i^+}\right) + \left(\frac{1}{T} + g_{i^{++},i^+}\right)^2$$

$$0$$


$\square$

*Second Proof of Theorem 1, "Invariance".* After the Upright Reidemeister Theorem (Theorem 3) which sets out what we need to do, and Propositions 8, 9, and 10 which prove invariance under R3, R2c, and R1l, it remains to show the invariance of  $\rho_1$  under R1r, R2b, and Sw<sup>+</sup>. This is done exactly as in the examples already shown, so in each case we show only the punch line:


 Simplify  $[R_1[1, i, i^+] + (g_{i^+, i^+} - 1/2)] // . \{ \quad (* R1r *)$   
  $g_{i, \beta} \mapsto \delta_{i, \beta} + T g_{i^+, \beta} + (1 - T) g_{i^{++}, \beta}, \quad g_{i^+, \beta} \mapsto \delta_{i^+, \beta} + g_{i^{++}, \beta},$   
 $g_{\alpha, i} \mapsto T^{-1} (g_{\alpha, i^+} - \delta_{\alpha, i^+}), \quad g_{\alpha, i^+} \mapsto T g_{\alpha, i^{++}} - (1 - T) \delta_{\alpha, i^+} - T \delta_{\alpha, i^{++}} \}$



(Note that the version of the  $g$ -rules we used above easily follows from (7)).

 Simplify  $[R_1[1, i, j] + R_1[-1, i^+, j^+]] // . \text{gRules}_{1, i, j} \cup \text{gRules}_{-1, i^+, j^+} \quad (* R2b *)$



  $(g_{i, i} - 1/2) + (g_{j, j} - 1/2) - (g_{i^+, i^+} - 1/2) - (g_{j^+, j^+} - 1/2) // . \text{gRules}_{1, i, j} \quad (* Sw^+ *)$



□

#### 4. SOME CONTEXT AND SOME MORALS

We would like to emphasize again that  $\rho_1$  seems very close to the Alexander polynomial, yet we have no topological interpretation for it. Until that changes, where is  $\rho_1$  coming from?

It comes via a lengthy path, which we will only sketch here. For a while now [BV1, BV3, BN2, BN3, BN4, BV2, BN5, BN6, BN8] we've been studying quantum invariants related to the Lie algebra  $sl_{2+}^\epsilon$ , the 4-dimensional Lie algebra with generators  $y, b, a, x$  and brackets

$$[b, x] = \epsilon x, \quad [b, y] = -\epsilon y, \quad [b, a] = 0, \quad [a, x] = x, \quad [a, y] = -y, \quad [x, y] = b + \epsilon a,$$

where  $\epsilon$  is a scalar. The beauty of this algebra stems from the following:

- It is a “classical double” of a two dimensional the Lie bialgebra  $\langle a, x \rangle$ , with

$$[a, x] = x, \quad \delta(a) = 0, \quad \delta(x) = \epsilon x \wedge a,$$

and hence quantization tools are available and are used below (e.g. [ES]).

- At invertible  $\epsilon$  it is isomorphic to  $sl_2 \oplus \langle t \rangle$ , where  $t = b - \epsilon a$  is a central element. Quantum topology tells us that the algebra  $sl_2$  is related to the Jones polynomial. In fact, the universal quantum invariant (see [La1, La2, Oh]) for the Lie algebra  $sl_2$  is equivalent to the coloured Jones polynomial of [Jo].
- At  $\epsilon = 0$  it becomes the diamond Lie algebra  $\diamond$ , a solvable algebra in which computations are easier. The algebra  $\diamond$  is the semi-direct product of the unique non-commutative 2D Lie algebra  $\mathfrak{a}$  with its dual, and quantum topology tells us that it is related to the Alexander polynomial [BN1, BD].

The last two facts taken together tell us that the Alexander polynomial is some limit of the coloured Jones polynomial (originally conjectured [MM, Ro1] and proven by other means [BNG]).

We can make this a bit more explicit. By using the Drinfel'd quantum double construction [Dr] we find that the universal enveloping algebra  $\mathcal{U}(sl_{2+}^\epsilon)$  has a quantization  $QU$ , which has an  $R$ -matrix solving the Yang-Baxter equation (meaning, satisfying the R3 move, in the



appropriate sense). These are given by:

$$QU = A\langle y, b, a, x \rangle \Big/ \left( \begin{array}{l} [b, a] = 0, \quad [b, x] = \epsilon x, \quad [b, y] = -\epsilon y, \\ [a, x] = x, \quad [a, y] = -y, \quad xy - qyx = \frac{1 - e^{-\hbar(b+\epsilon a)}}{\hbar} \end{array} \right),$$

where  $A\langle gens \rangle$  is the free associative algebra with generators  $gens$ , and  $q = e^{\hbar\epsilon}$ , and

$$R = \sum_{m, n \geq 0} \frac{y^n b^m \otimes (\hbar a)^m (\hbar x)^n}{m! [n]_q!} \quad \left( \text{where } [n]_q! = \prod_{k=1}^n \frac{1 - q^k}{1 - q} \text{ is a "quantum factorial"} \right).$$

Thus there is an associated universal quantum invariant of knots  $Z_\epsilon(K) \in QU$  (which, as stated, is equivalent to the coloured Jones polynomial). In our talks and papers we show that  $Z_\epsilon$  can be expanded as a power series in  $\epsilon$ , that at  $\epsilon = 0$  it is equivalent to the Alexander polynomial, and that in general, the coefficient  $Z_{(k)}$  of  $\epsilon^k$  in  $Z_\epsilon$  can be computed in polynomial time and is homomorphic, meaning that it leads to an ‘‘algebraic knot theory’’ in the sense of (say) [BN7]. We also know that the excess information in  $Z_{(k)}$  (beyond the information in  $\{Z_{(0)}, \dots, Z_{(k-1)}\}$ ) is contained in a single polynomial,  $\rho_k$ . The first of these polynomials is  $\rho_1$  of this paper.

But how did we arrive at the specific formulas of this paper? As often seen with quantizations,  $QU$  is isomorphic (though only as an algebra, not as a Hopf algebra) with  $\mathcal{U}(sl_{2+}^\epsilon)$ , and the latter can be represented into the Heisenberg algebra  $\mathbb{H} = A\langle p, x \rangle / ([p, x] = 1)$  via

$$y \rightarrow -tp - \epsilon \cdot xp^2, \quad b \rightarrow t + \epsilon \cdot xp, \quad a \rightarrow xp, \quad x \rightarrow x,$$

(abstractly,  $sl_{2+}^\epsilon$  acts on its Verma module  $\mathcal{U}(sl_{2+}^\epsilon) / (\mathcal{U}(sl_{2+}^\epsilon)\langle y, a, b - \epsilon a - t \rangle) \cong \mathbb{Q}[x]$  by differential operators, namely via  $\mathbb{H}$ ). So  $QU$ 's  $R$ -matrix can be expanded in powers of  $\epsilon$  and pushed to  $\mathcal{U}(sl_{2+}^\epsilon)$  and on to  $\mathbb{H}$ , resulting in  $\mathcal{R} = \mathcal{R}_0(1 + \epsilon\mathcal{R}_1 + \dots)$ , with  $\mathcal{R}_0 = e^{t(xp \otimes 1 - x \otimes p)}$  and  $\mathcal{R}_1$  a quartic polynomial in  $p$  and  $x$ . Now all the computations for  $\rho_1$  can be carried out by pushing around a rather small number of  $p$ 's and  $x$ 's (at most 4), and this can be done using the rules

$$\begin{aligned} (p \otimes 1)\mathcal{R}_0 &= \mathcal{R}_0(e^t(p \otimes 1) + (1 - e^t)(1 \otimes p)), \\ (1 \otimes p)\mathcal{R}_0 &= \mathcal{R}_0(1 \otimes p), \end{aligned}$$

which, after setting  $T = e^t$ , must remind the reader of Equation (1). When all the dust settles, the resulting formulas are similar to the ones in Equations (4) and (5) (but only similar, because we applied some ad hoc cosmetics to make the formulas appear nicer).

There are some morals to this story:

- (1) The definition of  $\rho_1$  in Section 1 and the proofs of its invariance in Section 3 are clearly much simpler than the origin story, as outlined above. So quite clearly, we still don't understand  $\rho_1$ . There ought to be a room for it directly within topology, that does not require that one would know anything about quantum algebra. (And better if that room is large enough to accommodate morals (2) and (6) below).
- (2) Like there is  $\rho_1$ , there are  $\rho_k$ . The origin story tells us that  $\rho_k$  should have a formula as a summation over choices of  $k$ -tuples of features of the knot (crossings and rotations), just as the formula for  $\rho_1$  is a single summation over these features. The summand for  $\rho_k$  will be a degree  $2k$  polynomial in the Green function  $g_{\alpha\beta}$  (compare with (4), which is quadratic). As a  $k$ -fold summation, after inverting  $A$ ,  $\rho_k$  should be computable in  $O(n^k)$  additions and multiplications of polynomials in  $T$ , where  $n$  is the crossing number.

- (3) These  $\rho_k$  should be equivalent to the invariants in our earlier works [BV1, BV3, BN2, BN3, BN4, BV2, BN5, BN6, BN8].
- (4) These  $\rho_k$  should be equivalent to the invariants studied earlier by Rozansky and Overbay [Ro1, Ro2, Ro3, Ov], as their quantum origin is essentially the same (though strictly speaking, we have not written proofs of that, and normalizations may differ). Our formulas are significantly simpler and faster to compute than the Rozansky-Overbay formulas, and in our language it is easier to see the behaviour of  $\rho_1$  under mirror reflection (Section 2.3 here, unproven in Rozansky-Overbay).
- (5) Like the Rozansky-Overbay invariants,  $\rho_k$  should be equivalent to the “higher diagonals” for the Melvin-Morton expansion (e.g. [Ro3]) and should be dominated by the “loop expansion” of the Kontsevich integral [Kr, GR].
- (6) The quantum algebra story extends to other Lie algebras, beyond  $sl_2$ . So there should be variants  $\rho_k^{\mathfrak{g}}$  of  $\rho_k$  at least for every semisimple Lie algebra  $\mathfrak{g}$ , given by more or less similar formulas. Quantum algebra suggests that  $\rho_k^{\mathfrak{g}}$  should be a polynomial in as many variables as the rank of  $\mathfrak{g}$ , and should in general be stronger than the “base”  $\rho_k$ . We have not seriously explored  $\rho_k^{\mathfrak{g}}$  yet, though some preliminary work was done by Schaveling [Sch].
- (7) It appears that  $QU$  has interesting traces and therefore that there should be a link version of  $\rho_1$ . We have not pursued this formally.
- (8)  $QU$  has a co-product and an antipode, and so the universal tangle invariant associated with  $QU$  has formulas for strand reversal and strand doubling (e.g. [BV3, BN6]). This implies (e.g., by following the ideas of [BN7]) that there should be formulas for  $\rho_1$  that start with a Seifert surface for the knot. We are pursuing such formulas now; we already know that the degree of  $\rho_1$  is bounded by  $2g$ , where  $g$  is the genus of a knot [BV3].
- (9) For the same reasons, for ribbon knots  $\rho_1$  should have a formula computable from a ribbon presentation, and its values might be restricted in a manner similar to the Fox-Milnor condition [FM]. We are pursuing this now.
- (10) The coloured Jones polynomial is invariant under mutation so we expect  $\rho_1$  to likewise be invariant under mutation (and indeed, also  $\rho_k$ ), yet we do not have a direct proof of that yet. Note that we can expect  $\rho_k^{\mathfrak{g}}$  for higher-rank  $\mathfrak{g}$  to no longer be invariant under mutation.

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