

```
CF[expr_] := (expr /. e^x->e^Simplify[x /. b_i->2Log[t_i]]) // Together // ExpandNumerator // ExpandDenominator
E /: CF[E[ω_, Q_]] := E[CF[ω], CF[Q]];
E /: E[ω1_, Q1_] E[ω2_, Q2_] := CF@E[ω1 ω2, Q1+Q2];
E[ω1_, Q1_] ≡ E[ω2_, Q2_] := CF[ω1==ω2∧Q1==Q2];
```

```
Nu_{i,c_j->k_}[E[ω_, Q_]] := CF[
E[ω, e^{-Yβ} u_{k+Y} c_k + (Q/.c_j | u_i->0)] /. {Y->∂_{c_j}Q, β->∂_{u_i}Q}];
Nw_{i,c_j->k_}[E[ω_, Q_]] := CF[
E[ω, e^{Yα} w_{k+Y} c_k + (Q/.c_j | w_i->0)] /. {Y->∂_{c_j}Q, α->∂_{w_i}Q}];
Nw_{i,u_j->k_}[E[ω_, Q_]] := CF[
E[ν ω, -b_r ν α β + ν β u_{k+ν} δ u_k w_{k+ν} α w_k + (Q/.w_i | u_j->0)] /. ν->(1+b_k δ)^{-1} /. {α->∂_{w_i}Q/.u_j->0, β->∂_{u_j}Q}];
```

```
m_{i,j->k_}[E[ω_, Q_]] := CF[Module[{x},
(E[ω, Q] // Nu_{i,c_j->x} // Nu_{i,c_x->x} // Nw_{i,u_j->x}) /. {c_i->c_k, w_j->w_k, y_x->y_k, b_i|j->b_k, t_i|j->t_k}]]]
```

```
R_{i,j_}[p_] := E[1, p b_i c_j + b_i^{-1} (e^{p b_i} - 1) u_i w_j] // CF;
R_{i,j_}^+ := R_{i,j}[1]; R_{i,j_}^- := R_{i,j}[-1];
ur_{i_} := E[t_i^{1/2}, 0];
ul_{i_} := E[t_i^{-1/2}, 0];
nr_{i_} := E[t_i^{-1/2}, 0];
nl_{i_} := E[t_i^{1/2}, 0];
```

```
E /: (e_E)^{σ[s___List]} := CF[e /. Flatten@Table[{
b_k->Total[b_#&/@{s}][[k]], c_k->Total[c_#&/@{s}][[k]], u_k->Total[u_#&/@{s}][[k]], w_k->Total[w_#&/@{s}][[k]], t_k->Total[t_#&/@{s}][[k]],
}, {k, 1, Length@{s}}]];
σ[l___, k_Integer, r___] := σ[l, {k}, r];
```

```
E /: e1_E**e2_E := Module[{λ, ν, e},
λ = (Union@Cases[e1, (b|t|c|u|w)_k_>k, ∞]) ∩ (Union@Cases[e2, (b|t|c|u|w)_k_>k, ∞]);
ν = Table[Unique[], {Length@λ}];
e = e1 (e2 /. Flatten@Table[{b_λ[[k]]->b_ν[[k]], t_λ[[k]]->t_ν[[k]], c_λ[[k]]->c_ν[[k]], u_λ[[k]]->u_ν[[k]], w_λ[[k]]->w_ν[[k]]}, {k, Length@λ}]);
Do[e = m_λ[[k]], ν[[k]]->λ[[k]] [e], {k, Length@λ}];
e
]
```

```
A[n_][e_E] := (e /. {c_n->-c_n, w_n->-w_n}) // Nu_{i,c_n->n} // Nw_{i,c_n->n} // Nw_{i,u_n->n};
A[] [e_E] := e;
A[n_, r___][e_E] := e // A[n] // A[r];
```

```
θ[i_][e_E] := (e /. {b_i->-b_i, c_i->-c_i, u_i->w_i, w_i->u_i, t_i->t_i^{-1}}) // CF // Nu_{i,u_i->i};
θ[] [e_E] := e;
θ[n_, r___][e_E] := e // θ[n] // θ[r];
```

```
S[n_][e_E] := (e /. {c_n->-c_n, w_n->-w_n, b_n->-b_n, u_n->-u_n, t_n->t_n^{-1}}) // Nu_{i,c_n->n} // Nw_{i,c_n->n} // Nw_{i,u_n->n};
S[] [e_E] := e;
S[n_, r___][e_E] := e // S[n] // S[r];
```

$$uR_{i,j} := (R_{1,2}[1/2] // m_{1,2 \rightarrow 1})^{\sigma[\{i,j\}]} ** (R_{1,2}[-1/2] // m_{1,2 \rightarrow 1})^{\sigma[\{i\}]} ** (R_{1,2}[-1/2] // m_{1,2 \rightarrow 1})^{\sigma[\{j\}]}$$

$$f_{11} = f_{21} = f_{22} = 0; f_{12} = 1/2;$$

$$V = \mathbb{E}[h[b_1, b_2, t_1, t_2], b_1 c_1 f_{11} + b_1 c_2 f_{12} + b_2 c_1 f_{21} + b_2 c_2 f_{22} + u_1 w_1 g_{11}[b_1, b_2, t_1, t_2] + u_1 w_2 g_{12}[b_1, b_2, t_1, t_2] + u_2 w_1 g_{21}[b_1, b_2, t_1, t_2] + u_2 w_2 g_{22}[b_1, b_2, t_1, t_2]];$$

$$\begin{aligned} \text{sol} = & \{g_{12}[b1_, b2_, t1_, t2_] \Rightarrow \\ & (b2 - b1 t1^2 - b2 t1^2 + b1 t1^2 t2^2 + b1^2 g_{11}[b1, b2, t1, t2] - b1^2 t1^2 t2^2 g_{11}[b1, b2, t1, t2]) / \\ & (b1 b2 (-1 + t1^2 t2^2)), g_{21}[b1_, b2_, t1_, t2_] \Rightarrow \\ & (-b2 + b1 t1 + b2 t1 - b1 t1 t2^2 - b2 t1 t2^2 + b2 t1^2 t2^2 + b2^2 g_{22}[b1, b2, t1, t2] - \\ & b2^2 t1^2 t2^2 g_{22}[b1, b2, t1, t2]) / (b1 b2 (-1 + t1^2 t2^2)), \\ g_{22}[b1_, b2_, t1_, t2_] \Rightarrow & \frac{1}{b2^2 (b1 + b2) (-1 + t1^2) (-1 + t1^2 t2^2)} \\ & (b2 (b1 + b2) - b2 (b1 + b2) t1 - b2 (b1 + b2) t1^2 + b1 (b1 + b2) t1^3 + b2 (b1 + b2) t1^3 + \\ & b2 (b1 + b2) t1 t2^2 - b2 (b1 + b2) t1^2 t2^2 - 2 b1 (b1 + b2) t1^3 t2^2 - b2 (b1 + b2) t1^3 t2^2 + \\ & b2 (b1 + b2) t1^4 t2^2 + b1 (b1 + b2) t1^3 t2^4 - b1 b2 (-1 + t1^2 t2^2)^2 h[b1, b2, t1, t2]^2 - \\ & b1^2 (b1 + b2) t1 (-1 + t2^2) (-1 + t1^2 t2^2) g_{11}[b1, b2, t1, t2]), \\ g_{11}[b1_, b2_, t1_, t2_] \Rightarrow & (t1 ((b1 + b2) t1 (-1 + t2^2) + (b2 - b2 t1^2 t2^2) h[b2, b1, t2, t1]^2)) / \\ & (b1 (b1 + b2) (-1 + t1^2 t2^2)), h[b1_, b2_, t1_, t2_] \Rightarrow \frac{\left(\frac{-1+t1^2}{b1}\right)^{1/4} \left(\frac{-1+t2^2}{b2}\right)^{1/4}}{\left(\frac{-1+t1^2 t2^2}{b1+b2}\right)^{1/4}} \}; \end{aligned}$$

CF[V //. sol]

$$E \left[\frac{\left(\frac{-1+t_1^2}{b_1} \right)^{1/4} \left(\frac{-1+t_2^2}{b_2} \right)^{1/4}}{\left(\frac{-1+t_1^2 t_2^2}{b_1+b_2} \right)^{1/4}}, \right.$$

$$\left(-b_1^3 b_2 c_2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} - b_1^2 b_2^2 c_2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} + b_1^3 b_2 c_2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} + b_1^2 b_2^2 c_2 t_1^2 t_2^2 \right.$$

$$\left. \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} + 2 b_2^2 t_1 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} u_1 w_1 - 2 b_2^2 t_1^3 \sqrt{\frac{-1+t_1^2}{b_1}} t_2^2 \sqrt{\frac{-1+t_2^2}{b_2}} u_1 w_1 - \right.$$

$$2 b_1 b_2 t_1^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_1 - 2 b_2^2 t_1^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_1 + 2 b_1 b_2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_1 +$$

$$2 b_2^2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_1 - 2 b_1 b_2 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} u_2 w_1 +$$

$$2 b_1 b_2 t_1^2 \sqrt{\frac{-1+t_1^2}{b_1}} t_2^2 \sqrt{\frac{-1+t_2^2}{b_2}} u_2 w_1 + 2 b_1^2 t_1 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_1 +$$

$$2 b_1 b_2 t_1 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_1 - 2 b_1^2 t_1 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_1 - 2 b_1 b_2 t_1 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_1 -$$

$$2 b_1 b_2 t_1 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} u_1 w_2 + 2 b_1 b_2 t_1^3 \sqrt{\frac{-1+t_1^2}{b_1}} t_2^2 \sqrt{\frac{-1+t_2^2}{b_2}} u_1 w_2 +$$

$$2 b_1 b_2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_2 + 2 b_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_2 - 2 b_1 b_2 t_1^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_2 -$$

$$2 b_2^2 t_1^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_1 w_2 + 2 b_1^2 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} u_2 w_2 - 2 b_1^2 t_1^2 \sqrt{\frac{-1+t_1^2}{b_1}} t_2^2 \sqrt{\frac{-1+t_2^2}{b_2}} u_2 w_2 -$$

$$2 b_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 - 2 b_1 b_2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 + 2 b_1^2 t_1 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 +$$

$$2 b_1 b_2 t_1 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 - 2 b_2^2 t_1 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 - 2 b_1 b_2 t_1 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 +$$

$$2 b_1^2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 + 2 b_1 b_2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} u_2 w_2 \Big) /$$

$$\left(-2 b_1^2 b_2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} - 2 b_1 b_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} + 2 b_1^2 b_2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} + \right.$$

$$\left. 2 b_1 b_2^2 t_1^2 t_2^2 \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} \right)]$$

Table[FullSimplify[∂_{u_i} ∂_{w_j} (V[[2]] //. sol)], {i, 2}, {j, 2}] // MatrixForm

$$\begin{pmatrix} \frac{t_1 \left(t_1 (-1+t_2^2) - b_2 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}} \right)}{b_1 (-1+t_1^2 t_2^2)} & \frac{1-t_1^2+b_1 t_1 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}}}{b_1 (-1+t_1^2 t_2^2)} \\ \frac{-t_1 (-1+t_2^2) + b_2 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}}}{b_2 (-1+t_1^2 t_2^2)} & \frac{-1+t_1 (1+(-1+t_1) t_2^2) - b_1 \sqrt{\frac{-1+t_1^2}{b_1}} \sqrt{\frac{-1+t_2^2}{b_2}} \sqrt{\frac{-1+t_1^2 t_2^2}{b_1+b_2}}}{b_2 (-1+t_1^2 t_2^2)} \end{pmatrix}$$

