Pensieve header: Testing the commutativity of two independent Nwu's.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Vo"];
```

NOE-0

0R

```
R⁺_{0,i_,j_} := 𝔼[b_i c_j + b_i^{-1} (e^{b_i} - 1) u_i w_j]; R⁻_{0,i_,j_} := 𝔼[-b_i c_j + b_i^{-1} (e^{-b_i} - 1) u_i w_j];
```

0Util

```
CF[ω_ . 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_ . 𝔼[Q1_] ≡ ω2_ . 𝔼[Q2_] := Simplify[ω1 == ω2 ⋀ Q1 == Q2];
```

0NO

```
N_{u_i_ c_j_→k_}[ω_ . 𝔼[Q_]] := CF[
    ω 𝔼[e^{-γ} β u_k + γ c_k + (Q /. c_j | u_i → 0)]] /. {γ → ∂_{c_j} Q, β → ∂_{u_i} Q}];
N_{w_i_ c_j_→k_}[ω_ . 𝔼[Q_]] := CF[
    ω 𝔼[e^{γ} α w_k + γ c_k + (Q /. c_j | w_i → 0)]] /. {γ → ∂_{c_j} Q, α → ∂_{w_i} Q}];
N_{w_i_ u_j_→k_}[ω_ . 𝔼[Q_]] := CF[
    ν ω 𝔼[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (Q /. w_i | u_j → 0)]] /. ν → (1 + b_k δ)^{-1} /.
     {α → ∂_{w_i} Q /. u_j → 0, β → ∂_{u_j} Q /. w_i → 0, δ → ∂_{w_i,u_j} Q}];
```

0m

```
m_{i_,j_→k_}[ω_ . 𝔼[Q_]] := CF[Module[{x},
    (ω 𝔼[Q] /. b_{i|j} → b_k // N_{w_i c_j→x} // N_{u_i c_x→x} // N_{w_x u_j→x}) /. {c_i → c_k, w_j → w_k, y_x :→ y_k}]]
```

0Q0

```
Q0 = 𝔼[Sum[f_{i,j} u_i w_j, {i, 3}, {j, 3}]]
```

0Q0

$$\mathbb{E}[u_1 w_1 f_{1,1} + u_1 w_2 f_{1,2} + u_1 w_3 f_{1,3} + u_2 w_1 f_{2,1} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + u_3 w_1 f_{3,1} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}]$$

```
Q0 // N_{w_1 u_1→1}
```

$$\frac{1}{1+b_1 f_{1,1}}\mathbb{E}\Big[\frac{u_1 w_1 f_{1,1}}{1+b_1 f_{1,1}} + \frac{u_1 (w_2 f_{1,2} + w_3 f_{1,3})}{1+b_1 f_{1,1}} + u_2 w_2 f_{2,2} + u_2 w_3 f_{2,3} + \frac{w_1 (u_2 f_{2,1} + u_3 f_{3,1})}{1+b_1 f_{1,1}} - \frac{b_1 (w_2 f_{1,2} + w_3 f_{1,3})(u_2 f_{2,1} + u_3 f_{3,1})}{1+b_1 f_{1,1}} + u_3 w_2 f_{3,2} + u_3 w_3 f_{3,3}\Big]$$

```
t1 = Q0 // N_{w_1 u_1→1} // N_{w_2 u_2→2}
```

$$\mathbb{E}\Big[\big(u_1 (w_2 f_{1,2} + w_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1}(1 + b_2 f_{2,2})) + w_3 (f_{1,3}(1 + b_2 f_{2,2}) - b_2 f_{1,2} f_{2,3})) + u_2 (w_1 f_{2,1} + w_2 f_{2,2} + w_3 f_{2,3} + b_1 (w_2 (-f_{1,2} f_{2,1} + f_{1,1} f_{2,2}) + w_3 (-f_{1,3} f_{2,1} + f_{1,1} f_{2,3}))) + u_3 (w_2 f_{3,2} - b_2 w_3 f_{2,3} f_{3,2} + w_1 ((1 + b_2 f_{2,2}) f_{3,1} - b_2 f_{2,1} f_{3,2}) + w_3 f_{3,3} + b_2 w_3 f_{2,2} f_{3,3} + b_1 (w_2 (-f_{1,2} f_{3,1} + f_{1,1} f_{3,2}) + w_3 (-f_{1,3}((1 + b_2 f_{2,2}) f_{3,1} - b_2 f_{2,1} f_{3,2}) + f_{1,1} f_{3,3} + b_2 (f_{1,2}(f_{2,3} f_{3,1} - f_{2,1} f_{3,3}) + f_{1,1}(-f_{2,3} f_{3,2} + f_{2,2} f_{3,3}))))))\big) / (1 + b_2 f_{2,2} + b_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1}(1 + b_2 f_{2,2})))\Big] / (1 + b_2 f_{2,2} + b_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1}(1 + b_2 f_{2,2})))$$

```
t2 = Q0 // N_{w_2 u_2→2} // N_{w_1 u_1→1}
```

$$\mathbb{E}\Big[\big(u_1 (w_2 f_{1,2} + w_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1} (1 + b_2 f_{2,2})) + w_3 (f_{1,3}(1 + b_2 f_{2,2}) - b_2 f_{1,2} f_{2,3})) + u_2 (w_1 f_{2,1} + w_2 f_{2,2} + w_3 f_{2,3} + b_1 (w_2 (-f_{1,2} f_{2,1} + f_{1,1} f_{2,2}) + w_3 (-f_{1,3} f_{2,1} + f_{1,1} f_{2,3}))) + u_3 (w_2 f_{3,2} - b_2 w_3 f_{2,3} f_{3,2} + w_1 ((1 + b_2 f_{2,2}) f_{3,1} - b_2 f_{2,1} f_{3,2}) + w_3 f_{3,3} + b_2 w_3 f_{2,2} f_{3,3} + b_1 (w_2 (-f_{1,2} f_{3,1} + f_{1,1} f_{3,2}) + w_3 (-f_{1,3}((1 + b_2 f_{2,2}) f_{3,1} - b_2 f_{2,1} f_{3,2}) + f_{1,1} f_{3,3} + b_2 (f_{1,2}(f_{2,3} f_{3,1} - f_{2,1} f_{3,3}) + f_{1,1}(-f_{2,3} f_{3,2} + f_{2,2} f_{3,3}))))))\big) / (1 + b_2 f_{2,2} + b_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1}(1 + b_2 f_{2,2})))\Big] / (1 + b_2 f_{2,2} + b_1 (-b_2 f_{1,2} f_{2,1} + f_{1,1}(1 + b_2 f_{2,2})))$$

```
t1 ≡ t2
```

```
True
```