```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, h_, Collect[#, t_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (h | t)_a_ ⧴ a, ∞];
   M = Outer[Factor[∂_{h_#1 t_#2} λ] &, S, S];
   M = Prepend[M, t_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[h_# & /@ S, ω]];
   M // MatrixForm];
```

```
Γ /: Γ[ω1_, λ1_] Γ[ω2_, λ2_] := Γ[ω1 * ω2, λ1 + λ2];
m_{a_ b_→c_}[Γ[ω_, λ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
```

$$\begin{pmatrix} \alpha & \beta & \theta \\ \gamma & \delta & \epsilon \\ \phi & \psi & \Xi \end{pmatrix} = \begin{pmatrix} \partial_{t_a,h_a}\lambda & \partial_{t_a,h_b}\lambda & \partial_{t_a}\lambda \\ \partial_{t_b,h_a}\lambda & \partial_{t_b,h_b}\lambda & \partial_{t_b}\lambda \\ \partial_{h_a}\lambda & \partial_{h_b}\lambda & \lambda \end{pmatrix} \; /. \; (t \mid h)_{a|b} \to 0;$$

$$\Gamma\left[(\mu = 1 - \beta)\,\omega,\ \{t_c, 1\}.\begin{pmatrix} \gamma + \alpha\,\delta/\mu & \epsilon + \delta\,\theta/\mu \\ \phi + \alpha\,\psi/\mu & \Xi + \psi\,\theta/\mu \end{pmatrix}.\{h_c, 1\}\right.$$

$$\left. /.\ \{T_a \to T_c,\ T_b \to T_c\} \ //\ \Gamma\text{Collect}\right];$$

$$Rp_{a\_b\_} := \Gamma\left[1,\ \{t_a, t_b\}.\begin{pmatrix} T_a & 0 \\ 1 - T_a & 1 \end{pmatrix}.\{h_a, h_b\}\right];$$

$$Rm_{a\_b\_} := Rp_{ab}\ /.\ T_a \to 1\big/T_a;$$

$$\xi = \Gamma\left[\omega,\ \{t_1, t_2, t_3, t_S\}.\begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}.\{h_1, h_2, h_3, h_S\}\right]$$

```
(ξ // m_{12→1} // m_{13→1}) == (ξ // m_{23→2} // m_{12→1})
```

$$\begin{pmatrix} \omega & h_1 & h_2 & h_3 & h_S \\ t_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ t_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ t_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ t_S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{pmatrix}$$

True

```
{Rm_{51} Rm_{62} Rp_{34} // m_{14→1} // m_{25→2} // m_{36→3},
  Rp_{61} Rm_{24} Rm_{35} // m_{14→1} // m_{25→2} // m_{36→3}}
```

$$\left\{\begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & 1 & \frac{-1+T_2}{T_2} & -\frac{-1+T_3}{T_2} \\ t_2 & 0 & \frac{1}{T_2} & \frac{-1+T_3}{T_2} \\ t_3 & 0 & 0 & 1 \end{pmatrix},\ \begin{pmatrix} 1 & h_1 & h_2 & h_3 \\ t_1 & 1 & \frac{-1+T_2}{T_2} & -\frac{-1+T_3}{T_3} \\ t_2 & 0 & \frac{1}{T_2} & \frac{-1+T_3}{T_3} \\ t_3 & 0 & 0 & 1 \end{pmatrix}\right\}$$

```
z = Rm_{12,1} Rm_{27} Rm_{83} Rm_{4,11} Rp_{16,5} Rp_{6,13} Rp_{14,9} Rp_{10,15};
Do[z = z // m_{1k→1}, {k, 2, 16}];
z
```

$$\begin{pmatrix} 11 - \frac{1}{T_1^3} + \frac{4}{T_1^2} - \frac{8}{T_1} - 8\,T_1 + 4\,T_1^2 - T_1^3 & h_1 \\ t_1 & 1 \end{pmatrix}$$

```
trc_[Γ[ω_, λ_]] := Module[{α, θ, ψ, Ξ},
  (α θ; ψ Ξ) = (∂_{t_c,h_c} λ  ∂_{t_c} λ; ∂_{h_c} λ  λ) /. (t | h)_c → 0;
  Γ[ω (1 - α), Ξ + ψ * θ / (1 - α)] // ΓCollect];
(ξ // m12→1 // tr1) == (ξ // m21→1 // tr1)
```

True

```
(Rp12 Rp45 Rp63 Rp78 Rm109 // m24→2 // m26→2 // m28→2 // m29→2 // m102→2 // m37→3 // m51→1) /.
 {T1 → T, T2 → T, T3 → T}
```

$$\begin{pmatrix} T-(-1+T)\,T & h_1 & h_2 & h_3 \\ t_1 & -\frac{1}{-2+T} & \frac{-1+T}{(-2+T)\,T} & 0 \\ t_2 & \frac{(-1+T)\,(1-T+T^2)}{-2+T} & -\frac{-1+2\,T}{(-2+T)\,T} & 1-T \\ t_3 & -\frac{(-1+T)^2\,T}{-2+T} & \frac{-1+T}{-2+T} & T \end{pmatrix}$$

```
UnitaryQ[Γ_, Ep_] := Module[{n, P, B, F, Tp, Bs, Tps, JB, JT, τ, Id, Blist, Tplist},
  n = Length[Ep] / 2;
  Id = IdentityMatrix[n];
  P = Partition[Ep, n];
  B = P[[1]]; Tp = Reverse[P[[2]]];
  Bs = B /. {x_ → 1, y_ → -1};
  Tps = Tp /. {x_ → -1, y_ → 1};
  Bi[l_] := Which[l[[1]] < l[[2]], 2 - (T + T^-1), l[[1]] > l[[2]], -2 + (T + T^-1), True, 1];
  BiF[l_] := (F = Map[Bi, Table[{i, j}, {i, 1, n}, {j, 1, n}], {2}];
    Do[If[l[[i]] > 0, F[[i, i]] = T - T^-1, F[[i, i]] = T^-1 - T], {i, 1, n}]; F);
  JB = BiF[Bs]; JT = BiF[Tps];
  Blist = B /. {x_i_ → {x, i}, y_j_ → {y, j}};
  Do[If[Blist[[i, 1]] === x, Blist[[i]] = Id[[Blist[[i, 2]]]], Blist[[i]] = -Γ[[Blist[[i, 2]]]]],
   {i, 1, Length[Blist]}];
  Tplist = Tp /. {x_i_ → {x, i}, y_j_ → {y, j}};
  Do[If[Tplist[[i, 1]] === x, Tplist[[i]] = -Id[[Tplist[[i, 2]]]],
    Tplist[[i]] = Γ[[Tplist[[i, 2]]]]], {i, 1, Length[Tplist]}];
  τ = Join[Blist, Tplist];
  ContainsOnly[Flatten[
    Transpose[τ].ArrayFlatten[{{-JB, 0}, {0, JT}}].(τ /. T → T^-1) // Simplify], {0}]
 ]
```

$$P = \begin{pmatrix} -\frac{1}{-2+T} & \frac{-1+T}{(-2+T)\,T} & 0 \\ \frac{(-1+T)\,(1-T+T^2)}{-2+T} & -\frac{-1+2\,T}{(-2+T)\,T} & 1-T \\ -\frac{(-1+T)^2\,T}{-2+T} & \frac{-1+T}{-2+T} & T \end{pmatrix}$$

$$\left\{\left\{-\frac{1}{-2+T}, \frac{-1+T}{(-2+T)\,T}, 0\right\}, \left\{\frac{(-1+T)\,(1-T+T^2)}{-2+T}, -\frac{-1+2\,T}{(-2+T)\,T}, 1-T\right\}, \left\{-\frac{(-1+T)^2\,T}{-2+T}, \frac{-1+T}{-2+T}, T\right\}\right\}$$

```
UnitaryQ[P, {y1, x1, x3, y3, x2, y2}]
```

True

```
UnitaryQ[P, {y2, y1, x1, x3, y3, x2}]
```

True

```
UnitaryQ[P, {x2, y2, y1, x1, x3, y3}]
```

True

$$(Rp_{9,1}\, Rp_{3,8}\, Rp_{4,11}\, Rp_{12,5}\, Rp_{6,13}\, Rp_{10,2} \,//\, m_{1,3\to1} \,//\, m_{1,4\to1} \,//\, m_{1,5\to1} \,//\, m_{1,6\to1} \,//\, m_{2,8\to2} \,//\, m_{2,9\to2} \,//\, m_{2,10\to2} \,//\, m_{2,11\to2} \,//\, m_{2,12\to2} \,//\, m_{2,13\to2}) \;/.\; \{T_{i\_} \to T\} \,//\, \text{Simplify}$$

$$\begin{pmatrix} T & h_1 & h_2 \\ t_1 & T\left(1-T+T^2-T^3+T^4\right) & -(-1+T)\,T\left(1+T^2\right) \\ t_2 & -(-1+T)\left(1-T+T^2\right)\left(1+T+T^2\right) & 1-T+T^2-T^3+T^4 \end{pmatrix}$$

$$Q = \begin{pmatrix} T\left(1-T+T^2-T^3+T^4\right) & -(-1+T)\,T\left(1+T^2\right) \\ -(-1+T)\left(1-T+T^2\right)\left(1+T+T^2\right) & 1-T+T^2-T^3+T^4 \end{pmatrix};$$

```
UnitaryQ[Q, {x1, x2, y1, y2}]
```

True