Pensieve header: Testing the commutativity of two independent Nwu's.

```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Vo"];
```

NOE-0

0R

```
R+[i_,j_] := 𝔼[b_i c_j + b_i^-1 (e^b_i - 1) u_i w_j]; R-[i_,j_] := 𝔼[-b_i c_j + b_i^-1 (e^-b_i - 1) u_i w_j];
```

0Util

```
CF[ω_ . 𝔼[Q_]] := Simplify[ω] 𝔼[Simplify[Q]];
𝔼 /: 𝔼[Q1_] 𝔼[Q2_] := CF@𝔼[Q1 + Q2];
ω1_ . 𝔼[Q1_] ≡ ω2_ . 𝔼[Q2_] := Simplify[ω1 == ω2 ∧ Q1 == Q2];
```

0NO

```
N[u_i_ c_j_→k_][ω_ . 𝔼[Q_]] := CF[
   ω 𝔼[e^-γ β u_k + γ c_k + (Q /. c_j | u_i → 0)]] /. {γ → ∂[c_j] Q, β → ∂[u_i] Q}];
N[w_i_ c_j_→k_][ω_ . 𝔼[Q_]] := CF[
   ω 𝔼[e^γ α w_k + γ c_k + (Q /. c_j | w_i → 0)]] /. {γ → ∂[c_j] Q, α → ∂[w_i] Q}];
N[w_i_ u_j_→k_][ω_ . 𝔼[Q_]] := CF[
   ν ω 𝔼[-b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (Q /. w_i | u_j → 0)]] /. ν → (1 + b_k δ)^-1 /.
    {α → ∂[w_i] Q /. u_j → 0, β → ∂[u_j] Q /. w_i → 0, δ → ∂[w_i,u_j] Q}];
```

0m

```
m[i_,j_→k_][ω_ . 𝔼[Q_]] := CF[Module[{x},
   (ω 𝔼[Q] /. b_i|j → b_k // N[w_i c_j→x] // N[u_i c_x→x] // N[w_x u_j→x]) /. {c_i → c_k, w_j → w_k, y_x :→ y_k}]]
```

```
Z0 = ((R+[1,2] // m[1,2→1]) /. x_1 :→ x_1 + x_2) (R+[3,5] // m[3,5→3]) (R+[4,6] // m[4,6→1])
```

$$\mathbb{E}\left[b_1 c_1 + \left(b_1 + b_2\right)\left(c_1 + c_2\right) + b_3 c_3 + \frac{e^{-b_1}\left(-1 + e^{b_1}\right) u_1 w_1}{b_1} + \frac{e^{-b_1-b_2}\left(-1 + e^{b_1+b_2}\right)\left(u_1 + u_2\right)\left(w_1 + w_2\right)}{b_1 + b_2} + \frac{e^{-b_3}\left(-1 + e^{b_3}\right) u_3 w_3}{b_3}\right]$$

```
Z0 // m[1,3→1] // m[2,4→2]
```

$$\mathbb{E}\Big[\frac{1}{b_1\left(b_1 + b_2\right)} e^{-2 b_1 - b_2}\left(2 e^{2 b_1 + b_2} b_1^2 b_2\left(2 c_1 + c_2\right) + e^{2 b_1 + b_2} b_1^3\left(3 c_1 + c_2\right) + e^{b_2}\left(-1 + e^{2 b_1}\right) b_2 u_1 w_1 + b_1\left(e^{2 b_1 + b_2} b_2^2\left(c_1 + c_2\right) + e^{b_1}\left(-1 + e^{b_1 + b_2}\right) u_2\left(w_1 + w_2\right) + u_1\left(\left(-1 - e^{b_2} + e^{b_1 + b_2} + e^{2 b_1 + b_2}\right) w_1 + \left(-1 + e^{b_1 + b_2}\right) w_2\right)\right)\right)\Big]$$

```
Z0 // m[3,1→1] // m[4,2→2]
```

$$\mathbb{E}\Big[\frac{1}{b_1\left(b_1 + b_2\right)} e^{-2 b_1 - b_2}\left(2 e^{2 b_1 + b_2} b_1^2 b_2\left(2 c_1 + c_2\right) + e^{2 b_1 + b_2} b_1^3\left(3 c_1 + c_2\right) + e^{b_2}\left(-1 + e^{2 b_1}\right) b_2 u_1 w_1 + b_1\left(e^{2 b_1 + b_2} b_2^2\left(c_1 + c_2\right) + e^{b_1}\left(-1 + e^{b_1 + b_2}\right) u_2\left(w_1 + w_2\right) + u_1\left(\left(-1 - e^{b_2} + e^{b_1 + b_2} + e^{2 b_1 + b_2}\right) w_1 + \left(-1 + e^{b_1 + b_2}\right) w_2\right)\right)\right)\Big]$$

```
R[p_, i_, j_] := 𝔼[p b_i c_j + b_i^-1 (e^p b_i - 1) u_i w_j];
```

```
R[-1, i, j]
```

$$\mathbb{E}\left[-b_i c_j + \frac{\left(-1 + e^{-b_i}\right) u_i w_j}{b_i}\right]$$

```
R[1/2, 1, 2] R[1/2, 3, 4] // m[1,3→1] // m[2,4→2]
```

$$\mathbb{E}\left[\frac{b_1^2 c_2 + \left(-1 + e^{b_1}\right) u_1 w_2}{b_1}\right]$$