Alexander Invariants of Tangles via Expansions

by

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Abstract

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In this thesis we describe a method to extend the Alexander polynomial to tangles. It is based on a technology known as expansions, which is inspired by the Taylor expansion and the Kontsevich integral. Our main object of study is the space of w-tangles, which contains usual tangles, but has a much simpler expansion. To study w-tangles, we introduce an algebraic structure called meta-monoids. An expansion of w-tangles together with the choice of a Lie algebra gives us a particular meta-monoid called Γ-calculus that recovers the Alexander polynomial. Using the language of Γ-calculus, we rederive certain important properties of the Alexander polynomial, most notably the Fox-Milnor condition on the Alexander polynomials of ribbon knots [Lic97, FM66]. We argue that our proof has some potential for generalization which may help tackle the slice-ribbon conjecture. In a sense this thesis is an extension of [BNS13].

Acknowledgements

I am grateful to my supervisor, Prof Dror Bar-Natan for his patient guidance and willingness to answer my questions regarding the subject. His knowledge has benefited me greatly throughout the project. I would like to thank all the staff in the Mathematics department at UofT for their help and service during my Phd. Thanks also go to my fellow graduate students at UofT for many interesting discussions, especially the regular members of the Knot-at-Lunch group: Ester Dalvit, Travis Ens, and Iva Halacheva. Finally, I thank my Mom for her constant love and unending support.

Contents

Chapter 1

Introduction

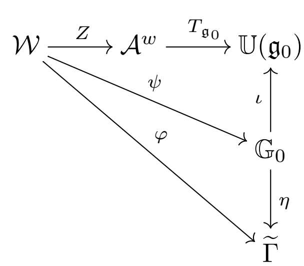
The Alexander polynomial is one of the most important invariants in knot theory. Originally discovered by Alexander in 1928 [Ale28], many things are known about the polynomial. For instance, it has a topological description [Lic97], an interpretation as a quantum invariant [Oht02, KS91], and recently has been categorified via Heegaard-Floer homology [OS04]. To compute the Alexander polynomial of a knot with many crossings, one strategy would be to break the knot into smaller pieces called tangles, find an appropriate extension of the Alexander polynomial to tangles, compute the said extension for each constituent tangle, and then "glue" the results together. One can obtain an Alexander invariant of tangles in several ways, which are roughly based on two perpestives: from the quantum invariant point of view [Oht02, Sar15] or from the topological/combinatorial point of view [CT05, Arc10, Pol10, BNS13, BCF15, DF16].

One important aspect of knot theory is its implementation on a computer. For that purpose, two definitions of the Alexander polynomial are particularly useful: in terms of R-matrix, i.e. quantum invariant [Oht02], or in terms of Fox derivatives [Arc10]. These two formulations come from two ways of viewing a tangle. One approach views a tangle as a morphism in a category and the other views a tangle in terms of planar/circuit algebra [BND14]. We introduce yet another way to view a tangle, as an element of a meta-monoid (Section 2.1) [BNS13, Hal16]. Namely, we can just decompose a tangle into a disjoint union of crossings and then stitch the strands together to recover the tangle, with the condition that we cannot stitch the same strand to itself, in order not to produce closed components. Hence one can think of meta-monoids as restrictions of circuit algebras where we do not allow closed components. An advantage of seeing things in this way as opposed to the usual categorical approach is that one no longer needs to divide the endpoints of a tangle into the bottom and the top. Moreover, virtual crossings are automatically included in the meta-monoid structure, so we can talk about the bigger class of virtual tangles (or more precisely w-tangle).

On the algebraic side the meta-monoid that gives us a tangle invariant is called Gassner calculus or Γ-calculus. Roughly speaking, Γ-calculus assigns to a tangle with n open components a Laurent polynomial and an $n \times n$ matrix whose entries are rational functions. In the case where a tangle has only one component, we recover the Alexander polynomial. One can obtain a topological interpretation of Γ-calculus along the lines of the arguments in [CT05] and [DF16] but we will not pursue it in this thesis. On a computer, Γ-calculus is quite simple to implement (see Section 3.1) and it also runs faster than current algorithms that compute the Alexander polynomial. Furthermore, its complexity is a polynomial in n, whereas in the quantum approach, the complexity tends to be exponential (since one needs to look at tensor products of representations). One can think of Γ-calculus as a generalization of the Gassner-Burau representation [KT08, BN14] to tangles (compare also with [KLW01]). In essence, we break the determinant formula of the Alexander polynomial into a step-by-step gluing instruction with each step involving some simple algebraic manipulations. This approach may play a role if one wants to categorify the invariant, which might lead to a simpler way to approach the formidable Heegaard-Floer homology.

In Chapter 4 and Chapter 5 we explain the general framework that produces Γ-calculus as the end result. Although one can define Γ-calculus simply by giving the formulas, it is always instructive to know where these formulas come from. The construction is based on two fundamental ideas: expansions and the relationship between knot theory and Lie algebras. The concept of an expansion is inspired mainly by Taylor expansions and the Kontsevich integral (see [Oht02, BN95, CDM12]) in knot theory. Roughly speaking an expansion converts an object to a graded object. Graded objects are more desirable to work with since we can test or solve for solutions degree by degree. In the case of Taylor expansions, we turn an analytic function into a power series which is graded by the power of the variable. In the case of the Kontsevich integral, we turn a knot into a series in chord diagrams which is graded by the number of chords. An expansion maps a space to its associated graded space, which is our main object of study. For the space of isotopy classes of knots, its associated graded is the space of chord diagrams (see [BN95, CDM12]). The formula of the Kontsevich integral is highly non-trivial and its extension to tangles requires the use of a sophisticated technology known as a Drinfeld associator (see [BN97, Oht02]). In this thesis we will perform a similar analysis for w-tangles (Section 2.2). For w-tangles, its associated graded space is the space of arrow diagrams (Section 4.2). Although w-knots contain usual knots, an expansion of w-knots is much more straightforward. Moreover the procedure naturally extends to w-tangles, without the necessity of a Drinfeld associator.

Let us briefly outline the main steps of our construction. We first look at the associated graded space of w-tangles, which is the meta-monoid of arrow diagrams. An expansion of w-tangles is given simply by an exponential map. Using the relationship between arrow diagrams and Lie algebras we obtain the corresponding image in the meta-monoid of Lie algebras. Finally, choosing a particular Lie algebra (in our case the non-abelian two-dimensional Lie algebra) we obtain Γ-calculus. In summary we have a commutative diagram



We will go over the above commutative diagram in details in Chapter 5. We will also study the compatibility of the diagram with the operations orientation reversal and strand doubling. The framework we just describe has been partially generalized to give a more powerful invariant (see [BNV17, BN17, BN16a]).

The bulk of the thesis is devoted to rederiving the Fox-Milnor condition for the Alexander polynomials of ribbon knots [Lic97, FM66], which simply says that the Alexander polynomial of a slice knot factors as a product of two Laurent polynomials $f(t)f(t^{-1})$, in the framework of Γ-calculus. Our ultimate goal is to say something about the ribbon-slice conjecture [GST10], which asks whether every slice knot is also ribbon. Let us give a brief overview of our approach (see [BN17] for more details). First of all, given a tangle T_{2n} with $2n$ components, there are two closure operations, denoted by τ and κ (Section 6.1), which gives an n-component tangle T_n and a one-component tangle T_1, i.e. a long knot, respectively

$$T_n \xleftarrow{\tau} T_{2n} \xrightarrow{\kappa} T_1.$$

Now we have the following characterization of ribbon knots (Proposition 6.1), namely a knot K is ribbon if and only if there exists a 2n-component tangle T_{2n} such that $\kappa(T_{2n}) = K$ and $\tau(T_{2n})$ is the trivial tangle. More succinctly, if we denote the set of all m-component tangles by $\mathcal{T}_m$, then

$$\{\text{ribbon knots}\} = \bigcup_{n=1}^{\infty} \Big\{\kappa(T_{2n}) : T_{2n} \in \mathcal{T}_{2n} \text{ and } \tau(T_{2n}) = U_n \in \mathcal{T}_n\Big\},$$

where U_n denote the trivial n-component tangle. Therefore if we have an invariant $Z : \mathcal{T}_k \to \mathcal{A}_k$ of tangles, where $\mathcal{A}_k$ is some algebraic space which is well-understood (think of matrices of polynomials), together with the corresponding closure operations $\tau_{\mathcal{A}}$ and $\kappa_{\mathcal{A}}$ which intertwine with τ and κ:

$$Z(\kappa(T_{2n})) = \kappa_{\mathcal{A}}(Z(T_{2n})), \quad Z(\tau(T_{2n})) = \tau_{\mathcal{A}}(Z(T_{2n})),$$

then we have an "algebraic criterion" to determine if a given knot K is ribbon. Specifically, if a knot K is ribbon then there exist some n and an element $\zeta \in \mathcal{A}_{2n}$ such that $Z(K) = \kappa_{\mathcal{A}}(\zeta)$ and $\tau_{\mathcal{A}}(\zeta) = \mathrm{Id}_n \in \mathcal{A}_n$, or more simply

$$Z(K) \in \bigcup_{n=1}^{\infty} \kappa_{\mathcal{A}}(\tau_{\mathcal{A}}^{-1}(\mathrm{Id}_n)). \tag{1.1}$$

We denote the set on the right hand side by $\mathcal{R}_{\mathcal{A}}$. Of course to have any practical values, we need to make sure that $\mathcal{R}_{\mathcal{A}}$ is strictly smaller than $\mathcal{A}_1$. Then a knot K is not ribbon if $Z(K) \notin \mathcal{R}_{\mathcal{A}}$.

In [GST10] the authors propose several potential counter-examples to the ribbon-slice conjecture. These are knots with a high number of crossings. Our long term goal is to construct a class of invariants of tangles which are computable in polynomial time and behave well under various operations in order to test these counter-examples in the framework proposed above (see partial progress in [BN16a]). The simplest example of such invariants is Γ-calculus, and condition (1.1) yields the familiar Fox-Milnor condition, as to be expected since Γ-calculus is an extension of the Alexander polynomial to tangles. Although the original proof of the Fox-Milnor condition [FM66] is quite short and elegant, we believe our proof offers several advantages as summarized below.

- The original proof uses homology and so cannot distinguish between slice and ribbon properties, whereas our proof follows quite mechanically from the characterization of ribbon knots in terms of tangles. Furthermore, the bulk of our proof uses just elementary linear algebra, which is more accessible to a novice.

- In our proof the function f appears naturally as the invariant of a tangle obtained from a tangle presentation (Propostion 6.1) of a ribbon knot.

- We believe our proof could be generalized to a stronger class of invariants and we hope these better invariants will give a much stronger condition for ribbon knots.

So as it stands this paper serves as a warm-up step in a long project and we hope that it will whet the readers' appetite to join our quest.

The paper is organized as follows. In Chapter 2 we give the main definitions and properties of meta-monoids as well as some main examples. In Chapter 3 we describe our main meta-monoid: Γ-calculus and derive various formulae therein. In Chapter 4 we introduce the general algebraic framework that gives rise to Γ-calculus: algebraic structures and expansions. In Chapter 5 we explain the connection between arrow diagrams and Lie algebras and tie everything together. Chapter 6 is the main main part of this paper where we introduce ribbon knots and prove the Fox-Milnor condition. Finally in Chapter 7 we show how one can extend the scalar part of Γ-calculus to links and derive the classic Alexander-Conway skein relation. Notice that Mathematica codes are given at various places throughout this thesis. We emphasize again that this is one advantage of Γ-calculus, where we can verify certain properties simply by using Mathematica.

Notations. In this paper we use the Mathematica notation // to denote compositions of functions because we find that it is more natural to read composition in this way and also easier to convert the formulas to Mathematica commands. Specifically,

$$f \,/\!/\, g := g \circ f.$$

Also a bold-faced ***letter*** will general denote a matrix, or in particular a vector, whose dimension depends on the context.

Chapter 2

Meta-Monoids

2.1 Definitions

A *meta-monoid* $\mathcal{G}$ (see [BNS13, BN15, Hal16]) is a collection of sets $(\mathcal{G}^X)$ indexed by finite sets X, i.e. for a finite set of *labels* X[1] we have a set of objects $\mathcal{G}^X$, together with the following operations:

- "stitching" $m^{x,y}_z : \mathcal{G}^{\{x,y\}\cup X} \to \mathcal{G}^{\{z\}\cup X}$, where $\{x,y,z\} \cap X = \emptyset$ and $x \neq y$,
- "identity" $e_x : \mathcal{G}^X \to \mathcal{G}^{\{x\}\cup X}$, where $x \notin X$,
- "deletion" $\eta_x : \mathcal{G}^{X\cup\{x\}} \to \mathcal{G}^X$, where $x \notin X$,
- "disjoint union" $\sqcup : \mathcal{G}^X \times \mathcal{G}^Y \to \mathcal{G}^{X\cup Y}$, where $X \cap Y = \emptyset$,
- "renaming" $\sigma^x_z : \mathcal{G}^{X\cup\{x\}} \to \mathcal{G}^{X\cup\{z\}}$, where $\{x,z\} \cap X = \emptyset$.

These operations satisfy the following axioms[2]:

- *Monoid axioms:*

$$m^{x,y}_u \mathbin{/\!\!/} m^{u,z}_v = m^{y,z}_u \mathbin{/\!\!/} m^{x,u}_v \qquad (\textit{meta-associativity}),$$
$$e_a \mathbin{/\!\!/} m^{a,b}_c = \sigma^b_c \qquad (\textit{left identity}),$$
$$e_b \mathbin{/\!\!/} m^{a,b}_c = \sigma^a_c \qquad (\textit{right identity}),$$

- *Miscellaneous axioms:*

$$e_a \mathbin{/\!\!/} \sigma^a_b = e_b, \quad \sigma^a_b \mathbin{/\!\!/} \sigma^b_c = \sigma^a_c, \quad \sigma^a_b \mathbin{/\!\!/} \sigma^b_a = Id,$$
$$\sigma^a_b \mathbin{/\!\!/} \eta_b = \eta_a, \quad e_a \mathbin{/\!\!/} \eta_a = Id, \quad m^{a,b}_c \mathbin{/\!\!/} \eta_c = \eta_b \mathbin{/\!\!/} \eta_a,$$
$$m^{a,b}_c \mathbin{/\!\!/} \sigma^c_d = m^{a,b}_d, \quad \sigma^a_b \mathbin{/\!\!/} m^{b,c}_d = m^{a,c}_d.$$

We also require that operations with distinct labels commute, for instance $\eta_a /\!\!/ \eta_b = \eta_b /\!\!/ \eta_a$ or $m^{a,b}_c /\!\!/ m^{d,e}_f = m^{d,e}_f /\!\!/ m^{a,b}_c$ etc. Moreover, the disjoint union operation $\sqcup$ commutes with all other operations, for example $\sqcup \mathbin{/\!\!/} m^{a,b}_c \mathbin{/\!\!/} m^{d,e}_f = (m^{a,b}_c, m^{d,e}_f) \mathbin{/\!\!/} \sqcup$. Given two meta-monoids $\mathcal{G}$ and $\mathcal{H}$, a *meta-monoid homomorphism* is a collection of maps $(f^X : \mathcal{G}^X \to \mathcal{H}^X)$ that commute with the operations.

[1] To avoid set-theoretic issues we can require these set of labels to be subsets of some fixed set, say the natural numbers

[2] In this paper we use the notation $/\!\!/$ to denote function compositions, namely $f \mathbin{/\!\!/} g = g \circ f$.

In practice, usually the only non-trivial relation we have to check is meta-associativity. While the definition of a meta-monoid is quite lengthy, a couple of examples will make it clear how to think about meta-monoids and where the name comes from.

Example 2.1 (Monoids). Given a monoid G with identity e (or an algebra), one obtains a meta-monoid G as follows. Set

$$G^X := \{\text{functions } f : X \to G\}.$$

We write an element of G^X as $(x \to g_x, \dots)$, where $x \in X$ and $g_x \in G$. In the following operations, $\dots$ denotes the remaining entries, which stay unchanged:

$$(x \to g_x, y \to g_y, \dots) \,/\!/\, m_z^{x,y} = (z \to g_x g_y, \dots),$$

$$(y \to g_y, \dots) \,/\!/\, e_x = (x \to e, y \to g_y, \dots),$$

$$(x \to g_x, y \to g_y, \dots) \,/\!/\, \eta_x = (y \to g_y, \dots),$$

$$(x \to g_x, \dots) \sqcup (y \to g_y, \dots) = (x \to g_x, \dots, y \to g_y, \dots),$$

$$(x \to g_x, \dots) \,/\!/\, \sigma_z^x = (z \to g_x, \dots).$$

Let us check meta-associativity. Suppose $\Omega \in G^{X \cup \{x,y,z\}}$ and we only write the relevant entries, the others are left unchanged:

$$\Omega = (x \to g_x, y \to g_y, z \to g_z).$$

Then

$$\Omega \,/\!/\, m_z^{x,y} \,/\!/\, m_v^{u,z} = (v \to (g_x g_y) g_z),$$

and

$$\Omega \,/\!/\, m_u^{y,z} \,/\!/\, m_v^{x,u} = (v \to g_x (g_y g_z)).$$

Thus we see that meta-associativity follows from the associativity of multiplication $(g_x g_y) g_z = g_x (g_y g_z)$. Similarly the left identity and right identity are consequences of $eg = ge = g$ for all $g \in G$. The other axioms are straightforward to verify. In general, $m_z^{x,y} \neq m_z^{y,x}$, unless G is commutative. This meta-monoid also satisfies the following property:

$$\Omega = (\Omega \,/\!/\, \eta_y) \sqcup (\Omega \,/\!/\, \eta_x), \quad \Omega \in G^{\{x,y\}}. \tag{2.1}$$

Indeed if $\Omega = (x \to g_x, y \to g_y)$, then $\Omega \,/\!/\, \eta_y = (x \to g_x)$ and $\Omega \,/\!/\, \eta_x = (y \to g_y)$ and so the right hand side is exactly Ω. Most examples of meta-monoids will not satisfy this property, and so we see that not every meta-monoid comes from a monoid in the above manner. ♣

Example 2.2 (Groups (see also [BNS13])). Consider the meta-monoid $\mathcal{G}$ given as follows. For a finite set of labels X let $\mathcal{G}^X$ consist of triples of the form (F, μ, λ), where F is a finitely presented group and $\mu : X \to F$, $x \mapsto \mu_x$ and $\lambda : X \to F$, $x \mapsto \lambda_x$ are functions $X \to F$ (μ is called a *meridian map* and λ is called a *longitude map*, which we also allow up to conjugation). Now the operations are

$$(F, \mu, \lambda) \,/\!/\, m_z^{x,y} = (F / \left\langle \mu_y = \lambda_x^{-1} \mu_x \lambda_x \right\rangle, \mu \setminus \{x \mapsto \mu_x, y \mapsto \mu_y\} \cup \{z \mapsto \mu_x\},$$
$$\lambda \setminus \{x \mapsto \lambda_x, y \mapsto \lambda_y\} \cup \{z \mapsto \lambda_x \lambda_y\}),$$

$$(F,\mu,\lambda) \mathbin{/\!\!/} e_x = (F * \langle x \rangle, \mu \cup \{x \mapsto x\}, \lambda \cup \{x \to e\}),$$

$$(F,\mu,\lambda) \mathbin{/\!\!/} \eta_x = (F/\langle \mu_x = 1 \rangle, \mu \setminus \{x \mapsto \mu_x\}, \lambda \setminus \{x \mapsto \lambda_x\}),$$

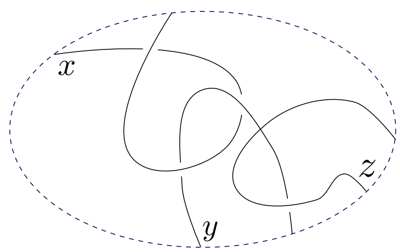
$$(F,\mu,\lambda) \sqcup (F',\mu',\lambda') = (F * F', \mu \cup \mu', \lambda \cup \lambda'),$$

$$(F,\mu,\lambda) \mathbin{/\!\!/} \sigma^x_z = (F, \mu \setminus \{x \mapsto \mu_x\} \cup \{z \mapsto \mu_x\}, \lambda \setminus \{x \mapsto \lambda_x\} \cup \{z \mapsto \lambda_x\}).$$

We leave the verification of the axioms to the reader. Notice that property (2.1) does not hold here, for instance if $F = \mathbb{Z} \oplus \mathbb{Z}$, then $(F \mathbin{/\!\!/} \eta_x) \sqcup (F \mathbin{/\!\!/} \eta_y)$ is the free product $\mathbb{Z} * \mathbb{Z}$. ♣

2.2 The meta-monoid of w-tangles

A *w-tangle diagram* is a finite collection of oriented arcs (or *components*) smoothly drawn on a plane, with finitely many intersections, divided into *virtual crossings*, *positive crossings*, and *negative crossings*; and regarded up to planar isotopy. We also require distinct components to be labeled by distinct labels from some set of labels X. An example of a w-tangle diagram is



In the figure the boundary circle is drawn in dashed line for ease of visualization, but is not part of the data. In particular, we do not care about the position of the endpoints. So for instance

represent the same diagram.

A *w-tangle* is an equivalence class of w-tangle diagrams, modulo the equivalence generated by the Reidemeister 2 and 3 moves ($R2$, and $R3$), the virtual Reidemeister 1 through 3 moves ($VR1$, $VR2$, $VR3$), the mixed relations (M), and the *overcrossings commute* relations (OC).

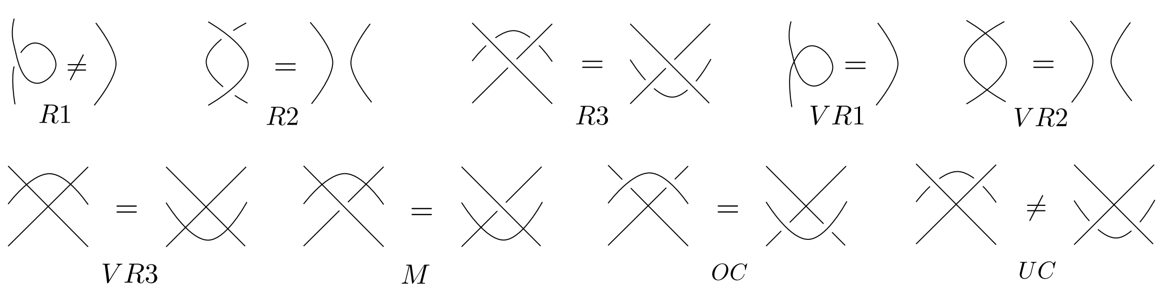
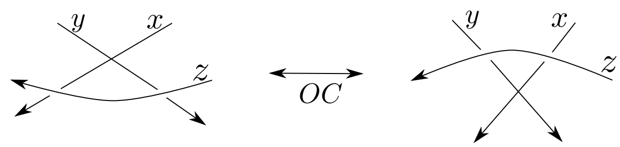


Figure 2.1: The relations defining w-tangles.

Note that we do not mod out by the Reidemeister 1 move ($R1$) nor by the undercrossings commute relations (UC). Also we do not allow closed components. For a topological interpretation of w-tangles, see [BND16, BN15].

Let us describe the OC relations in a bit more details.

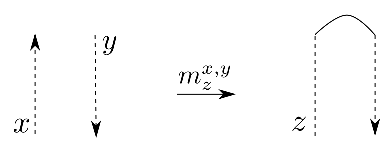


Notice that in our description of w-tangles, virtual crossings play no essential role. However the OC relations are not trivial, since it says that from the perspective of strand z, going over strand y and then strand x is the same as going over strand x and then strand y.

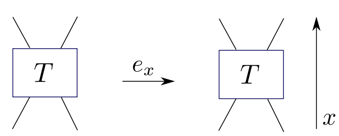
When a w-tangle has only one component, we obtain a theory of long w-knots. It is well-known that *ordinary long knots* (which are the same as closed knots, see for example [JF13]) inject into long w-knots [BND16]. In other words, two knots are isotopic (as ordinary knots) if and only if they are isotopic as w-knots. Thus an invariant of long w-knots yields a knot invariant. However note that long w-knots are not equivalent to closed w-knots (see Example 3.1).

Now we would like to introduce our main object of study: the *meta-monoid* $\mathcal{W}$ *of w-tangles*. Specifically, for a finite set X, let $\mathcal{W}^X$ be the collection of w-tangles with $|X|$ components which are labeled by the elements of X. It is important to note that we do not allow closed components (embeddings of S^1) in the definition of $\mathcal{W}^X$. Now let us specify the meta-monoid operations, which all have explicit geometric descriptions in this context.

- Stitching $m_z^{x,y}$ means connecting the head of strand x to the tail of strand y and calling the resulting strand z. Note that if strand x and strand y are far away, we can always bring them together via virtual crossings. Also from now on we use the convention that a dashed line means it can be knotted freely.

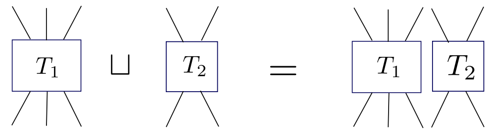


- Identity e_x means adding a trivial strand labeled x which does not cross any other strand.



- Deletion η_x means deleting strand x from the w-tangle.

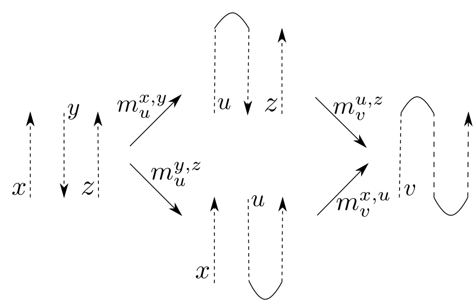
- Disjoint union $\sqcup$ means putting the two w-tangles side by side, again by the nature of virtual crossings it does not matter how we put these two tangles together. To simplify notation, we abbreviate $T_1 \sqcup T_2$ as just T_1T_2.



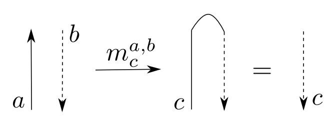
- Renaming σ_z^x means relabeling strand x to strand z.

Then we can verify the meta-monoid axioms visually as follows.

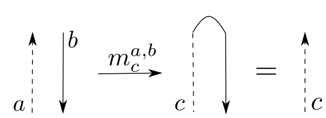
- The meta-associativity relation:



- The left identity relation:

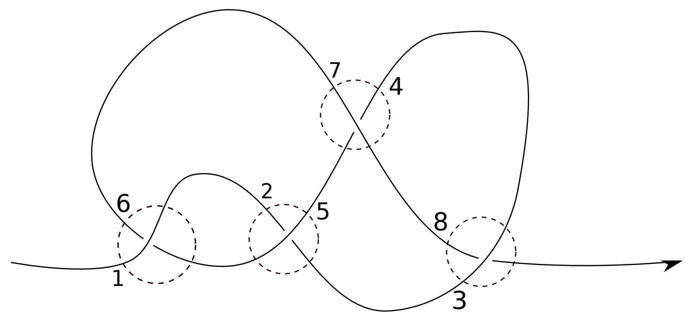


- The right identity relation:



In the framework of meta-monoids, a w-tangle can be described in terms of generators and relations. Specifically, given a w-tangle, we can first decompose it into a disjoint union of positive crossings $R^+_{i,j}$ and negative crossings $R^-_{i,j}$:

Here i and j are the labels of the incoming strands and i is the label of the overstrand. Then we obtain the original w-tangle by stitching the crossings appropriately. For a concrete example, let us look at the long figure-eight knot:



We can label the long figure-eight knot as in the above figure, namely we label the incoming strand with 1 and every time we go over or under a strand, we increase the label. Then we can write the long figure-eight knot as a disjoint union of crossings:

$$R^+_{1,6}R^+_{5,2}R^-_{3,8}R^-_{7,4}.$$

The long figure-eight knot consists of four crossings, two positive and two negative. Then we stitch strand 1 to strand 2 through to strand 8. Therefore the long figure-eight knot is given by

$$R^+_{1,6}R^+_{5,2}R^-_{3,8}R^-_{7,4} \,/\!\!/\, m_1^{1,2} \,/\!\!/\, m_1^{1,3} \,/\!\!/\, m_1^{1,4} \,/\!\!/\, m_1^{1,5} \,/\!\!/\, m_1^{1,6} \,/\!\!/\, m_1^{1,7} \,/\!\!/\, m_1^{1,8}.$$

We can then summarize the above observation in the following proposition.

Proposition 2.1. *The meta-monoid $\mathcal{W}$ of w-tangles is* generated *by the crossings $R^+_{i,j}$ and $R^-_{i,j}$, i.e. all expressions that can be formed using the crossings and the meta-monoid operations, modulo the relations $R2$, $R3$ and OC (Figure 2.1).*

Remark 2.1. To describe a w-tangle using the categorical language, one would have to use positive crossings, negative crossings and virtual crossings. One would also need to include all the relations involving virtual crossings. In the context of meta-monoids, the relations $VR1$, $VR2$, $VR3$, M are automatically true. Therefore meta-monoids give a more succinct way to talk about w-tangles. In particular meta-monoids provide a shift in perspective where one focuses on the strands instead of the endpoints (as opposed to the categorical language where we need to split the endpoints into the top and the bottom).

The next proposition will not be needed in the rest of the paper. It is a generalization of knot groups to w-tangles.

Proposition 2.2 ([BN15])**.** *There is a meta-monoid homomorphism from the meta-monoid of w-tangles $\mathcal{W}$ to the meta-monoid of groups $\mathcal{G}$ given in Example 2.2.*

Proof. An observant reader will realize that (F, μ, λ) is simply the *peripheral system* of a tangle. Given a w-tangle T, we can compute its fundamental group $F = \pi_1(T)$ using the Wirtinger presentation [Rol03] (ignore virtual crossings). Then μ and λ are the images of the meridians and longitudes in F. More specifically, taking as the basepoint our eyes, for a strand labeled x, μ_x is the loop starting from the basepoint to the right of the tail of strand x, going perpendicularly to the left under strand x and then back to the base point. For the longitudes, let λ_x be the loop starting from the basepoint to the right of the tail of strand x and then going along a parallel copy of strand x to the head of strand x and then back to the basepoint. For example, the image of the positive crossing $R^+_{i,j}$ is

$$R^+_{i,j} \mapsto (\langle \mu_i \rangle * \langle \mu_j \rangle, \{i \mapsto \mu_i, j \mapsto \mu_j\}, \{i \mapsto 1, j \mapsto \mu_i\}).$$

We leave it to the readers to verify the operations and the axioms. □

Chapter 3

The Gassner Calculus Γ

3.1 Definition and Properties of Gassner Calculus

In this section we introduce a meta-monoid that will serve as the target space of an algebraic invariant for w-tangles. Let Γ be the meta-monoid given as follows. For a finite set X, let R_X be $\mathbb{Q}(\{t_i : i \in X\})$, the field of rational functions in the variables t_i, $i \in X$, and $M_{X\times X}(R_X)$ be the collection of $|X| \times |X|$ *labeled matrices* with rows and columns labeled by the elements of X. Suppose that the set X has the form $X = \{a, b\} \cup S$, where $S \cap \{a, b\} = \emptyset$. An element of $R_X \times M_{X\times X}(R_X)$ is a pair consisting of an element ω in R_X, which we call the *scalar part*, and an element in $M_{X\times X}(R_X)$, which we call the *matrix part*, can be represented as

$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right).$$

Let us explain a bit about the notations. Here θ and ϵ are row vectors (notice the horizontal line in each letter), whereas ϕ and ψ are column vectors (notice the vertical line in each letter) and Ξ is a square matrix (as evident from the shape of the letter Ξ). In most cases the rows and columns of a labeled matrix have the same labels, but occasionally we also allow permutations of the labels. If the labels are clear from the context sometimes we will omit the labels to simplify notations.

Now let Γ^X be the subset of $R_X \times M_{X\times X}(R_X)$ satisfying the condition

$$\left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right)_{t_i \to 1} = \left(\begin{array}{c|c} 1 & X \\ \hline X & I \end{array}\right).$$

Here $t_i \to 1$ means substituting all the variables t_i by 1 for $i \in X$ and I is the identity matrix. In particular, we see that the matrix part is always invertible (since the determinant is not identically 0). Then the operations in a meta-monoid are given by, where $t_a \to t_b$ means substituting t_a by t_b:

- identity: $\left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right) /\!\!/ e_a = \left(\begin{array}{c|cc} \omega & a & X \\ \hline a & 1 & \mathbf{0} \\ X & \mathbf{0} & M \end{array}\right)$, where $a \notin X$,

- disjoint union: $\left(\begin{array}{c|c} \omega_1 & X_1 \\ \hline X_1 & M_1 \end{array}\right) \sqcup \left(\begin{array}{c|c} \omega_2 & X_2 \\ \hline y_{X_2} & M_2 \end{array}\right) = \left(\begin{array}{c|cc} \omega_1\omega_2 & X_1 & X_2 \\ \hline X_1 & M_1 & \mathbf{0} \\ X_2 & \mathbf{0} & M_2 \end{array}\right)$, where $X_1 \cap X_2 = \emptyset$,

- deletion: $\left(\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array}\right) /\!\!/ \eta_a = \left(\begin{array}{c|c} \omega & S \\ \hline S & \Xi \end{array}\right)_{t_a \to 1}$,

- renaming: $\left(\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array}\right) /\!\!/ \sigma^a_b = \left(\begin{array}{c|cc} \omega & b & S \\ \hline b & \alpha & \theta \\ S & \phi & \Xi \end{array}\right)_{t_a \to t_b}$,

- stitching:

$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right) /\!\!/ m^{a,b}_c = \left(\begin{array}{c|cc} (1-\gamma)\omega & c & S \\ \hline c & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array}\right)_{t_a, t_b \to t_c} . \tag{3.1}$$

Here by **0** we denote a matrix of zeros with size depending on the context.

Lemma 3.1. *The above operations are well-defined, i.e. it makes sense to divide by* $1-\gamma$ *and when all the variables are set to 1, we obtain the identity matrix and* $\omega = 1$.

Proof. The only non-trivial operation to check is the stitching operation. First of all, observe that since $(1-\gamma)|_{t_i \to 1} = 1$, it makes sense to divide by $1-\gamma$. It also follows that $(1-\gamma)\omega|_{t_i \to 1} = \omega|_{t_i \to 1} = 1$. Now when all the variables are set to 1, we have $\alpha = \delta = 1$, $\beta, \theta, \gamma, \epsilon, \phi, \psi$ all vanish, and Ξ is the identity matrix. Plugging these into the matrix after stitching we obtain the identity matrix, as required. ☐

The stitching formula may seem mysterious at first. Nevertheless it has an elementary interpretation in terms of linear algebra. Specifically we can think of the matrix part of

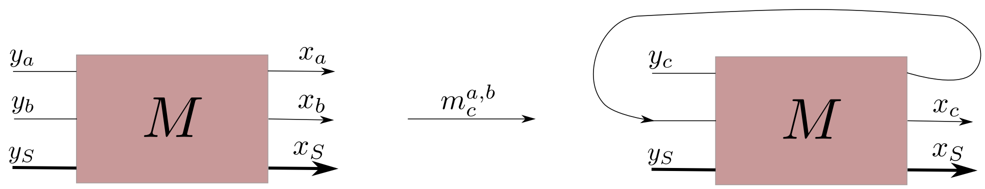
$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right)$$

as an operator with input strands labeled by y_a, y_b, y_S and output strands labeled by x_a, x_b, x_S. In other words, the strands are labeled by $\{a, b\} \cup S$. We label the tail of strand a by y_a and the head of strand a by x_a.

In the language of linear algebra we have a system of equations

$$\begin{cases} y_a = \alpha x_a + \beta x_b + \theta x_S, \\ y_b = \gamma x_a + \delta x_b + \epsilon x_S, \\ y_S = \phi x_a + \psi x_b + \Xi x_S. \end{cases}$$

Now the stitching operation $m_c^{a,b}$ can be interpreted as connecting the head of strand a to the tail of strand b and labeling the resulting strand c



In terms of linear algebra we obtain the extra equation $y_b = x_a$. Plugging it in the second equation we obtain

$$x_a = \gamma x_a + \delta x_b + \epsilon x_S, \quad \text{i.e.} \quad x_a = \frac{\delta}{1-\gamma} x_b + \frac{\epsilon}{1-\gamma} x_S.$$

It follows that

$$\begin{cases} y_a = \left(\beta + \frac{\alpha\delta}{1-\gamma}\right) x_b + \left(\theta + \frac{\alpha\epsilon}{1-\gamma}\right) x_S \\ y_S = \left(\psi + \frac{\delta\phi}{1-\gamma}\right) x_b + \left(\Xi + \frac{\phi\epsilon}{1-\gamma}\right) x_S \end{cases}$$

Finally, since the new strand is labeled c, we need to rename the variables on strand a and strand b, namely substituting t_a and t_b by t_c, and changing y_a to y_c, x_b to x_c:

$$\begin{cases} y_c = \left(\beta + \frac{\alpha\delta}{1-\gamma}\right)_{t_a, t_b \to t_c} x_c + \left(\theta + \frac{\alpha\epsilon}{1-\gamma}\right)_{t_a, t_b \to t_c} x_S \\ y_S = \left(\psi + \frac{\delta\phi}{1-\gamma}\right)_{t_a, t_b \to t_c} x_c + \left(\Xi + \frac{\phi\epsilon}{1-\gamma}\right)_{t_a, t_b \to t_c} x_S \end{cases}$$

which is precisely the stitching formula for the matrix part.

It will be useful to have a formula for stitching many strands as the same time, provided we do not stitch the same strand to itself (for instance we cannot stitch strand 1 to strand 2 and then strand 2 to strand 1). A priori the order in which these stitching operations are carried out matters. For instance, stitching strand 1 to strand 2 and then strand 2 to strand 3 may not be the same as stitching strand 2 to strand 3 and then strand 1 to strand 2.

Consider an element ζ of Γ^X given by

$$\zeta = \left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right)$$

and given two vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ where $a_i, b_j \in X$. Suppose we want to stitch strand a_1 to b_1, strand a_2 to b_2, …, strand a_n to b_n in that order, where a_i and b_j are chosen in such a way to ensure that we do not stitch the same strand to itself. We denote these operations simply by $m^{\boldsymbol{a},\boldsymbol{b}}$ (note that if we relabel the strands then we also have to rename the variables t_i accordingly, but this can be carried out easily after all the stitchings are done). In order to describe the result it is convenient to rearrange the matrix part as follows. Let $\boldsymbol{c} = X \setminus \boldsymbol{a}$ and $\boldsymbol{d} = X \setminus \boldsymbol{b}$, we can then rewrite ζ as

$$\left(\begin{array}{c|cc} \omega & \boldsymbol{a} & \boldsymbol{c} \\ \hline \boldsymbol{b} & \gamma & \epsilon \\ \boldsymbol{d} & \phi & \Xi \end{array}\right).$$

We record the stitching-in-bulk formula in the next proposition.

Proposition 3.1 (Stitching in Bulk). *With the above data we have*

$$\left(\begin{array}{c|cc} \omega & \boldsymbol{a} & \boldsymbol{c} \\ \hline \boldsymbol{b} & \gamma & \epsilon \\ \boldsymbol{d} & \phi & \Xi \end{array}\right) \xrightarrow{m^{\boldsymbol{a},\boldsymbol{b}}} \left(\begin{array}{c|c} \omega \det(I-\gamma) & \boldsymbol{c} \\ \hline \boldsymbol{d} & \Xi + \phi(I-\gamma)^{-1}\epsilon \end{array}\right), \tag{3.2}$$

where I denotes the $n \times n$ identity matrix.

Note that if we relabel the strands after doing the stitching operations then we need to relabel the vectors $\boldsymbol{c}$, $\boldsymbol{d}$ and rename the variables. But we can just do this at the end.

Proof. We will prove the formula by induction on the number n of strands being stitched. When $n = 1$, let us show that we recover the stitching formula (3.1). Suppose we want to stitch strand a to strand b, we first rearrange the matrix part as follows.

$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline b & \gamma & \delta & \epsilon \\ a & \alpha & \beta & \theta \\ S & \phi & \psi & \Xi \end{array}\right).$$

Then under $m^{a,b}$ we have $\omega \mapsto \omega(1-\gamma)$, and

$$\left(\begin{array}{c|cc} \omega(1-\gamma) & b & S \\ \hline \begin{array}{c} a \\ S \end{array} & \multicolumn{2}{c}{\begin{pmatrix} \beta & \theta \\ \psi & \Xi \end{pmatrix} + \begin{pmatrix} \alpha \\ \phi \end{pmatrix} (1-\gamma)^{-1} \begin{pmatrix} \delta & \epsilon \end{pmatrix}} \end{array}\right) = \left(\begin{array}{c|cc} \omega(1-\gamma) & b & S \\ \hline a & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array}\right).$$

Then if we label the resulting strand c we need to make the substitutions $a \to c$, $b \to c$, $t_a \to t_c$, $t_b \to t_c$, which yields the stitching formula (3.1). Now for the induction step, we write $\boldsymbol{a} = (\boldsymbol{a}', a)$ and $\boldsymbol{b} = (\boldsymbol{b}', b)$, then from the inductive hypothesis $m^{\boldsymbol{a}',\boldsymbol{b}'}$ is given by

$$\left(\begin{array}{c|ccc} \omega & \boldsymbol{a}' & a & \boldsymbol{c} \\ \hline \boldsymbol{b}' & \gamma_1 & \gamma_2 & \epsilon_1 \\ b & \gamma_3 & \gamma_4 & \epsilon_2 \\ \boldsymbol{d} & \phi_1 & \phi_2 & \Xi \end{array}\right) \xrightarrow{m^{\boldsymbol{a}',\boldsymbol{b}'}} \left(\begin{array}{c|cc} \omega \det(I-\gamma_1) & a & \boldsymbol{c} \\ \hline b & \gamma_4 + \gamma_3(I-\gamma_1)^{-1}\gamma_2 & \epsilon_2 + \gamma_3(I-\gamma_1)^{-1}\epsilon_1 \\ \boldsymbol{d} & \phi_2 + \phi_1(I-\gamma_1)^{-1}\gamma_2 & \Xi + \phi_1(I-\gamma_1)^{-1}\epsilon_1 \end{array}\right).$$

To obtain $m^{\boldsymbol{a},\boldsymbol{b}}$ we stitch strand a to strand b using formula (3.1) and the result is

$$\left(\begin{array}{c|c} \omega(1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2)\det(I-\gamma_1) & \boldsymbol{c} \\ \hline \boldsymbol{d} & \Xi + \phi_1(I-\gamma_1)^{-1}\epsilon_1 + \frac{(\phi_2+\phi_1(I-\gamma_1)^{-1}\gamma_2)(\epsilon_2+\gamma_3(I-\gamma_1)^{-1}\epsilon_1)}{1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2} \end{array}\right).$$

To finish the induction step we need to show that the above is the same as

$$\left(\begin{array}{c|c} \omega \det\left[I - \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix}\right] & \boldsymbol{c} \\ \hline \boldsymbol{d} & \Xi + \begin{pmatrix}\phi_1 & \phi_2\end{pmatrix}\left(I - \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix}\right)^{-1}\begin{pmatrix}\epsilon_1\\ \epsilon_2\end{pmatrix}\end{array}\right).$$

For that we record the following elementary result from linear algebra (see [Pow11])

Lemma 3.2. *Consider the block matrix*

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}$$

where A and D are square matrices not necessarily of the same size and D is invertible. Then

$$\det\begin{pmatrix} A & B\\ C & D\end{pmatrix} = \det(A - BD^{-1}C)\det(D).$$

Proof of lemma. It is easy to check that

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} I & 0\\ -D^{-1}C & I\end{pmatrix} = \begin{pmatrix} A - BD^{-1}C & B\\ 0 & D\end{pmatrix}.$$

Now taking the determinant of both sides and using the fact that the determinant of a block triangular matrix is the product of the determinants of the diagonal blocks (one can prove this by induction) we obtain the required identity. □

Back to our proof, from the above lemma we have that

$$\begin{aligned}\det\left[I - \begin{pmatrix}\gamma_1 & \gamma_2\\ \gamma_3 & \gamma_4\end{pmatrix}\right] &= \det\begin{pmatrix} I-\gamma_1 & -\gamma_2\\ -\gamma_3 & 1-\gamma_4\end{pmatrix} = \det\begin{pmatrix} 1-\gamma_4 & -\gamma_3\\ -\gamma_2 & I-\gamma_1\end{pmatrix}\\ &= \det(1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2)\det(I-\gamma_1),\end{aligned}$$

which agrees with the scalar part. Now for the matrix part, we have to show that

$$\begin{pmatrix} I-\gamma_1 & -\gamma_2\\ -\gamma_3 & 1-\gamma_4\end{pmatrix}^{-1} = \begin{pmatrix} (I-\gamma_1)^{-1} + \frac{(I-\gamma_1)^{-1}\gamma_2\gamma_3(I-\gamma_1)^{-1}}{1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2} & \frac{(I-\gamma_1)^{-1}\gamma_2}{1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2}\\ \frac{\gamma_3(I-\gamma_1)^{-1}}{1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2} & \frac{1}{1-\gamma_4-\gamma_3(I-\gamma_1)^{-1}\gamma_2}\end{pmatrix},$$

which we can verify directly by computing the products of matrices. □

Remark 3.1. Let us present a shortcut to arrive at formula (3.2). Using the linear algebra interpretation the matrix part gives us the system of equations

$$\begin{cases} y_{\boldsymbol{b}} = \gamma x_{\boldsymbol{a}} + \epsilon x_{\boldsymbol{c}},\\ y_{\boldsymbol{d}} = \phi x_{\boldsymbol{a}} + \Xi x_{\boldsymbol{c}}.\end{cases}$$

Now the stitching instruction yields the equation $y_{\boldsymbol{b}} = x_{\boldsymbol{a}}$. Thus the first equation becomes

$$x_{\boldsymbol{a}} = \gamma x_{\boldsymbol{a}} + \epsilon x_{\boldsymbol{c}}, \quad \text{or} \quad x_{\boldsymbol{a}} = (I-\gamma)^{-1}\epsilon x_{\boldsymbol{c}}.$$

Plugging it in the second equation we obtain

$$y_{\boldsymbol{d}} = (\Xi + \phi(I-\gamma)^{-1}\epsilon)x_{\boldsymbol{c}},$$

as required.

As a corollary, when $a_i \neq b_j$ for $1 \leq i, j \leq n$, we have the following stitching formula

$$\left(\begin{array}{c|ccc} \omega & \boldsymbol{a} & \boldsymbol{b} & S \\ \hline \boldsymbol{a} & \alpha & \beta & \theta \\ \boldsymbol{b} & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right) \xrightarrow{m^{\boldsymbol{a},\boldsymbol{b}}_{\boldsymbol{c}}} \left(\begin{array}{c|cc} \det(I-\gamma)\omega & \boldsymbol{c} & S \\ \hline \boldsymbol{c} & \beta + \alpha(I-\gamma)^{-1}\delta & \theta + \alpha(I-\gamma)^{-1}\epsilon \\ S & \psi + \phi(I-\gamma)^{-1}\delta & \Xi + \phi(I-\gamma)^{-1}\epsilon \end{array}\right)_{t_{\boldsymbol{a}}, t_{\boldsymbol{b}} \to t_{\boldsymbol{c}}}. \tag{3.3}$$

Here $t_{\boldsymbol{a}} = (t_{a_1}, \dots, t_{a_n})$ and similarly for $t_{\boldsymbol{b}}$ and $t_{\boldsymbol{c}}$. This is a straightforward application of formula (3.2) and we leave the details to the readers.

Proposition 3.2. *The order in which one performs the stitching operations does not matter.*

Proof. From formula (3.2) we see that switching two stitching operations amounts to switching the corresponding labels in $\boldsymbol{b}$ and $\boldsymbol{a}$, which in turn will switch the corresponding columns of γ and ϵ and the corresponding rows of γ and ϕ. The matrix Ξ stays unchanged. Therefore

$$\Xi + \phi(I-\gamma)^{-1}\epsilon$$

will be invariant. For the scalar part, since we switch the rows and columns of γ of the same indices, we preserve I and the determinant is unchanged. (One can make the argument more precise using permutation matrices.) □

Let us illustrate the above proposition in a concrete case to show meta-associativity. Let

$$\zeta = \left(\begin{array}{c|cccc} \omega & 1 & 2 & 3 & S \\ \hline 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ 2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ 3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{array}\right).$$

To stitch strand 1 to strand 2 and strand 2 to strand 3 we rewrite ζ as

$$\left(\begin{array}{c|cccc} \omega & 1 & 2 & 3 & S \\ \hline 2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2 \\ 3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3 \\ 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1 \\ S & \phi_1 & \phi_2 & \phi_3 & \Xi \end{array}\right).$$

Then $\zeta \mathbin{/\!/} m_1^{1,2} \mathbin{/\!/} m_1^{1,3}$ is given by

$$\left(\begin{array}{c|c} \omega \det\begin{pmatrix} 1-\alpha_{21} & -\alpha_{22} \\ -\alpha_{31} & 1-\alpha_{32} \end{pmatrix} & \begin{matrix} 1 & \qquad\qquad\qquad S \end{matrix} \\ \hline \begin{matrix} 1 \\ S \end{matrix} & \begin{pmatrix} \alpha_{13} & \theta_1 \\ \phi_3 & \Xi \end{pmatrix} + \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} 1-\alpha_{21} & -\alpha_{22} \\ -\alpha_{31} & 1-\alpha_{32} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{23} & \theta_2 \\ \alpha_{33} & \theta_3 \end{pmatrix} \end{array}\right)_{t_2,t_3\to t_1}.$$

Similarly $\zeta \mathbin{/\!/} m_2^{2,3} \mathbin{/\!/} m_1^{1,2}$ is given by

$$\left(\begin{array}{c|c} \omega \det\begin{pmatrix} 1-\alpha_{32} & -\alpha_{31} \\ -\alpha_{22} & 1-\alpha_{21} \end{pmatrix} & \begin{matrix} 1 & \qquad\qquad\qquad S \end{matrix} \\ \hline \begin{matrix} 1 \\ S \end{matrix} & \begin{pmatrix} \alpha_{13} & \theta_1 \\ \phi_3 & \Xi \end{pmatrix} + \begin{pmatrix} \alpha_{12} & \alpha_{11} \\ \phi_2 & \phi_1 \end{pmatrix} \begin{pmatrix} 1-\alpha_{32} & -\alpha_{31} \\ -\alpha_{22} & 1-\alpha_{21} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{33} & \theta_3 \\ \alpha_{23} & \theta_2 \end{pmatrix} \end{array}\right)_{t_2,t_3\to t_1}.$$

Observe that

$$\begin{aligned} &\begin{pmatrix} \alpha_{12} & \alpha_{11} \\ \phi_2 & \phi_1 \end{pmatrix} \begin{pmatrix} 1-\alpha_{32} & -\alpha_{31} \\ -\alpha_{22} & 1-\alpha_{21} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{33} & \theta_3 \\ \alpha_{23} & \theta_2 \end{pmatrix} \\ =&\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1-\alpha_{32} & -\alpha_{31} \\ -\alpha_{22} & 1-\alpha_{21} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_{23} & \theta_2 \\ \alpha_{33} & \theta_3 \end{pmatrix} \\ =&\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \phi_1 & \phi_2 \end{pmatrix} \begin{pmatrix} 1-\alpha_{21} & -\alpha_{22} \\ -\alpha_{31} & 1-\alpha_{32} \end{pmatrix}^{-1} \begin{pmatrix} \alpha_{23} & \theta_2 \\ \alpha_{33} & \theta_3 \end{pmatrix}. \end{aligned}$$

Thus it follows that

$$\zeta \mathbin{/\!/} m_1^{1,2} \mathbin{/\!/} m_1^{1,3} = \zeta \mathbin{/\!/} m_2^{2,3} \mathbin{/\!/} m_1^{1,2}.$$

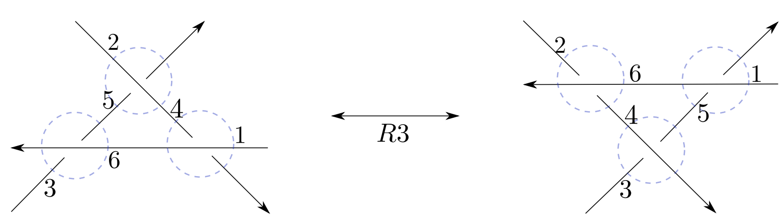
This establishes the meta-associativity property. The other axioms of a meta-monoid are straightforward to verify. Thus Γ is indeed a meta-monoid. The meta-monoid Γ is called the *Gassner Calculus* or Γ-*Calculus*, for reasons which will be clear below (Proposition 3.7).

Our interpretation of stitching suggests a relationship between the meta-monoids $\mathcal{W}$ of w-tangles and Γ-calculus. From Proposition 2.1 in order to define a meta-monoid homomorphism $\varphi : \mathcal{W} \to \Gamma$ we only need to specify the images of the crossings in Γ-calculus and verify the relations $R2$, $R3$ and OC.

Proposition 3.3. *There is a meta-monoid homomorphism φ from the meta-monoid $\mathcal{W}$ of w-tangles to Γ-calculus given by*

$$\varphi(R^{\pm}_{a,b}) = \left(\begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a^{\pm 1} \\ b & 0 & t_a^{\pm} \end{array}\right).$$

Proof. Let us check the Reidemeister $R3$ move and leave the other relations as exercises



In the language of meta-monoids we need to show that

$$\varphi(R^+_{1,4}R^+_{2,5}R^-_{6,3}) \mathbin{/\!/} m_1^{1,6} \mathbin{/\!/} m_2^{2,4} \mathbin{/\!/} m_3^{3,5} = \varphi(R^-_{1,5}R^+_{4,3}R^+_{6,2}) \mathbin{/\!/} m_1^{1,6} \mathbin{/\!/} m_2^{2,4} \mathbin{/\!/} m_3^{3,5}.$$

Let us first compute the left hand side. The image of $R^+_{1,4}R^+_{2,5}R^-_{6,3}$ under φ is

$$\left(\begin{array}{c|cccccc} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 1-t_1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1-t_2 & 0 \\ 3 & 0 & 0 & t_6^{-1} & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & t_1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & t_2 & 0 \\ 6 & 0 & 0 & 1-t_6^{-1} & 0 & 0 & 1 \end{array}\right).$$

To perform all the stitching operations at once we rearrange the rows and columns as follows.

$$\left(\begin{array}{c|cccccc} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 6 & 0 & 0 & 1-t_6^{-1} & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & t_1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & t_2 & 0 \\ 1 & 1 & 0 & 0 & 1-t_1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1-t_2 & 0 \\ 3 & 0 & 0 & t_6^{-1} & 0 & 0 & 0 \end{array}\right).$$

Then according to formula (3.2), the left hand side is given by

$$\left(\begin{array}{c|ccc} 1 & 4 & 5 & 6 \\ \hline 1 & 1-t_1 & t_2-\frac{t_2}{t_6} & 1 \\ 2 & t_1 & 1-t_2 & 0 \\ 3 & 0 & \frac{t_2}{t_6} & 0 \end{array}\right).$$

According to the relabeling we relabel $4 \to 2$, $5 \to 3$, $6 \to 1$ and $t_4 \to t_2$, $t_5 \to t_3$, $t_6 \to t_1$ to obtain

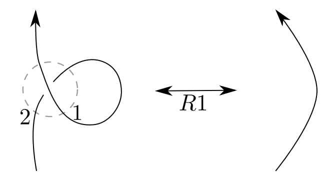
$$\left(\begin{array}{c|ccc} 1 & 2 & 3 & 1 \\ \hline 1 & 1-t_1 & t_2-\frac{t_2}{t_1} & 1 \\ 2 & t_1 & 1-t_2 & 0 \\ 3 & 0 & \frac{t_2}{t_1} & 0 \end{array}\right).$$

Finally we rearrange the columns

$$\left(\begin{array}{c|ccc} 1 & 1 & 2 & 3 \\ \hline 1 & 1 & 1-t_1 & t_2-\frac{t_2}{t_1} \\ 2 & 0 & t_1 & 1-t_2 \\ 3 & 0 & 0 & \frac{t_2}{t_1} \end{array}\right).$$

We leave it as an exercise to show that the right hand side also yields the same result. □

Remark 3.2. Note that the Reidemeister move $R1$



does not hold because the left hand side is given by

$$\varphi(R^-_{1,2}) \,/\!/\, m_1^{2,1} = \left(\begin{array}{c|c} t_1^{-1} & 1 \\ \hline 1 & 1 \end{array}\right)$$

whereas the right hand side is trivial.

Mathematica®. One advantage of Γ-calculus is its easy implementation on a computer. Our implementation is done through Mathematica. A reader with Mathematica can just get the entire notebook from here and run it directly. The version of Γ-calculus that we present is a slightly modified form of the original program, which can be found here. Let us briefly go through the program. First we write a subroutine that will display Γ-calculus in a nice format. This is mostly for aesthetic purpose.

```
ΓCollect[Γ[ω_, λ_]] := Γ[Simplify[ω],
   Collect[λ, x_, Collect[#, y_, Factor] &]];
Format[Γ[ω_, λ_]] := Module[{S, M},
   S = Union@Cases[Γ[ω, λ], (x | y)_a_ :→ a, ∞];
   M = Outer[Factor[∂_{x_#1 y_#2} λ] &, S, S];
   M = Prepend[M, y_# & /@ S] // Transpose;
   M = Prepend[M, Prepend[x_# & /@ S, ω]];
   M // MatrixForm];
```

The subroutine Γ takes as input a rational function ω and a matrix λ. Here λ is given in the form

$$\lambda = \{y_{\boldsymbol{a}}\} \cdot \text{matrix} \cdot \{x_{\boldsymbol{a}}\}$$

where the vector $\boldsymbol{a}$ is the labels of the strands. Note that here we use y to label the rows and x to label the columns. So for instance, the following input

```
Γ[ω, {y_a, y_b}.(g11 g12
                 g21 g22).{x_a, x_b}]
```

produces

$$\begin{pmatrix} \omega & x_a & x_b \\ y_a & g_{11} & g_{12} \\ y_b & g_{21} & g_{22} \end{pmatrix}$$

Now we include the main bulk of the program, which is the subroutine that executes stitching together with the definitions of the crossings.

$$\Gamma\ /\!:\ \Gamma[\omega1\_,\ \lambda1\_]\ \Gamma[\omega2\_,\ \lambda2\_]\ :=\ \Gamma[\omega1 * \omega2,\ \lambda1 + \lambda2];$$

$$\mathbf{m}_{a\_,e\_\to c\_}[\Gamma[\omega\_,\ \lambda\_]]\ :=\ \mathbf{Module}\Big[\{\alpha,\beta,\gamma,\delta,\theta,\epsilon,\phi,\psi,\Xi,\mu\},$$

$$\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix} = \begin{pmatrix}\partial_{y_a,x_a}\lambda & \partial_{y_a,x_e}\lambda & \partial_{y_a}\lambda\\ \partial_{y_e,x_a}\lambda & \partial_{y_e,x_e}\lambda & \partial_{y_e}\lambda\\ \partial_{x_a}\lambda & \partial_{x_e}\lambda & \lambda\end{pmatrix}\ /.\ (\mathbf{y}\mid\mathbf{x})_{a\mid e}\to 0;$$

$$\Gamma\Big[(\mu = 1-\gamma)\ \omega,\ \{\mathbf{y}_c,\ 1\}.\begin{pmatrix}\beta+\alpha\,\delta/\mu & \theta+\alpha\,\epsilon/\mu\\ \psi+\delta\,\phi/\mu & \Xi+\phi\,\epsilon/\mu\end{pmatrix}.\{\mathbf{x}_c,\ 1\}\Big]$$

$$/.\ \{\mathbf{t}_a\to\mathbf{t}_c,\ \mathbf{t}_e\to\mathbf{t}_c\}\ //\ \Gamma\mathbf{Collect}\Big];$$

$$\mathbf{R}^{+}_{a\_,e\_}\ :=\ \Gamma\Big[1,\ \{\mathbf{y}_a,\ \mathbf{y}_e\}.\begin{pmatrix}1 & 1-\mathbf{t}_a\\ 0 & \mathbf{t}_a\end{pmatrix}.\{\mathbf{x}_a,\ \mathbf{x}_e\}\Big];$$

$$\mathbf{R}^{-}_{a\_,e\_}\ :=\ \mathbf{R}^{+}_{a,e}\ /.\ \mathbf{t}_a\to 1/\mathbf{t}_a;$$

Let us check the meta-associativity condition. Meta-associativity involves three strands in a tangle, so we input a matrix with a 3×3 minor singled out together with the meta-associativity equation

$$\zeta = \Gamma\Big[\omega,\ \{\mathbf{y}_1,\ \mathbf{y}_2,\ \mathbf{y}_3,\ \mathbf{y}_S\}.\begin{pmatrix}\alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}.\{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_S\}\Big]$$

$$(\zeta\ //\ \mathbf{m}_{1,2\to1}\ //\ \mathbf{m}_{1,3\to1})\ ==\ (\zeta\ //\ \mathbf{m}_{2,3\to2}\ //\ \mathbf{m}_{1,2\to1})$$

The output is

$$\begin{pmatrix}\omega & x_1 & x_2 & x_3 & x_S\\ y_1 & \alpha_{11} & \alpha_{12} & \alpha_{13} & \theta_1\\ y_2 & \alpha_{21} & \alpha_{22} & \alpha_{23} & \theta_2\\ y_3 & \alpha_{31} & \alpha_{32} & \alpha_{33} & \theta_3\\ y_S & \phi_1 & \phi_2 & \phi_3 & \Xi\end{pmatrix}$$

True

as expected. Next we check the Reidemeister III relation. Its left hand side is

$$\mathbf{R}^{+}_{1,4}\ \mathbf{R}^{+}_{2,5}\ \mathbf{R}^{-}_{6,3}\ //\ \mathbf{m}_{1,6\to1}\ //\ \mathbf{m}_{2,4\to2}\ //\ \mathbf{m}_{3,5\to3}$$

and the output is

$$\begin{pmatrix}1 & x_1 & x_2 & x_3\\ y_1 & 1 & 1-t_1 & \frac{(-1+t_1)\,t_2}{t_1}\\ y_2 & 0 & t_1 & 1-t_2\\ y_3 & 0 & 0 & \frac{t_2}{t_1}\end{pmatrix}$$

Its right hand side is

$$\mathbf{R}^{-}_{1,4}\ \mathbf{R}^{+}_{5,2}\ \mathbf{R}^{+}_{6,3}\ //\ \mathbf{m}_{1,5\to1}\ //\ \mathbf{m}_{2,6\to2}\ //\ \mathbf{m}_{3,4\to3}$$

and the output is

$$\begin{pmatrix}1 & x_1 & x_2 & x_3\\ y_1 & 1 & 1-t_1 & \frac{(-1+t_1)\,t_2}{t_1}\\ y_2 & 0 & t_1 & 1-t_2\\ y_3 & 0 & 0 & \frac{t_2}{t_1}\end{pmatrix}$$

as expected. For the Reidemeister II move we look at

$$R^{+}_{i,j}\,R^{-}_{k,l}\ //\ m_{i,k\to i}\ //\ m_{j,l\to j}$$

which yields

$$\begin{pmatrix} 1 & x_i & x_j \\ y_i & 1 & 0 \\ y_j & 0 & 1 \end{pmatrix}$$

as required. For the OC relation we want to verify

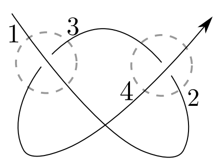
$$R^{+}_{4,2}\,R^{+}_{1,3}\ //\ m_{1,4\to 1} == R^{+}_{4,3}\,R^{+}_{1,2}\ //\ m_{1,4\to 1}$$

Both sides yield

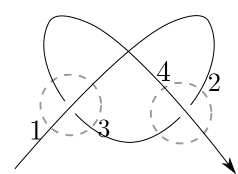
$$\begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ y_1 & 1 & 1-t_1 & 1-t_1 \\ y_2 & 0 & t_1 & 0 \\ y_3 & 0 & 0 & t_1 \end{pmatrix}$$

as expected.

Example 3.1. In this example we show that long w-knots and closed w-knots are not equivalent. Consider the long w-knots L and L' given by



L



L'

In the language of meta-monoids, L has the description

$$L = R^{-}_{1,3}R^{+}_{4,2} /\!\!/ m_1^{1,2} /\!\!/ m_1^{1,3} /\!\!/ m_1^{1,4}.$$

Then its invariant in Γ-calculus is

$$\varphi(L) = \left(\begin{array}{c|c} 2-t_1^{-1} & 1 \\ \hline 1 & 1 \end{array}\right).$$

In the language of meta-monoids, L' has the description

$$L' = R^{+}_{1,3}R^{-}_{4,2} /\!\!/ m_1^{1,2} /\!\!/ m_1^{1,3} /\!\!/ m_1^{1,4}.$$

So its invariant in Γ-calculus is

$$\varphi(L') = \left(\begin{array}{c|c} 2-t_1 & 1 \\ \hline 1 & 1 \end{array}\right).$$

Thus L and L' are not isotopic as long w-knots and are non-trivial. However when we close L and L' we obtain the trivial (closed) knot. ♣

Observe that Proposition 2.1 gives an inductive framework to prove properties for w-tangles. Namely, one first check the property for the crossings, and then show that the property still holds under disjoint union and stitching. Let us illustrate this method with an important property of w-tangles.

Proposition 3.4. *Let T be a w-tangle whose components are labeled by the set X and*

$$\varphi(T) = \left(\begin{array}{c|c} \omega & X \\ \hline X & M \end{array}\right).$$

Then the sum of the entries in each column of M is 1.

Proof. The property clearly holds for crossings and is preserved under disjoint union. So we only need to show that it is invariant under stitching:

$$\left(\begin{array}{c|ccc} \omega & a & b & S \\ \hline a & \alpha & \beta & \theta \\ b & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right) \xrightarrow{m_c^{a,b}} \left(\begin{array}{c|cc} (1-\gamma)\omega & c & S \\ \hline c & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array}\right)_{t_a, t_b \to t_c}.$$

Assume that the property is true for the matrix on the left, i.e.

$$\begin{cases} \alpha + \gamma + \langle \phi \rangle = 1 \\ \beta + \delta + \langle \psi \rangle = 1 \\ \theta + \epsilon + \langle \Xi \rangle = \mathbf{1}, \end{cases}$$

where $\mathbf{1}$ denotes a row vector whose each entry is 1 and $\langle \boldsymbol{c} \rangle$ of a column vector $\boldsymbol{c}$ means taking the sum of the entries. For the case of Ξ, we apply $\langle \rangle$ to each column to obtain a row vector. Then we have

$$\beta + \frac{\delta\alpha}{1-\gamma} + \langle \psi \rangle + \frac{\delta \langle \phi \rangle}{1-\gamma} = 1 - \delta + \frac{\delta(\alpha + \langle \phi \rangle)}{1-\gamma} = 1 - \delta + \frac{\delta(1-\gamma)}{1-\gamma} = 1,$$

and

$$\theta + \frac{\alpha\epsilon}{1-\gamma} + \langle \Xi \rangle + \frac{\langle \phi \rangle \epsilon}{1-\gamma} = \mathbf{1} - \epsilon + \frac{(\alpha + \langle \phi \rangle)\epsilon}{1-\gamma} = \mathbf{1} - \epsilon + \frac{(1-\gamma)\epsilon}{1-\gamma} = \mathbf{1},$$

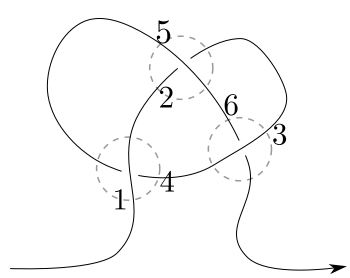
as required. □

As a corollary we have that when K is a long w-knot the matrix part is 1, so only the scalar part is interesting, i.e.

$$\varphi(K) = \left(\begin{array}{c|c} \omega_K & 1 \\ \hline 1 & 1 \end{array}\right),$$

where we denote the scalar part by ω_K.

Example 3.2. Let us look at the long trefoil K



Its meta-monoid description is given by

$$R^+_{1,4}R^+_{5,2}R^+_{3,6} \mathbin{/\!\!/} m_1^{1,2} \mathbin{/\!\!/} m_1^{1,3} \mathbin{/\!\!/} m_1^{1,4} \mathbin{/\!\!/} m_1^{1,5} \mathbin{/\!\!/} m_1^{1,6}.$$

The image of $R^+_{1,4}R^+_{5,2}R^+_{3,6}$ under φ is

$$\left(\begin{array}{c|cccccc} 1 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 1-t_1 & 0 & 0 \\ 2 & 0 & t_5 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1-t_3 \\ 4 & 0 & 0 & 0 & t_1 & 0 & 0 \\ 5 & 0 & 1-t_5 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & t_3 \end{array}\right).$$

After we perform all the stitching operations the matrix part is 1 and the scalar part by formula (3.2) is the determinant of the matrix $I-\gamma$, where γ is obtained by removing the first row and the last column of the above matrix, i.e.

$$\omega_K = \det \begin{pmatrix} 1 & -t_5 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -t_1 & 0 \\ 0 & t_5-1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{t_5 \to t_1} = 1 - t + t^2,$$

which one recognizes to be the Alexander polynomial of the trefoil (Proposition 3.8). ♣

As another application of the stitching-in-bulk formula (3.2), observe that a priori, the scalar ω and the matrix entries are rational functions. However, it turns out that for a w-tangle ω is a Laurent polynomial, as shown in the following proposition.

Proposition 3.5. *Let T be a w-tangle with scalar part ω and matrix part M, then ω is a Laurent polynomial and ωM is a matrix whose entries are Laurent polynomials.*

Proof. One can obtain T starting with a collection of crossings and then stitching all these crossings at once using formula (3.2). Observe that when we take the disjoint union of crossings, the matrix part consists of Laurent polynomials (since each crossing is) and the scalar part is 1. Then after stitching the scalar part becomes $\det(I-\gamma)$ where γ is specified by the stitching instruction. Since γ consists of Laurent polynomials, $\det(I-\gamma)$ is a Laurent polynomials, thus establishes the polynomiality of ω. Now for the other property, we look at

$$\omega \det(I-\gamma)(\Xi + \phi(I-\gamma)^{-1}\epsilon).$$

All the matrices have Laurent polynomial entries, except for $(I-\gamma)^{-1}$. Recall that $(I-\gamma)^{-1}$ can be computed by dividing its adjugate (which are Laurent polynomials) by $\det(I-\gamma)$. Therefore multiplying with $\det(I-\gamma)$ removes the denominator, and so the resulting entries are Laurent polynomials. □

Example 3.3. Let us compute the invariant for the tangle T given by.

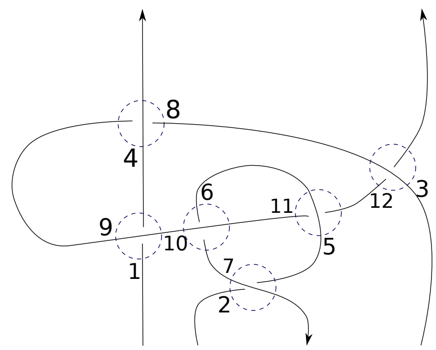


Figure 3.1: A tangle.

As a disjoint union of crossings, T is given as follows

$$R^+_{7,2}R^-_{10,6}R^-_{5,11}R^-_{3,12}R^+_{4,8}R^+_{9,1} \,/\!\!/\, m_1^{1,4} \,/\!\!/\, m_2^{2,5} \,/\!\!/\, m_2^{2,6} \,/\!\!/\, m_2^{2,7} \,/\!\!/\, m_3^{3,8} \,/\!\!/\, m_3^{3,9} \,/\!\!/\, m_3^{3,10} \,/\!\!/\, m_3^{3,11} \,/\!\!/\, m_3^{3,12}.$$

Using Mathematica we obtain its invariant in Γ-calculus:

$$\left(\begin{array}{c|ccc} \left(\frac{t_2-1}{t_3}+1\right)(t_3-t_1(t_3-1)) & 1 & 2 & 3 \\ \hline 1 & -\frac{t_3}{t_3t_1-t_1-t_3} & \frac{(t_1-1)(t_3-1)t_3}{(t_2+t_3-1)(t_3t_1-t_1-t_3)} & -\frac{(t_1-1)(t_3t_2-t_2-2t_3+1)}{(t_2+t_3-1)(t_3t_1-t_1-t_3)} \\ 2 & 0 & \frac{t_2}{t_2+t_3-1} & \frac{t_2-1}{t_2+t_3-1} \\ 3 & \frac{t_1(t_3-1)}{t_3t_1-t_1-t_3} & -\frac{t_1(t_3-1)}{(t_2+t_3-1)(t_3t_1-t_1-t_3)} & \frac{t_1t_3^2-t_3^2-3t_1t_3+t_1t_2t_3-t_2t_3+2t_3+t_1-t_1t_2+t_2-1}{(t_2+t_3-1)(t_3t_1-t_1-t_3)} \end{array}\right).$$

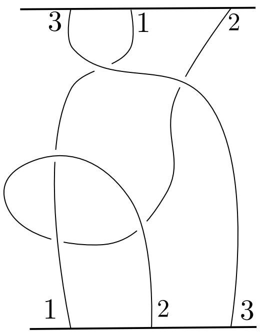
If we multiply the matrix part with the scalar part then we get

$$\left(\begin{array}{ccc} -1+t_2+t_3 & -1+t_1+t_3-t_1t_3 & t_2t_1-\frac{t_2t_1}{t_3}+\frac{t_1}{t_3}-2t_1-t_2+\frac{t_2}{t_3}-\frac{1}{t_3}+2 \\ 0 & -t_1t_2+\frac{t_1t_2}{t_3}+t_2 & -t_2t_1+\frac{t_2t_1}{t_3}-\frac{t_1}{t_3}+t_1+t_2-1 \\ -t_2t_1-t_3t_1+\frac{t_2t_1}{t_3}-\frac{t_1}{t_3}+2t_1 & t_1-\frac{t_1}{t_3} & -t_2t_1-t_3t_1+\frac{t_2t_1}{t_3}-\frac{t_1}{t_3}+3t_1+t_2+t_3-\frac{t_2}{t_3}+\frac{1}{t_3}-2 \end{array}\right).$$

The fact that each entry is a Laurent polynomial suggests that it might be possible to categorify the invariant. ♣

3.2 The Gassner Representation of String Links

In this section we restrict Γ-calculus to string links (compare with [KLW01]). Given a positive integer n, fix n points in the interior of the 2-disk $p_1, \dots, p_n$. A *string link of n components* is a smooth, proper, oriented 1-dimensional submanifold of $D^2 \times [0,1]$ homeomorphic to the disjoint union of n intervals such that the initial point of each interval coincides with some $p_i \times \{0\}$ and the endpoint coincides with $p_j \times \{1\}$. Two string links are *isotopic* (the same) if there is a smooth family of string links interpolating between the two. In our setting the string links are *labeled*, i.e. each component is labeled with an element from some set of labels X. An example of a string link is as follows.



In the figure the orientation is such that the components run from the bottom to the top of the diagram. Given a labeled string link β, the labels of the components yield a labeling of the bottom endpoints and top endpoints. Suppose that the bottom endpoints of β are labeled by $a_1, a_2, \ldots, a_n$ and the top endpoints of β are labeled by $b_1, b_2, \ldots, b_n$ (where we read the endpoints from left to right). The labeling of the endpoints yields a permutation ρ given by

$$a_i \mathbin{/\!\!/} \rho = b_i, \quad 1 \le i \le n.$$

Note that here permutations act on the right. We call ρ the *permutation induced by* β. To simplify notation, we denote

$$\boldsymbol{a}\rho := \boldsymbol{a} \mathbin{/\!\!/} \rho = (a_1\rho, a_2\rho, \ldots, a_n\rho) = (b_1, b_2, \ldots, b_n).$$

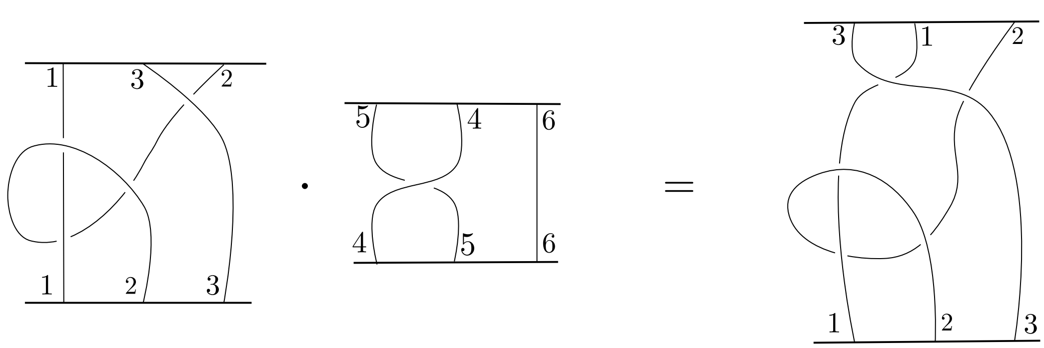
For instance in the above figure the string link induces the permutation $(1 \mapsto 3, 2 \mapsto 1, 3 \mapsto 2)$.

Correspondingly, if $\varphi(\beta)$ is the image of β in Γ-calculus, we can rearrange the columns and rows of the matrix part of $\varphi(\beta)$ as follows

$$\varphi(\beta) = \left(\begin{array}{c|ccc} \omega & a_1 & \cdots & a_n \\ \hline a_1 & & & \\ \vdots & & M & \\ a_n & & & \end{array}\right) \xrightarrow[\text{according to } \rho]{\text{permute the columns}} \left(\begin{array}{c|ccc} \omega & a_1\rho & \cdots & a_n\rho \\ \hline a_1 & & & \\ \vdots & & M^\rho & \\ a_n & & & \end{array}\right). \tag{3.4}$$

In other words column j of M^ρ is column $a_j\rho$ of M.

Let β_1 and β_2 be string links with n components. There is a *composition* or *multiplication* of string links $(\beta_1, \beta_2) \mapsto \beta_1 \cdot \beta_2$ obtained by stacking β_2 on top of β_1. Note that we also identify the labels of the top endpoints of β_1 and the labels of the bottom endpoints of β_2. So for instance in the following



we identify the label 4 with 1, 5 with 3, and 6 with 2. In terms of meta-monoids, the composition $\beta_1 \cdot \beta_2$ can be described by the sequence of stitching

$$(\beta_1\beta_2) \,/\!\!/\, m_1^{1,4} \,/\!\!/\, m_3^{3,5} \,/\!\!/\, m_2^{2,6}.$$

Let us find out the permutation induced by $\beta_1 \cdot \beta_2$. Suppose that the bottom endpoints of β_1 are labeled by $\boldsymbol{a} = (a_1, \dots, a_n)$ and the bottom endpoints of β_2 are labeled by $\boldsymbol{b} = (b_1, \dots, b_n)$, where $a_i \neq b_j$ for $1 \leq i, j \leq n$. If ρ_1 is the permutation induced by β_1 and ρ_2 is the permutation induced by β_2, then the top endpoints of β_1 are labeled by $\boldsymbol{a}\rho_1 = (a_1\rho_1, \dots, a_n\rho_1)$, and the top endpoints of β_2 are labeled by $\boldsymbol{b}\rho_2 = (b_1\rho_2, \dots, b_n\rho_2)$. In the composition $\beta_1 \cdot \beta_2$ we relabel b_i to $a_i\rho_1$. Therefore the labels of the top endpoints of $\beta_1 \cdot \beta_2$ is $a_1\rho_1\rho_2, \dots, a_n\rho_1\rho_2$. In other words, the permutation induced by $\beta_1 \cdot \beta_2$ is $\rho_1\rho_2$, where recall that in our notations $\rho_1\rho_2 = \rho_2 \circ \rho_1$.

Assume that the images of β_1 and β_2 in Γ-calculus are given by

$$\varphi(\beta_1) = \left(\begin{array}{c|c} \omega_1 & \boldsymbol{a} \\ \hline \boldsymbol{a} & M_1 \end{array}\right) \quad \text{and} \quad \varphi(\beta_2) = \left(\begin{array}{c|c} \omega_2 & \boldsymbol{b} \\ \hline \boldsymbol{b} & M_2 \end{array}\right),$$

then we have the following result.

Proposition 3.6. *In Γ-calculus, the composition $\beta_1 \cdot \beta_2$ is given by*

$$\left(\begin{array}{c|c} \omega_1\omega_2 & \boldsymbol{a}\rho_1\rho_2 \\ \hline \boldsymbol{a} & M_1^{\rho_1} M_2^{\rho_2} \end{array}\right)_{t_{\boldsymbol{b}} \to t_{\boldsymbol{a}\rho_1}}.$$

Proof. In the stitching language, the composition $\beta_1 \cdot \beta_2$ is obtained by stitching the strands $a_i\rho_1$ to the strands b_i. By formula (3.2) we obtain

$$\left(\begin{array}{c|cc} \omega_1\omega_2 & \boldsymbol{a}\rho_1 & \boldsymbol{b}\rho_2 \\ \hline \boldsymbol{b} & \mathbf{0} & M_2^{\rho_2} \\ \boldsymbol{a} & M_1^{\rho_1} & \mathbf{0} \end{array}\right) \xrightarrow{m^{\boldsymbol{a}\rho_1, \boldsymbol{b}}} \left(\begin{array}{c|c} \omega_1\omega_2 & \boldsymbol{b}\rho_2 \\ \hline \boldsymbol{a} & M_1^{\rho_1} M_2^{\rho_2} \end{array}\right).$$

Then identifying the labels b_i with the labels $a_i\rho_1$ we obtain

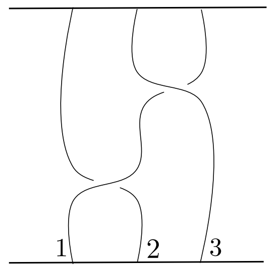
$$\left(\begin{array}{c|c} \omega_1\omega_2 & \boldsymbol{a}\rho_1\rho_2 \\ \hline \boldsymbol{a} & M_1^{\rho_1} M_2^{\rho_2} \end{array}\right)_{t_{\boldsymbol{b}} \to \boldsymbol{a}\rho_1},$$

as required. □

Now let us explain where the name Gassner calculus comes from. When β is a *labeled braid*, recall that its *Gassner representation* (see [BN14]) is given by

$$R^+_{a,b} \mapsto \begin{pmatrix} 1 - t_a & 1 \\ t_a & 0 \end{pmatrix}, \qquad R^-_{a,b} \mapsto \begin{pmatrix} 0 & t_a^{-1} \\ 1 & 1 - t_a^{-1} \end{pmatrix}$$

and extends by the identity matrix. For instance the following braid



has the Gassner representation

$$\begin{pmatrix} 1-t_1 & 1 & 0 \\ t_1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & t_3^{-1} \\ 0 & 1 & 1-t_3^{-1} \end{pmatrix} = \begin{pmatrix} 1-t_1 & 0 & t_3^{-1} \\ t_1 & 0 & 0 \\ 0 & 1 & 1-t_3^{-1} \end{pmatrix},$$

as required.

Proposition 3.7. *Let β be a labeled braid with n components and induced permutation ρ. Suppose that*

$$\varphi(\beta) = \left(\begin{array}{c|ccc} \omega & a_1 & \cdots & a_n \\ \hline a_1 & & & \\ \vdots & & M & \\ a_n & & & \end{array}\right),$$

then $\omega = 1$ and M^ρ is the Gassner representation *of β.*

Proof. We first look at the standard generators of the braid groups $\sigma_i^{\pm 1}$, $1 \le i \le n-1$. Notice that the permutation induced by each generator is a transposition. Ignoring the identity part, we have

$$\varphi(R^+_{a,b}) = \left(\begin{array}{c|cc} 1 & a & b \\ \hline a & 1 & 1-t_a \\ b & 0 & t_a \end{array}\right) \xrightarrow[\text{according to the permutation}]{\text{permute the columns}} \left(\begin{array}{c|cc} 1 & b & a \\ \hline a & 1-t_a & 1 \\ b & t_a & 0 \end{array}\right),$$

and

$$\varphi(R^-_{a,b}) = \left(\begin{array}{c|cc} 1 & b & a \\ \hline b & t_a^{-1} & 0 \\ a & 1-t_a^{-1} & 1 \end{array}\right) \xrightarrow[\text{according to the permutation}]{\text{permute the columns}} \left(\begin{array}{c|cc} 1 & a & b \\ \hline b & 0 & t_a^{-1} \\ a & 1 & 1-t_a^{-1} \end{array}\right).$$

We see that the right hand sides are exactly the Gassner representation. From Proposition 3.6, compositions of braids correspond to products of matrices. Thus M^ρ agrees with the Gassner representation of β. Furthermore, since the scalar part of each generator is 1, the scalar part of β is still 1. □

3.3 The Alexander Polynomial

In this section we relate Γ-calculus and the Alexander polynomial. First it is well-known that for a (usual) knot K, the operation of cutting K open is well-defined, i.e. the isotopy class of K as a long knot does not depend on where we cut K. The same result holds for links, provided we cut the same component. For an intuitive explanation of this fact the readers can refer to [JF13]. Note that the proof will not work if we allow virtual crossings.

Proposition 3.8. *Let K be a long knot and suppose that*

$$\varphi(K) = \left(\begin{array}{c|c} \omega & 1 \\ \hline 1 & 1 \end{array}\right).$$

Then $\omega \doteq \Delta_{\widetilde{K}}(t)$. Here $\Delta_{\widetilde{K}}(t)$ is the Alexander polynomial (see [MK99]) of $\widetilde{K}$, where $\widetilde{K}$ is the closed knot obtained by closing the open component of K trivially and $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$.

Proof. By Alexander's Theorem (see [KT08]) $\widetilde{K}$ is the closure of a braid β. Then the Alexander polynomial of $\widetilde{K}$ (see [MK99]) is given by

$$\Delta_{\widetilde{K}}(t) \doteq \det([I - f(\beta)]_1^1).$$

Here $f(\beta)$ denotes the Burau representation of β, i.e. the Gassner representation when we set all the variables to t and $[A]_i^j$ denotes the matrix obtained from A by removing the ith row and the jth column. From Proposition 3.7 we know that $f(\beta)$ agrees with (a permutation of) the matrix part of $\varphi(\beta)$. Now if we take the closure of β except for the first strand, then we obtain a long knot K_1. Proposition 3.1 says that the scalar part of K_1 is

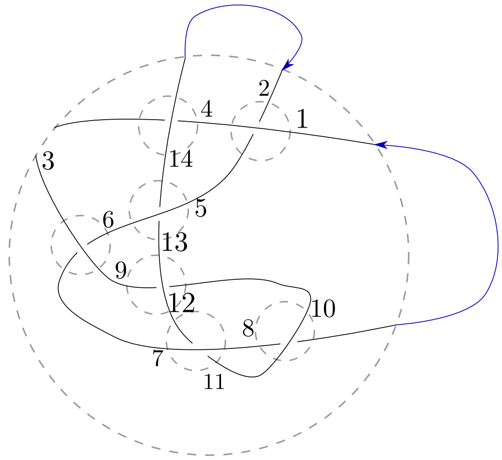
$$\det([I - f(\beta)]_1^1).$$

To finish off, we observe that K_1 is isotopic to K because they are the results of cutting $\widetilde{K}$ at two different places. Therefore the scalar parts of K_1 and K must agree since they are both invariants. In other words,

$$\omega \doteq \Delta_{\widetilde{K}}(t),$$

as required. □

Thus we see that Γ-calculus gives us an extension of the Alexander polynomial to w-tangles, which include usual tangles. In the case of one component, we obtain an invariant of long w-knots, which contains the Alexander polynomials of usual knots. (Note that our theory yields a trivial invariant for closed w-knots.) We can compute the Alexander polynomial by taking the closure of an arbitrary tangle (not necessarily a braid). For instance, consider the long knot 7_7 in the Knot Atlas obtained as the closure of the following tangle



In terms of meta-monoids the tangle is given by

$$R^+_{1,2}R^+_{14,4}R^-_{5,13}R^-_{3,6}R^-_{12,9}R^+_{7,11}R^+_{10,8}/\!\!/m_1^{1,4}/\!\!/m_2^{2,5}/\!\!/m_2^{2,6}/\!\!/m_2^{2,7}/\!\!/m_2^{2,8}/\!\!/m_3^{3,9}/\!\!/m_3^{3,10}/\!\!/m_3^{3,11}/\!\!/m_3^{3,12}/\!\!/m_3^{3,13}/\!\!/m_3^{3,14}.$$

Suppose that its invariant in Γ-calculus has the form

$$\left(\begin{array}{c|ccc} \omega & 1 & 2 & 3 \\ \hline 1 & \alpha_{11} & \alpha_{12} & \alpha_{13} \\ 2 & \alpha_{21} & \alpha_{22} & \alpha_{23} \\ 3 & \alpha_{31} & \alpha_{32} & \alpha_{33} \end{array}\right).$$

Then by stitching strand 2 to strand 1 and strand 3 to strand 2 the invariant of the long knot is given by

$$\omega \det\left(I - \begin{pmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{pmatrix}\right)\Bigg|_{t_2,t_3\to t}.$$

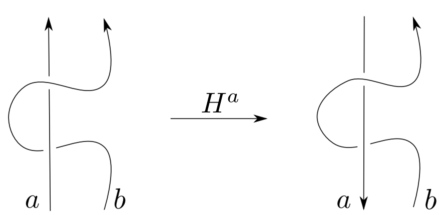
Doing the calculation one obtain

$$t^{-2} - 5t^{-1} + 9 - 5t + t^2,$$

which one can check to be the Alexander polynomial of the knot.

3.4 Orientation Reversal

For subsequent sections, it is useful to have a formula to reverse the orientation of a strand of a w-tangle in Γ-calculus.



We denote the operation of *reversing the orientation* of strand a of a w-tangle by H^a. To proceed, let us introduce another meta-monoid σ, called *σ-calculus*, defined as follows. For a finite set X, let σ^X be the set $\{\sum_{x\in X} s_x \boldsymbol{v}_x\}$, where s_x is a monomial in the variables t_z, $z \in X$ and $\{\boldsymbol{v}_x : x \in X\}$ is a linearly independent set of vectors. Let us record the operations below:

- identity $\quad \left(\sum_{x\in X} s_x\boldsymbol{v}_x\right) /\!\!/ e_a = \left(\sum_{x\in X} s_x\boldsymbol{v}_x\right) + \boldsymbol{v}_a$, where $a \notin X$,
- disjoint union $\quad \left(\sum_{x\in X} s_x\boldsymbol{v}_x\right) \sqcup \left(\sum_{y\in Y} s_y\boldsymbol{v}_y\right) = \sum_{z\in X\cup Y} s_z\boldsymbol{v}_z$, where $X \cap Y = \emptyset$,
- deletion $\quad \left(\sum_{x\in X} s_x\boldsymbol{v}_x + s_a\boldsymbol{v}_a\right) /\!\!/ \eta_a = \sum_{x\in X} (s_x)_{t_a\to 1}\boldsymbol{v}_x$, where $a \notin X$,
- renaming $\quad \left(\sum_{x\in X} s_x\boldsymbol{v}_x + s_a\boldsymbol{v}_a\right) /\!\!/ \sigma^a_b = \sum_{x\in X} (s_x)_{t_a\to t_b}\boldsymbol{v}_x + (s_a)_{t_a\to t_b}\boldsymbol{v}_b$, where $\{a,b\} \cap X = \emptyset$,
- stitching $\quad \left(s_a\boldsymbol{v}_a + s_b\boldsymbol{v}_b + \sum_{x\in S} s_x\boldsymbol{v}_x\right) /\!\!/ m^{a,b}_c = \sum_{x\in S} (s_x)_{t_a,t_b\to t_c}\boldsymbol{v}_x + (s_a s_b)_{t_a,t_b\to t_c}\boldsymbol{v}_c$.

It is easy to see that these operations satisfy the meta-monoid axioms. There is a meta-monoid homomorphism from w-tangles to σ-calculus, which we also denote by φ, given by

$$R^{\pm}_{a,b} \mapsto \boldsymbol{v}_a + t_a^{\pm 1}\boldsymbol{v}_b.$$

One checks readily that the Reidemeister relations are satisfied. So we obtain a w-tangle invariant. Given a w-tangle, one sees that s_a of the strand labeled a is given by

$$\prod t_b^{\pm 1},$$

where the product is over all crossings such that a is the understrand and b is the overstrand (including a itself) and ± 1 is the sign of the crossing. For example, the tangle given in Figure 3.1 has value

$$\sigma = t_3\boldsymbol{v}_1 + t_2 t_3^{-1}\boldsymbol{v}_2 + t_1 t_2^{-1} t_3^{-1}\boldsymbol{v}_3.$$

To describe the operation H^a properly we need to extend Γ-calculus. Let $\widetilde{\Gamma}$ be the meta-monoid given as follows. For a finite set X of labels,

$$\widetilde{\Gamma}^X = (\Gamma^X, \sigma^X).$$

We call $\widetilde{\Gamma}$ *extended* Γ*-calculus*. From the above discussion there is a meta-monoid homomorphism $\varphi : \mathcal{W} \to \widetilde{\Gamma}$ defined componentwise.

Mathematica®. Let us briefly discuss how we can implement $\widetilde{\Gamma}$-calculus in Mathematica. A reader with Mathematica can get the notebook from here. This will be very similar to the Γ-calculus program. First we write a subroutine to display $\widetilde{\Gamma}$ in a nice format

```
eΓCollect[eΓ[ω_, λ_, σ_]] := eΓ[Simplify[ω],
   Collect[λ, x_, Collect[#, y_, Factor] &], σ];
Format[eΓ[ω_, λ_, σ_]] := Module[{S, M},
   S = Union@Cases[eΓ[ω, λ, σ], (x | y)a_ :→ a, ∞];
   M = Outer[Factor[∂x#1 y#2 λ] &, S, S];
   M = Prepend[M, y# & /@ S] // Transpose;
   M = Prepend[M, Prepend[x# & /@ S, ω]];
   {M // MatrixForm, σ}];
eΓ[ω1_, λ1_, σ1_] ≡ eΓ[ω2_, λ2_, σ2_] :=
  Simplify[PowerExpand[ω1 == ω2 ∧ λ1 == λ2 ∧ σ1 == σ2]];
```

Here we call the subroutine $e\Gamma$ to distinguish it from the Γ subroutine. It will take as input a scalar ω, a labeled matrix λ and a σ element, for instance

```
In[21]:= eΓ[1, {y_a, y_e}.(1  1 - t_a
                            0  t_a    ).{x_a, x_e}, {s_a, s_e}.{v_a, v_e}]
```

and returns

```
Out[21]= {( 1    x_a  x_e
            y_a  1    1 - t_a
            y_e  0    t_a     ), s_a v_a + s_e v_e}
```

Notice also that we use $\equiv$ to compare two elements in eΓ. Then we include the stitching subroutine together with the definitions of the crossings

```
eΓ /: eΓ[ω1_, λ1_, σ1_] eΓ[ω2_, λ2_, σ2_] := eΓ[ω1 * ω2, λ1 + λ2, σ1 + σ2];
em_{a_,e_→c_}[eΓ[ω_, λ_, σ_]] := Module[{α, β, γ, δ, θ, ϵ, ϕ, ψ, Ξ, μ},
  (α β θ         (∂_{y_a,x_a} λ  ∂_{y_a,x_e} λ  ∂_{y_a} λ
   γ δ ϵ     =    ∂_{y_e,x_a} λ  ∂_{y_e,x_e} λ  ∂_{y_e} λ
   ϕ ψ Ξ)         ∂_{x_a} λ      ∂_{x_e} λ      λ        ) /. (y | x)_{a|e} → 0;
  eΓ[(μ = 1 - γ) ω, {y_c, 1}.(β + α δ / μ  θ + α ϵ / μ
                              ψ + δ ϕ / μ  Ξ + ϕ ϵ / μ).{x_c, 1},
     (σ /. v_{a|e} → 0) + v_c (∂_{v_a} σ) (∂_{v_e} σ)]
   /. {t_a → t_c, t_e → t_c, b_a → b_c, b_e → b_c} // eΓCollect];
eR⁺_{a_,e_} := eΓ[1, {y_a, y_e}.(1  1 - t_a
                                 0  t_a    ).{x_a, x_e}, v_a + t_a v_e];
eR⁻_{a_,e_} := eR⁺_{a,e} /. t_a → t_a^-1;
```

Note that here we denote the stitching operation by $em_{a,e\to c}$ and the crossings by $eR^{\pm}_{a,e}$ to distinguish them from the ones in Γ-calculus.

Proposition 3.9. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{W}^{\{a\}\cup S} & \xrightarrow{H^a} & \mathcal{W}^{\{a\}\cup S} \\ \downarrow \varphi & & \downarrow \varphi \\ \widetilde{\Gamma}^{\{a\}\cup S} & \xrightarrow{dH^a} & \widetilde{\Gamma}^{\{a\}\cup S} \end{array}$$

where the operation dH^a is described as follows. For an element of $\widetilde{\Gamma}^{\{a\}\cup S}$ given by

$$\left(\left(\begin{array}{c|cc} \omega & a & S \\ \hline a & \alpha & \theta \\ S & \phi & \Xi \end{array}\right), s_a \boldsymbol{v}_a + \sum_{x\in S} s_x \boldsymbol{v}_x\right),$$

its image under dH^a is

$$\left(\left(\begin{array}{c|cc} \alpha\omega/s_a & a & S \\ \hline a & 1/\alpha & \theta/\alpha \\ S & -\phi/\alpha & (\alpha\Xi - \phi\theta)/\alpha \end{array}\right), s_a^{-1} \boldsymbol{v}_a + \sum_{x\in S} s_x \boldsymbol{v}_x\right)_{t_a \to t_a^{-1}}.$$

Mathematica®. Before presenting the proof let us describe our implementation of dH^a in Mathematica

```
dH[a_][eΓ[ω_, λ_, σ_]] := Module[{α, θ, ϕ, Ξ, sa},
  (α θ; ϕ Ξ) = (∂_{y_a,x_a} λ  ∂_{y_a} λ; ∂_{x_a} λ  λ) /. (y | x)_a → 0;
sa = ∂_{v_a} σ;
   eΓ[α ω / sa, {y_a, 1}.(1/α  θ/α; -ϕ/α  (α Ξ - ϕ θ)/α).{x_a, 1},
     v_a/sa + (σ /. {v_a → 0})] /. {t_a :→ 1/t_a, b_a :→ -b_a} // eΓCollect];
```

The subroutine $dH[a]$ reverses the orientation of strand a in $\widetilde{\Gamma}$-calculus.

Proof. We want to show that

$$H^a \,/\!/\, \varphi = \varphi \,/\!/\, dH^a. \tag{3.5}$$

The meta-monoid structure allows us to use an "inductive" proof as follows. Given a w-tangle T, to reverse the orientation of strand a, we first decompose T into a disjoint union of crossings, reverse the orientations of the crossings that contain the strand, and then stitch them together. For the base step, we need to check the crossings:

$$R^{\pm}_{1,2} \,/\!/\, \varphi \,/\!/\, dH^1 = R^{\pm}_{1,2} \,/\!/\, H^1 \,/\!/\, \varphi = R^{\mp}_{1,2} \,/\!/\, \varphi, \tag{3.6}$$

$$R^{\pm}_{1,2} \,/\!/\, \varphi \,/\!/\, dH^2 = R^{\pm}_{1,2} \,/\!/\, H^2 \,/\!/\, \varphi = R^{\mp}_{1,2} \,/\!/\, \varphi, \tag{3.7}$$

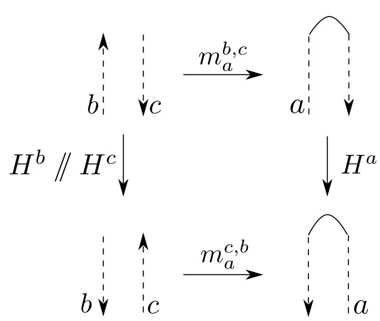
where recall that here the image lies in $\widetilde{\Gamma}$-calculus

$$R^{\pm}_{1,2} \,/\!/\, \varphi = \left(\left(\begin{array}{c|cc} 1 & 1 & 2 \\ \hline 1 & 1 & 1 - t_1^{\pm 1} \\ 2 & 0 & t_1^{\pm 1} \end{array} \right), \boldsymbol{v}_1 + t_1^{\pm 1} \boldsymbol{v}_2 \right).$$

For the "induction" step the relevant equation to check is

$$\varphi \,/\!/\, m_a^{b,c} \,/\!/\, dH^a = \varphi \,/\!/\, dH^b \,/\!/\, dH^c \,/\!/\, m_a^{c,b}. \tag{3.8}$$

We can visualize the above equation as follows



To see why equation (3.8) implies equation (3.5), suppose that strand a is obtained by stitching strand b to strand c. Then to reverse the orientation of strand a we can reverse the orientations of strands b and c and then stitch them, i.e. $H^a \,/\!/\, \varphi$ is given by

$$H^b \,/\!/\, H^c \,/\!/\, m_a^{c,b} \,/\!/\, \varphi = H^b \,/\!/\, H^c \,/\!/\, \varphi \,/\!/\, m_a^{c,b}$$

where we can commute φ and $m_a^{c,b}$ because φ is a meta-monoid homomorphism. From the "induction hypothesis" suppose that we already have

$$H^b \,/\!/\, \varphi = \varphi \,/\!/\, dH^b, \qquad H^c \,/\!/\, \varphi = \varphi \,/\!/\, dH^c.$$

Then

$$\begin{aligned} H^a \,/\!/\, \varphi &= H^b \,/\!/\, H^c \,/\!/\, \varphi \,/\!/\, m_a^{c,b} = H^b \,/\!/\, \varphi \,/\!/\, dH^c \,/\!/\, m_a^{c,b} = \varphi \,/\!/\, dH^b \,/\!/\, dH^c \,/\!/\, m_a^{c,b} = \varphi \,/\!/\, m_a^{b,c} \,/\!/\, dH^a \\ &= m_a^{b,c} \,/\!/\, \varphi \,/\!/\, dH^a = \varphi \,/\!/\, dH^a, \end{aligned}$$

as required. Now equations (3.6), (3.7) and (3.8) are simple enough that we can just check them by hand. However it is much faster to use Mathematica. For equations (3.6) and (3.7) the commands are

```
(eR⁺1,2 // dH[1]) ≡ (eR⁻1,2)
(eR⁺1,2 // dH[2]) ≡ (eR⁻1,2)
(eR⁻1,2 // dH[1]) ≡ (eR⁺1,2)
(eR⁻1,2 // dH[1]) ≡ (eR⁺1,2)
```

For equation (3.8) we define an arbitrary element ζ and apply both sides to ζ. The command is

$$\zeta = \mathrm{e}\Gamma\left[\omega,\ \{y_b, y_c, y_S\}.\begin{pmatrix}\alpha & \beta & \theta\\ \gamma & \delta & \epsilon\\ \phi & \psi & \Xi\end{pmatrix}.\{x_b, x_c, x_S\},\ s_b v_b + s_c v_c + s_S v_S\right]$$

```
(ζ // em_{b,c→a} // dH[a]) ≡ (ζ // dH[b] // dH[c] // em_{c,b→a})
```

When one runs these commands, they all return True, and that completes the proof. □

Again it is useful to have a formula to reverse the orientations of many strands at the same time. We record it in the next proposition.

Proposition 3.10. *Let T be a w-tangle and $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a vector where $a_i \neq a_j$ for $1 \leq i, j \leq n$. Suppose that the image of T in $\widetilde{\Gamma}$-calculus is*

$$T \,/\!/\, \varphi = \left(\left(\begin{array}{c|cc} \omega & \boldsymbol{a} & S \\ \hline \boldsymbol{a} & \alpha & \theta \\ S & \phi & \Xi \end{array}\right), \sum_{i=1}^{n} s_{a_i}\boldsymbol{v}_{a_i} + \sum_{x\in S} s_x \boldsymbol{v}_x\right).$$

Let $dH^{\boldsymbol{a}}$ denote the composition $dH^{a_1} \,/\!/\, \cdots \,/\!/\, dH^{a_n}$ then $\varphi \,/\!/\, dH^{\boldsymbol{a}}$ is given by

$$\left(\left(\begin{array}{c|cc} \frac{\omega \det(\alpha)}{\prod_{i=1}^{n} s_{a_i}} & \boldsymbol{a} & S \\ \hline \boldsymbol{a} & \alpha^{-1} & \alpha^{-1}\theta \\ S & -\phi\alpha^{-1} & \Xi - \phi\alpha^{-1}\theta \end{array}\right), \sum_{i=1}^{n} s_{a_i}^{-1}\boldsymbol{v}_{a_i} + \sum_{x\in S} s_x \boldsymbol{v}_x\right)_{t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}},$$

where $t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}$ denotes the sequence of substitution $t_{a_i} \to t_{a_i}^{-1}$ for $1 \leq i \leq n$.

Proof. We proceed by induction on n. The case when $n = 1$ is precisely dS^a. Now for the induction step, we write $\boldsymbol{a} = (\boldsymbol{a}', a_n)$ and

$$\varphi(T) = \left(\left(\begin{array}{c|ccc} \omega & \boldsymbol{a}' & a_n & S \\ \hline \boldsymbol{a}' & \alpha_1 & \alpha_2 & \theta_1 \\ a_n & \alpha_3 & \alpha_4 & \theta_2 \\ S & \phi_1 & \phi_2 & \Xi \end{array} \right), \sum_{i=1}^{n-1} s_{a_i} \boldsymbol{v}_{a_i} + s_{a_n} \boldsymbol{v}_{a_n} + \sum_{x \in S} s_x \boldsymbol{v}_x \right).$$

Then reversing the orientation of strands $\boldsymbol{a}'$, using the induction hypothesis, we obtain

$$\left(\left(\begin{array}{c|ccc} \frac{\omega \det(\alpha_1)}{\prod_{i=1}^{n-1} s_{a_i}} & \boldsymbol{a}' & a_n & S \\ \hline \boldsymbol{a}' & \alpha_1^{-1} & \alpha_1^{-1}\alpha_2 & \alpha_1^{-1}\theta_1 \\ a_n & -\alpha_3\alpha_1^{-1} & \alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2 & \theta_2 - \alpha_3\alpha_1^{-1}\theta_1 \\ S & -\phi_1\alpha_1^{-1} & \phi_2 - \phi_1\alpha_1^{-1}\alpha_2 & \Xi - \phi_1\alpha_1^{-1}\theta_1 \end{array} \right), \sum_{i=1}^{n-1} s_{a_i}^{-1} \boldsymbol{v}_{a_i} + s_{a_n} \boldsymbol{v}_{a_n} + \sum_{x \in S} s_x \boldsymbol{v}_x \right)_{t_{\boldsymbol{a}'} \to t_{\boldsymbol{a}'}^{-1}}.$$

Now we reverse the orientation of strand a_n to get

$$\left(\begin{array}{c|ccc} \widetilde{\omega} & a_n & \boldsymbol{a}' & S \\ \hline a_n & \frac{1}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & -\frac{\alpha_3\alpha_1^{-1}}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & \frac{\theta_2 - \alpha_3\alpha_1^{-1}\theta_1}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} \\ \boldsymbol{a}' & -\frac{\alpha_1^{-1}\alpha_2}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & \alpha_1^{-1} + \frac{\alpha_1^{-1}\alpha_2\alpha_3\alpha_1^{-1}}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & \alpha_1^{-1}\theta_1 - \frac{\alpha_1^{-1}\alpha_2(\theta_2 - \alpha_3\alpha_1^{-1}\theta_1)}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} \\ S & \frac{-\phi_2 + \phi_1\alpha_1^{-1}\alpha_2}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & -\phi_1\alpha_1^{-1} + \frac{(\phi_2 - \phi_1\alpha_1^{-1}\alpha_2)\alpha_3\alpha_1}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & \Xi - \phi_1\alpha_1^{-1}\theta_1 - \frac{(\phi_2 - \alpha_1^{-1}\alpha_2\phi_1)(\theta_2 - \alpha_3\alpha_1^{-1}\theta_1)}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} \end{array} \right)_{t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}},$$

where

$$\widetilde{\omega} = \left. \frac{\omega(\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2)\det(\alpha_1)}{\prod_{i=1}^{n} s_{a_i}} \right|_{t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}},$$

and the σ-part is given by

$$\sum_{i=1}^{n} (s_{a_i}^{-1})_{t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}} \boldsymbol{v}_{a_i} + \sum_{x \in S} (s_x)_{t_{\boldsymbol{a}} \to t_{\boldsymbol{a}}^{-1}} \boldsymbol{v}_x$$

Again by Lemma 3.2 we have

$$\det \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix} = \det \begin{pmatrix} \alpha_4 & \alpha_3 \\ \alpha_2 & \alpha_1 \end{pmatrix} = \det(\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2)\det(\alpha_1).$$

To finish off, we just need to show that

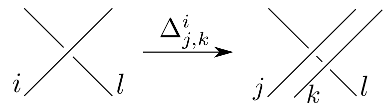
$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{pmatrix}^{-1} = \begin{pmatrix} \alpha_1^{-1} + \frac{\alpha_1^{-1}\alpha_2\alpha_3\alpha_1^{-1}}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & -\frac{\alpha_1^{-1}\alpha_2}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} \\ -\frac{\alpha_3\alpha_1^{-1}}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} & \frac{1}{\alpha_4 - \alpha_3\alpha_1^{-1}\alpha_2} \end{pmatrix},$$

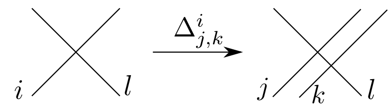
which one can easily check by performing matrix multiplications and we leave the details to the readers.

□

3.5 Strand Doubling

For convenience let us also describe the operation of *doubling* or *unzipping* a strand of a w-tangle, which is the operation of replacing a strand by two parallel copies of itself.





More concretely we denote the operation of doubling a strand labeled i to two strands labeled j and k by $\Delta^i_{j,k}$. So if X is a set of labels and $X = \{i\} \cup S$, where $\{i, j, k\} \cap S = \emptyset$, then

$$\Delta^i_{j,k} : \mathcal{W}^{\{i\}\cup S} \to \mathcal{W}^{\{j,k\}\cup S}.$$

We would like to investigate the effect of strand doubling on the image of a w-tangle in $\widetilde{\Gamma}$-calculus. Our framework will be similar to the case of orientation reversal.

Proposition 3.11. *We have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{W}^{\{i\}\cup S} & \xrightarrow{\Delta^i_{j,k}} & \mathcal{W}^{\{j,k\}\cup S} \\ \downarrow \varphi & & \downarrow \varphi \\ \widetilde{\Gamma}^{\{i\}\cup S} & \xrightarrow{q\Delta^i_{j,k}} & \widetilde{\Gamma}^{\{j,k\}\cup S} \end{array}$$

where the operation $q\Delta^i_{j,k}$ is described as follows. For an element of $\widetilde{\Gamma}^{\{i\}\cup S}$ given by

$$\left(\left(\begin{array}{c|cc} \omega & i & S \\ \hline i & \alpha & \theta \\ S & \phi & \Xi \end{array} \right), s_i \boldsymbol{v}_i + \sum_{x\in S} s_x \boldsymbol{v}_x \right),$$

its image under $q\Delta^i_{j,k}$ is

$$\left(\left(\begin{array}{c|ccc} \omega & j & k & S \\ \hline j & \frac{-\alpha+t_j t_k s_i + t_j \nu}{\mu} & \frac{(-1+t_j)\nu}{\mu} & \frac{(-1+t_j)\theta}{\mu} \\ k & \frac{t_j(-1+t_k)\nu}{\mu} & \frac{-s_i+t_j t_k \alpha - t_j \nu}{\mu} & \frac{t_j(-1+t_k)\theta}{\mu} \\ S & \phi & \phi & \Xi \end{array} \right), s_i(\boldsymbol{v}_j + \boldsymbol{v}_k) + \sum_{x\in S} s_x \boldsymbol{v}_x \right)_{t_i \to t_j t_k},$$

where $\mu = -1 + t_i$ and $\nu = \alpha - s_i$.

Mathematica®. Again before proving the proposition let us present our implementation of $q\Delta^i_{j,k}$ in Mathematica:

```
qΔ[i_, j_, k_][eΓ[ω_, λ_, σ_]] := Module[
  {α, θ, ϕ, Ξ, si, M, ti, μ, ν},
  (α θ; ϕ Ξ) = (∂_{y_i,x_i} λ  ∂_{y_i} λ; ∂_{x_i} λ  λ) /. (y | x)_i → 0 /. t_i → ti;
  si = ∂_{v_i} σ /. t_i → ti; μ = -1 + ti; ν = α - si;
  M = ( (-α+ti si+t_j ν)/μ   ((-1+t_j) ν)/μ      ((-1+t_j) θ)/μ
        (t_j (-1+t_k) ν)/μ   (-si+ti α-t_j ν)/μ  (t_j (-1+t_k) θ)/μ
        ϕ                    ϕ                   Ξ );
  eΓ[ω /. {t_i → t_j t_k}, {y_j, y_k, 1}.M.{x_j, x_k, 1} /. {ti → t_j t_k},
    (σ /. {v_i → 0}) + (v_j + v_k) si /. t_i | ti → t_j t_k] // eΓCollect
];
```

The subroutine $q\Delta[i,j,k]$ doubles strand i to strands j and k.

Proof. Our strategy will be to use an "induction" procedure analogous to the proof of orientation reversal. Given a w-tangle T, to double strand i, we first decompose T into a disjoint union of crossings, double the relevant strands, and then stitch them together. For the base case we have to check the following equations

$$\begin{aligned}
R^+_{1,3} \mathbin{/\!/} \varphi \mathbin{/\!/} q\Delta^1_{1,2} &= R^+_{1,3} \mathbin{/\!/} \Delta^1_{1,2} \mathbin{/\!/} \varphi = R^+_{2,3}R^+_{1,4} \mathbin{/\!/} m^{3,4}_3 \mathbin{/\!/} \varphi = R^+_{2,3}R^+_{1,4} \mathbin{/\!/} \varphi \mathbin{/\!/} m^{3,4}_3,\\
R^-_{1,3} \mathbin{/\!/} \varphi \mathbin{/\!/} q\Delta^1_{1,2} &= R^-_{1,3} \mathbin{/\!/} \Delta^1_{1,2} \mathbin{/\!/} \varphi = R^-_{1,3}R^-_{2,4} \mathbin{/\!/} m^{3,4}_3 \mathbin{/\!/} \varphi = R^-_{1,3}R^-_{2,4} \mathbin{/\!/} \varphi \mathbin{/\!/} m^{3,4}_3,\\
R^+_{1,2} \mathbin{/\!/} \varphi \mathbin{/\!/} q\Delta^2_{2,3} &= R^+_{1,2} \mathbin{/\!/} \Delta^2_{2,3} \mathbin{/\!/} \varphi = R^+_{1,2}R^+_{4,3} \mathbin{/\!/} m^{1,4}_1 \mathbin{/\!/} \varphi = R^+_{1,2}R^+_{4,3} \mathbin{/\!/} \varphi \mathbin{/\!/} m^{1,4}_1,\\
R^-_{1,2} \mathbin{/\!/} \varphi \mathbin{/\!/} q\Delta^2_{2,3} &= R^-_{1,2} \mathbin{/\!/} \Delta^2_{2,3} \mathbin{/\!/} \varphi = R^-_{1,3}R^-_{4,2} \mathbin{/\!/} m^{1,4}_1 \mathbin{/\!/} \varphi = R^-_{1,3}R^-_{4,2} \mathbin{/\!/} \varphi \mathbin{/\!/} m^{1,4}_1.
\end{aligned}$$

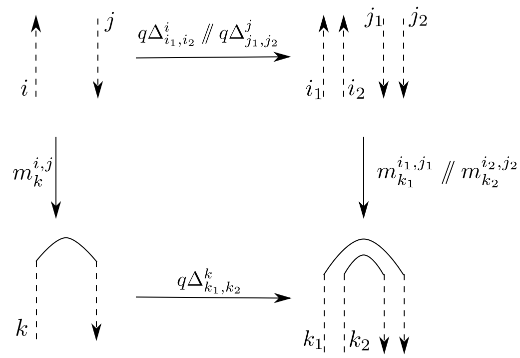
These equations are simple enough to be checked by hand, but it is more convenient to use Mathematica. The commands are

```
qΔ[1, 1, 2][eR⁺_{1,3}] ≡ (eR⁺_{2,3} eR⁺_{1,4} // em_{3,4→3})
qΔ[1, 1, 2][eR⁻_{1,3}] ≡ (eR⁻_{1,3} eR⁻_{2,4} // em_{3,4→3})
qΔ[2, 2, 3][eR⁺_{1,2}] ≡ (eR⁺_{1,2} eR⁺_{4,3} // em_{1,4→1})
qΔ[2, 2, 3][eR⁻_{1,2}] ≡ (eR⁻_{1,3} eR⁻_{4,2} // em_{1,4→1})
```

They all return True. For the "induction" step the equation we need to verify is

$$\varphi \mathbin{/\!/} q\Delta^i_{i_1,i_2} \mathbin{/\!/} q\Delta^j_{j_1,j_2} \mathbin{/\!/} m^{i_1,j_1}_{k_1} \mathbin{/\!/} m^{i_2,j_2}_{k_2} = \varphi \mathbin{/\!/} m^{i,j}_k \mathbin{/\!/} q\Delta^k_{k_1,k_2}. \tag{3.9}$$

We can visualize the above equation as follows



Finally we can verify equation (3.9) in Mathematica by applying both sides to an element ζ as follows

```
ζ = eΓ[ω, {y_i, y_j, y_S}.{{α, β, θ}, {γ, δ, ϵ}, {ϕ, ψ, Ξ}}.{x_i, x_j, x_S}, s_i v_i + s_j v_j + s_S v_S]
(ζ // qΔ[i, i_1, i_2] // qΔ[j, j_1, j_2] // em_{i_1,j_1→k_1} // em_{i_2,j_2→k_2}) ≡
 (ζ // em_{i,j→k} // qΔ[k, k_1, k_2])
```

The command returns True and that completes the proof. □

Chapter 4

Expansions of w-Tangles

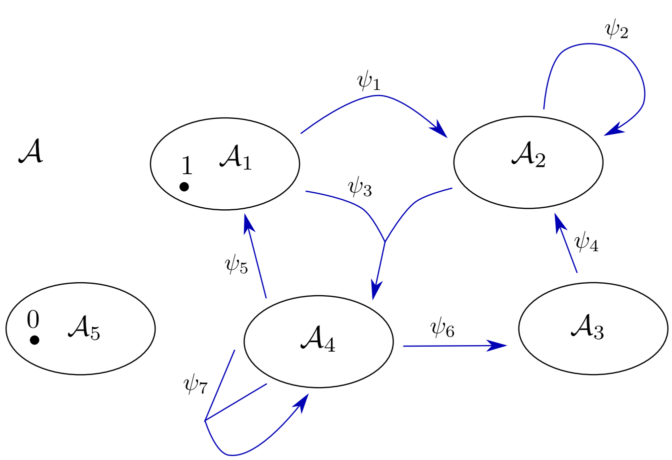
4.1 Algebraic Structures and Expansions

In this section we briefly describe the general algebraic framework that gives rise to an Alexander invariant of w-tangles. The reader can refer to [BND14, BN16b] for more details.

An *algebraic structure* $\mathcal{A}$ is some collection $(\mathcal{A}_\alpha)$ of sets of objects of different kinds, where the subscript α denotes the *kind* of the objects in $\mathcal{A}_\alpha$, along with some collection of *operations* ψ_β, where each ψ_β is a map

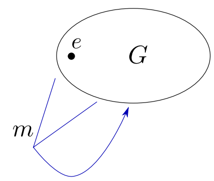
$$\psi_\beta : \mathcal{A}_{\alpha_1} \times \cdots \times \mathcal{A}_{\alpha_k} \to \mathcal{A}_{\alpha_0}.$$

Here k is a non-negative integer. So the operations may be unary or binary or multinary, but they always return a value of some fixed kind. When $k = 0$ it is the operation of naming *constants* within some $\mathcal{A}_{\alpha_0}$. The operations may or may not be subject to axioms–an *axiom* is an identity asserting that some composition of operations is equal to some other composition of operations. One can think of an algebraic structure $\mathcal{A}$ schematically as follows.



In the figure the algebraic structure $\mathcal{A}$ has five kinds of objects, five unary operations, two binary operations, and two 0-nary operations. Examples of algebraic structures abound in mathematics. For our purpose, we focus on two main examples: monoids and meta-monoids.

Example 4.1 (Monoids)**.** Let G be a monoid with identity e. Then as an algebraic structure G has one kind of object, one binary operation: multiplication, which we denote by m, and one 0-nary operation: the identity, which we denote by e.



The operations satisfy the following axioms:

$$(m \times \mathrm{Id}) \mathbin{/\!\!/} m = (\mathrm{Id} \times m) \mathbin{/\!\!/} m,$$
$$(\mathrm{Id} \times e) \mathbin{/\!\!/} m = (e \times \mathrm{Id}) \mathbin{/\!\!/} m = \mathrm{Id}.$$

The first axiom corresponds to associativity and the second axiom corresponds to the identity e. ♣

Example 4.2 (Meta-monoids)**.** Our main examples of algebraic structures will be meta-monoids (see Section 2.1). Note that the definition of a meta-monoid is already formulated in the language of algebraic structures. In this case we have infinitely many kinds of objects and infinitely many operations; the kinds are indexed by finite sets. ♣

Now given an algebraic structure $\mathcal{A}$, we first extend $\mathcal{A}$ to allow formal linear combinations of objects of the same kind (extending the operations in a linear or multi-linear manner). In other words, we consider the $\mathbb{Q}$-module $\mathbb{Q}\mathcal{A}_\alpha$ for objects of kind α (one can replace $\mathbb{Q}$ by any field with characteristic 0). In this manner, we can assume from now on that for an algebraic structure $\mathcal{A}$, each $\mathcal{A}_\alpha$ is a $\mathbb{Q}$-module.

Given two algebraic structures $\mathcal{A}$ and $\mathcal{B}$ with the same kinds of objects, then $\mathcal{A} \supseteq \mathcal{B}$ means that $\mathcal{B}_\alpha$ is a submodule of $\mathcal{A}_\alpha$ for all α. An algebraic structure $\mathcal{A}$ is called *filtered* if there exists a *filtration*

$$\mathcal{A} = \mathcal{A}^0 \supseteq \mathcal{A}^1 \supseteq \mathcal{A}^2 \supseteq \cdots$$

If an algebraic structure $\mathcal{A}$ is filtered, we define the *associated graded structure* of $\mathcal{A}$ (with respect to the given filtration) to be

$$\mathrm{gr}\mathcal{A} := \prod_{m=0}^{\infty} \mathcal{A}^m/\mathcal{A}^{m+1}.$$

Here again by the *quotient* $\mathcal{A}^m/\mathcal{A}^{m+1}$ we mean the algebraic structure consisting of $(\mathcal{A}^m_\alpha/\mathcal{A}^{m+1}_\alpha)$, the usual quotients of modules. The algebraic structure $\mathrm{gr}\mathcal{A}$ inherits the kinds of objects from $\mathcal{A}$, i.e.

$$(\mathrm{gr}\mathcal{A})_\alpha = \prod_{m=0}^{\infty} \mathcal{A}^m_\alpha/\mathcal{A}^{m+1}_\alpha,$$

where α is a kind of objects in $\mathcal{A}$. One advantage of working with $\mathrm{gr}\mathcal{A}$ is that it is *graded*. We denote the degree m piece $\mathcal{A}^m/\mathcal{A}^{m+1}$ of $\mathrm{gr}\mathcal{A}$ by $\mathrm{gr}_m\mathcal{A}$. So an element a of $\mathrm{gr}\mathcal{A}$ has the form

$$a = \sum_{n=1}^{\infty} a_n, \quad a_n \in \mathcal{A}^n_\alpha/\mathcal{A}^{n+1}_\alpha.$$

If the operations of $\mathcal{A}$ *preserve the filtration*, this means that an operation

$$\psi_\beta : \mathcal{A}_{\alpha_1} \times \cdots \times \mathcal{A}_{\alpha_k} \to \mathcal{A}_{\alpha_0}$$

satisfies

$$\psi_\beta : \mathcal{A}^{m_1}_{\alpha_1} \times \cdots \times \mathcal{A}^{m_k}_{\alpha_k} \to \mathcal{A}^{m_1+\cdots+m_k}_{\alpha_0}$$

for all $m_1, \ldots, m_k$, then $\mathrm{gr}\mathcal{A}$ inherits the operations from $\mathcal{A}$. However the induced operations on $\mathrm{gr}\mathcal{A}$ may or may not satisfy the axioms satisfied by the operations of $\mathcal{A}$.

Note that if an algebraic structure $\mathcal{A}$ is graded:

$$\mathcal{A} = \prod_{m=0}^{\infty} \mathcal{A}_m,$$

then it has a canonical filtration given by

$$\mathcal{A}^m = \prod_{n=m}^{\infty} \mathcal{A}_n.$$

With respect to the canonical filtration we have

$$\mathrm{gr}\mathcal{A} = \prod_{m=0}^{\infty} \mathcal{A}^m/\mathcal{A}^{m+1} = \prod_{m=0}^{\infty} \mathcal{A}_m = \mathcal{A}.$$

So in particular we have $\mathrm{gr}(\mathrm{gr}\mathcal{A}) = \mathrm{gr}\mathcal{A}$.

Consider two *filtered* algebraic structures $\mathcal{A}$ and $\mathcal{B}$ with the same kinds of objects, i.e. we have filtrations

$$\mathcal{A} = \mathcal{A}^0 \supseteq \mathcal{A}^1 \supseteq \mathcal{A}^2 \supseteq \cdots, \quad \mathcal{B} = \mathcal{B}^0 \supseteq \mathcal{B}^1 \supseteq \mathcal{B}^2 \supseteq \cdots.$$

A *map* $f : \mathcal{A} \to \mathcal{B}$ between two algebraic structures consists of a collection of module homomorphism $(f_\alpha : \mathcal{A}_\alpha \to \mathcal{B}_\alpha)$. A map f is called *filtered* if it preserves the filtration of $\mathcal{A}$ and $\mathcal{B}$:

$$f(\mathcal{A}^n) \subseteq f(\mathcal{B}^n), \quad n = 1, 2, \ldots$$

A filtered map f between two filtered algebraic structures induces a *graded map* $\mathrm{gr}f$ between their associated graded structures

$$\mathrm{gr}f : \mathrm{gr}\mathcal{A} \to \mathrm{gr}\mathcal{B}$$

given by $\mathrm{gr}f([a]_n) = [f(a)]_n$, where $a \in \mathcal{A}^n$ and $[a]_n$ denotes its equivalence class in $\mathcal{A}^n/\mathcal{A}^{n+1}$. Now we are ready to define the main construction of this paper:

Definition (Expansions)**.** An *expansion* is a filtered map Z from a filtered algebraic structure $\mathcal{A}$ to its associated graded $\mathrm{gr}\mathcal{A}$

$$Z : \mathcal{A} \to \mathrm{gr}\mathcal{A}$$

such that the induced graded map $\mathrm{gr}Z : \mathrm{gr}\mathcal{A} \to \mathrm{gr}(\mathrm{gr}\mathcal{A}) = \mathrm{gr}\mathcal{A}$ is the identity map.

Let us unpack the above definition. First of all, since the map Z is filtered, we have

$$Z(a) \in \prod_{m \geq n} \mathcal{A}^m/\mathcal{A}^{m+1}, \quad \text{for } a \in \mathcal{A}^n.$$

We can make the condition for Z more concrete as follows: let $[a]_n \in \mathcal{A}^n/\mathcal{A}^{n+1}$, we have

$$\text{gr}Z([a]_n) = [Z(a)]_n = [a]_n.$$

In other words, for $a \in \mathcal{A}^n$, we have

$$Z(a) = [a]_n + \text{higher order terms}. \tag{4.1}$$

When $\text{gr}\mathcal{A}$ inherits the operations from $\mathcal{A}$, we say that an expansion Z is *homomorphic* if it commutes with the operations of $\mathcal{A}$. We are interested in finding homomorphic expansions of various algebraic structures.

Example 4.3 (Taylor Expansions)**.** The prototypical example of an expansion is the Taylor expansion. Let $\mathcal{A}$ be the algebra over $\mathbb{R}$ of analytic functions $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{I}$ be the ideal

$$\mathcal{I} = \{f \in \mathcal{A} : f(0) = 0\}.$$

Then one obtains a filtration of $\mathcal{A}$ given by

$$\mathcal{A} = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \cdots,$$

where $\mathcal{I}^m$ denotes the product ideal. Then one obtains a filtration of $\mathcal{A}$ given by

$$\mathcal{A} = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \cdots$$

We can then define the associated graded

$$\text{gr}\mathcal{A} = \prod_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}.$$

In this case it is quite easy to figure out what the n-th degree piece of $\text{gr}\mathcal{A}$ should be. Specifically, an element of $\mathcal{I}^n$ has the form

$$f = x^n g, \quad \text{for some } g \in \mathcal{A}.$$

Therefore

$$\mathcal{I}^n/\mathcal{I}^{n+1} = \text{span}\{x^n\}.$$

Now an expansion $Z : \mathcal{A} \to \text{gr}\mathcal{A}$ is given by

$$Z(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

To check that Z is indeed an expansion is almost a tautology. If $f \in \mathcal{A}^n$, then f has the form

$$f = x^n g = x^n \left(g(0) + g'(0)x + \frac{g''(0)}{2!}x^2 + \cdots \right).$$

Thus

$$Z(f) = x^n g(0) + \text{higher ordered terms} = [f]_n + \text{higher ordered terms},$$

as expected, where $[f]_n$ just denotes the n-th degree term of f. Note that the expansion is clearly homomorphic:

$$Z(fg) = Z(f)Z(g)$$

since the Taylor series of a product is the product of Taylor series. ♣

Given an algebraic structure $\mathcal{A}$, we describe a procedure to obtain a filtration of $\mathcal{A}$ as follows. First we extend $\mathcal{A}$ to allow formal linear combinations of objects of the same kind and extend the operations in a linear or multi-linear manner. Then we define $\mathcal{I}$, the *augmentation ideal*, to be the *substructure* of $\mathcal{A}$ made out of all such combinations in which the sum of coefficients is 0, i.e. $\mathcal{I}$ consists of a collection of submodules $\mathcal{I}_\alpha \subseteq \mathcal{A}_\alpha$, where

$$\mathcal{I}_\alpha = \left\{ \sum_{k=1}^{n} a_k x_k : \sum_{k=1}^{n} a_k = 0 \text{ and } x_k \in \mathcal{A}_\alpha \right\}.$$

For $m \geq 0$, we let $\mathcal{I}^0 = \mathcal{A}$, and $\mathcal{I}^m$ be the substructure consisting of all outputs of *algebraic expressions*, i.e. arbitrary compositions of the operations in $\mathcal{A}$, that have at least m inputs in $\mathcal{I}$ and possibly, further inputs in $\mathcal{A}$ (note that the inputs are not necessarily of the same kinds). It is clear that $\mathcal{I}^m \supseteq \mathcal{I}^{m+1}$ for $m \geq 0$, meaning that $\mathcal{I}^m_\alpha \supseteq \mathcal{I}^{m+1}_\alpha$ for all kinds α. We then have a filtration of $\mathcal{A}$

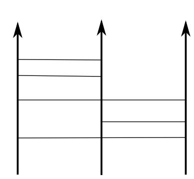
$$\mathcal{A} = \mathcal{I}^0 \supseteq \mathcal{I} \supseteq \mathcal{I}^2 \supseteq \cdots$$

and its associated graded

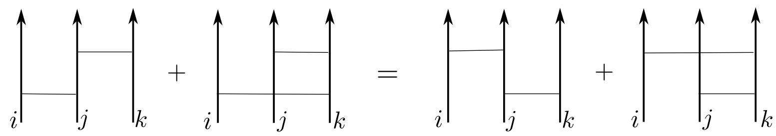
$$\mathrm{gr}\mathcal{A} = \prod_{n=0}^{\infty} \mathcal{I}^n / \mathcal{I}^{n+1}.$$

It is clear that for this particular filtration, the operations of $\mathcal{A}$ preserve the filtration, and therefore $\mathrm{gr}\mathcal{A}$ automatically inherits the operations of $\mathcal{A}$. When we write $\mathrm{gr}\mathcal{A}$ without specifying an explicit filtration, we mean the augmentation ideal filtration.

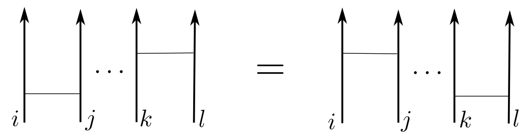
Example 4.4 (Braid groups)**.** Our prototypical example in topology is the expansion of the braid groups. Unfortunately this requires quite a bit of background, so we only introduce the procedure very briefly and refer the interested readers to [BN97] for more details. Consider the braid group on n strands B_n, its associated graded $\mathrm{gr}B_n$ with respect to the augmentation ideal is the algebra of *horizontal chord diagrams* on n strands:



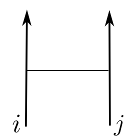
subject to the relations



and



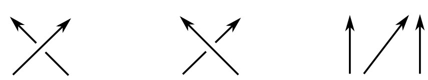
The grading is specified by the number of chords, so we see that the above relations preserve the grading. If we denote a diagram with a single chord by t_{ij}

$t_{ij} \quad =$ 

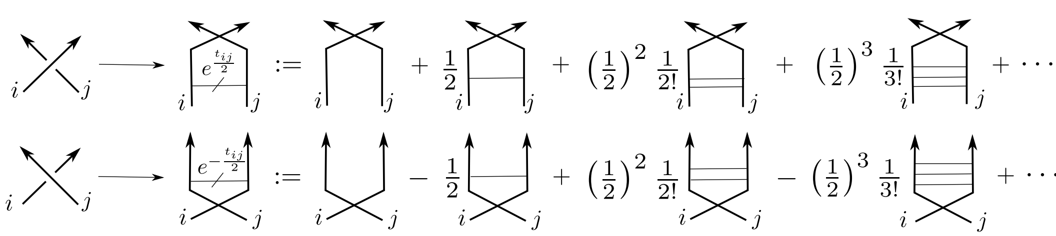
then we can write the above two relations as

$$[t_{ij} + t_{ik}, t_{jk}] = 0, \qquad [t_{ij}, t_{kl}] = 0.$$

Constructing a homomorphic expansion of B_n is quite non-trivial. First we need to equip B_n with an extra structure called a *parenthesization*, i.e. a way to group the endpoints at the bottom and the top of a braid. Equipped with a parenthesization we can decompose a braid into elementary pieces consisting of



Thus to construct a homomorphic expansion we just need to specify the images of the generators. We map the crossings to



The image of the third generator belongs to the algebra of chord diagrams on 3 strands, which we denote by Φ. The element Φ is required to satisfy the *pentagon equation*:

$$\Phi^{123} \cdot (1 \otimes \Delta \otimes 1)(\Phi) \cdot \Phi^{234} = (\Delta \otimes 1 \otimes 1)(\Phi) \cdot (1 \otimes 1 \otimes \Delta)(\Phi),$$

and the two *hexagon equations*:

$$(\Delta \otimes 1)(R^{\pm}) = \Phi^{123} \cdot (R^{\pm})^{23} \cdot (\Phi^{-1})^{132} \cdot (R^{\pm})^{13} \cdot \Phi^{312}.$$

Here Φ^{ijk} means putting Φ on strands labeled i, j, k; Δ is the *doubling map*, i.e. replacing a strand by two parallel strands and sum over all ways to put a chord on these strands; and recall that composition is read from left to right and drawn from bottom to top. A solution to the above equations is called an *associator*, which plays an important role in certain areas of mathematical physics. A closed-form formula for an associator is not known, however associators can be computed degree by degree, and it is known that there exists an associator with rational coefficients. Therefore it suffices to set our field to be $\mathbb{Q}$.

Note that the image of the *Yang-Baxter equation*

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$$

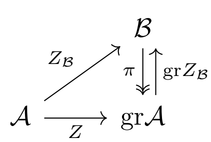
in the algebra of chord diagrams is

$$e^{t_{12}/2}e^{t_{13}/2}e^{t_{23}/2} = e^{t_{23}/2}e^{t_{13}/2}e^{t_{12}/2}.$$

The above equation does not hold because of commutativity issues (when $[x, y] \neq 0$ one does not have $e^x e^y = e^{x+y}$ but needs to appeal to the BCH formula). To fix the non-commutativity we have to introduce an associator. In the next section, we will see that this matter does not arise in the case of w-tangles, and therefore it is much easier to study w-tangles from an expansion point of view. ♣

In practice, to find the associated graded structure of an algebraic structure, the following proposition is useful.

Proposition 4.1. *Let $\mathcal{B}$ be a graded algebraic structure and $\mathcal{A}$ a filtered algebraic structure with the same multi-graph of spaces and operations and suppose we have a surjective graded map $\pi : \mathcal{B} \to \mathrm{gr}\mathcal{A}$. If we have a filtered map $Z_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ such that $(\mathrm{gr}Z_{\mathcal{B}}) \circ \pi : \mathcal{B} \to \mathcal{B}$ is the identity map, then $\pi\colon \mathcal{B} \to \mathrm{gr}\mathcal{A}$ is an isomorphism (of modules) and $Z = \pi \circ Z_{\mathcal{B}} : \mathcal{A} \to \mathrm{gr}\mathcal{A}$ is an expansion. In short we have the commutative diagram*



If $Z_{\mathcal{B}}$ is homomorphic, then Z is also homomorphic.

Proof. The map π is surjective by assumption, and the condition $(\mathrm{gr}Z_{\mathcal{B}}) \circ \pi = \mathrm{id}$ shows that it is also injective. To show that Z is an expansion, first of all note that Z is filtered because $Z_{\mathcal{B}}$ is filtered and π is graded. We can write the condition $(\mathrm{gr}Z_{\mathcal{B}}) \circ \pi = \mathrm{id}$ more explicitly as

$$Z_{\mathcal{B}}(\pi(b_n)) = b_n + \text{higher order terms}, \quad \text{for } b_n \in \mathcal{B}_n.$$

Now for $a \in \mathcal{A}^n$ and $[a]_n \in \mathcal{A}^n/\mathcal{A}^{n+1}$, there exists a unique $b_n \in \mathcal{B}_n$ such that $\pi(b_n) = [a]_n$. It follows that

$$\mathrm{gr}Z([a]_n) = [Z(a)]_n = [\pi \circ Z_B(\pi(b_n))]_n = [\pi(b_n)]_n = [a]_n.$$

Therefore Z is an expansion, as required. □

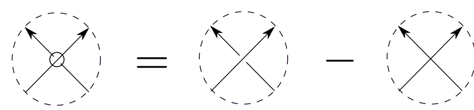
To summarize, to find the associated graded structure $\mathrm{gr}\mathcal{A}$ of a filtered algebraic structure $\mathcal{A}$ we need to construct a surjective graded map $\pi : \mathcal{B} \to \mathrm{gr}\mathcal{A}$ and a filtered map $Z_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ such that

$$Z_{\mathcal{B}}(\pi(b_n)) = b_n + \text{higher order terms}, \quad \text{for } b_n \in \mathcal{B}_n. \tag{4.2}$$

Then $\mathcal{B}$ is $\mathrm{gr}\mathcal{A}$ and $\pi \circ Z_{\mathcal{B}}$ is an expansion. We will illustrate this method in the concrete case of $\mathcal{W}$, the meta-monoid of w-tangles.

4.2 The Associated Graded Structure of w-Tangles

Consider the meta-monoid $\mathcal{W}$ of w-tangles. In this section we describe its associated graded structure $\mathrm{gr}\mathcal{W}$. We first extend $\mathcal{W}$ to $\mathbb{Q}\mathcal{W}$. Following the theory of finite type invariants (see [BND16, BN95]), we denote



We call the crossings on the left hand side *semi-virtual crossings*. Here again the rest of a w-tangle outside the crossings will stay the same. A w-tangle with semi-virtual crossings, which we also call *singular w-tangle*, is an element of $\mathbb{Q}\mathcal{W}$. In particular, a singular w-tangle with n semi-virtual crossings is a linear combination of 2^n w-tangles.

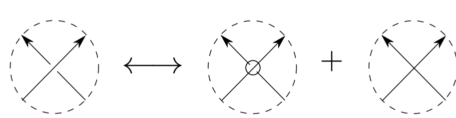
Recall that the augmentation ideal $\mathcal{I}$ consists of elements of the form

$$\sum_{k=1}^{n} a_k x_k, \quad \sum_{k=1}^{n} a_k = 0,\ x_k \in \mathcal{W}^X \quad \text{for } k = 1, \ldots, n,$$

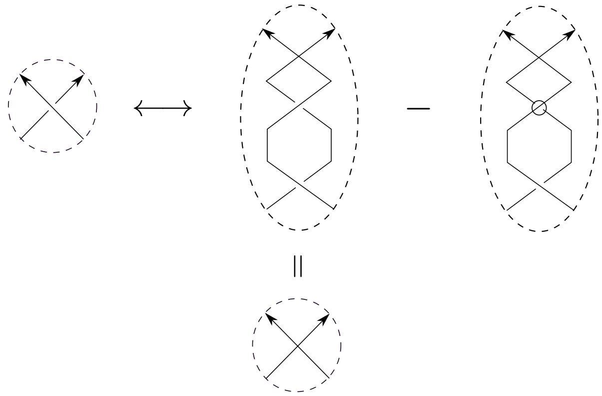
where X is a finite set of labels of the strands and can be thought of as a kind of the w-tangles. Since the sum of the coefficients is 0 we can rewrite the linear combination as

$$\sum_{k=1}^{n-1} a_k (x_k - x_n).$$

Therefore we see that $\mathcal{I}$ is generated by differences $x - y$, where $x, y \in \mathcal{W}^X$, and X again is a set of labels. Now for two w-tangles $x, y \in \mathcal{W}^X$, we can turn x into y provided we can turn a crossing to a virtual crossing and vice versa. Concretely, we can turn all the crossings of x to virtual crossings, and then turn the virtual crossings to the corresponding crossings of y. To turn a positive crossing to a virtual crossing and vice versa we proceed as follows.

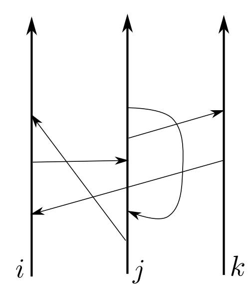


Similarly we can turn a negative crossing to a virtual crossing and using the following procedure

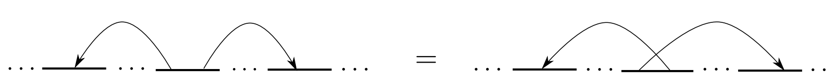


In each case the cost is a singular w-tangle with one semi-virtual crossing. Thus we can write $x - y$ as a linear combination of singular w-tangles each with one semi-virtual crossing. In other words, the ideal $\mathcal{I}$ is spanned by singular w-tangles with one semi-virtual crossing. It follows that $\mathcal{I}^m$ is spanned by singular w-tangles with m semi-virtual crossings.

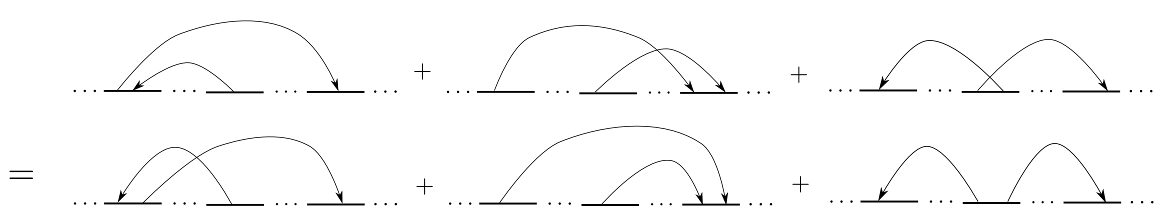
To proceed, we are going to define the meta-monoid of arrow diagrams, which we denote by $\mathcal{A}^w$. We will show that $\mathcal{A}^w$ is in fact $\text{gr}\mathcal{W}$. Let X be a finite set of labels, consider the collection of $|X|$ parallel directed lines labeled by X, which we also call a *skeleton*. Then an *arrow diagram* on a skeleton labeled by X is the skeleton together with a collection of arrows between the directed lines.



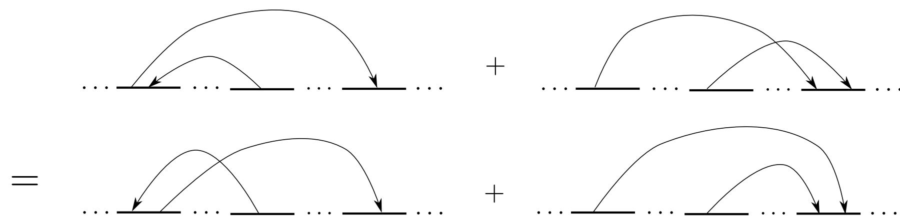
In the figure, we use thick lines to denote the skeleton and thin lines to denote the arrows. Then we let $(\mathcal{A}^w)^X$ be the space of formal linear combinations of arrow diagrams on the skeleta labeled by X modulo the TC (*tails commute*) relations



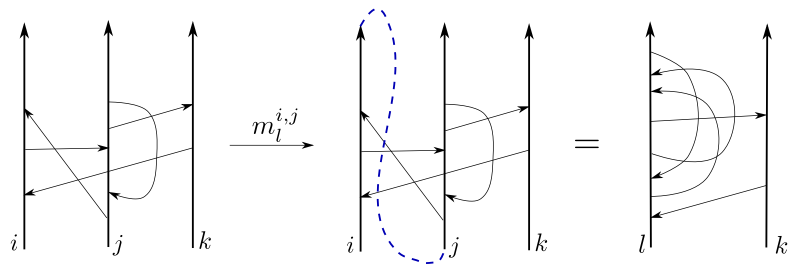
and the $6T$ relations



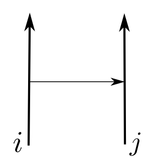
Note that the TC relations allow us to simplify the $6T$ relations to obtain the (directed) $\overrightarrow{4T}$ relations



Let us explain the pictures. Here the thick lines denote three disjoint parts of the skeleton, which may belong to different lines. The ...'s indicate the remaining parts of the diagrams, which stay unchanged on both sides. Note that we can have any number of arrows in the dotted parts. The space $\mathcal{A}^w$ forms a meta-monoid with the obvious stitching operation: $m_l^{i,j}$ means connecting the head of strand i to the tail of strand j combinatorially and calling the resulting strand l, for instance



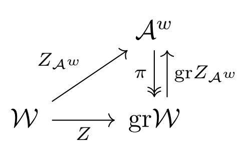
To delete a strand labeled a, if there are arrows connected to the strand, the result is 0, otherwise we can just remove the strand from the diagram. The other operations are straightforward. As a meta-monoid, it is clear that $\mathcal{A}^w$ is generated by the arrow diagrams with a single arrow, which we denote by a_{ij}

$a_{ij} \quad =$ 

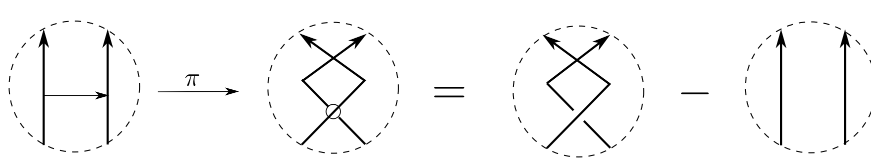
In other words, we can obtain any arrow diagram from a collection of arrow diagrams with a single arrow together with the disjoint union and stitching operations. The meta-monoid $\mathcal{A}^w$ is graded by the number of arrows in an arrow diagram.

Proposition 4.2. *The associated graded structure of $\mathcal{W}$ is $\mathcal{A}^w$.*

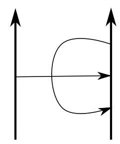
Proof. Following Proposition 4.1 we need to establish the following commutative diagram



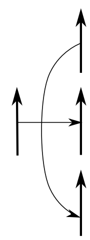
Let us first define the map $\pi : \mathcal{A}^w \to \mathrm{gr}\mathcal{W}$. Since $\mathcal{A}^w$ is finitely generated it suffices to define π on the generators. We set



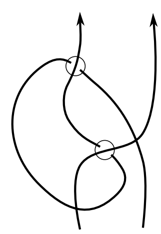
Notice that the arrow goes from the overstrand to the understrand. As an example, let us look at the image of the arrow diagram



We first break the diagram as follows



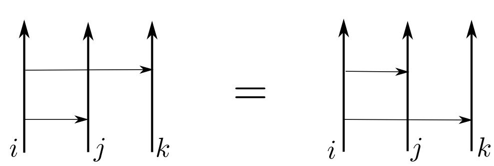
Then by mapping each single arrow to a semi-virtual crossing and then stitch the strands together, where if two strands are far apart we bring them together via virtual crossings, we obtain



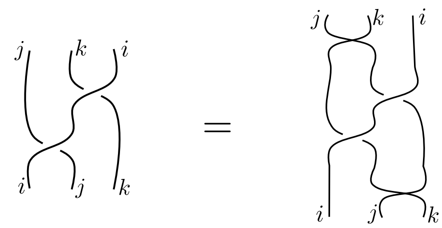
Using the notations for a single arrow and a crossing we get

$$\pi(a_{ij}) = R^{+}_{i,j} - 1 \quad \text{or} \quad R^{+}_{i,j} = \pi(a_{ij}) + 1.$$

The map π is clearly graded, we need to show that it is well-defined and is surjective. First of all observe that the image of an arrow diagram with m arrows lies in $\mathcal{I}^m/\mathcal{I}^{m+1}$. Two realizations of the same arrow diagram can be turn into one another at the cost of w-tangles with $m+1$ semi-virtual crossings, which are zero when we quotient out by $\mathcal{I}^{m+1}$. Now let us check the relations. We can rewrite the TC relations as



or in terms of equations $a_{ij}a_{ik} - a_{ik}a_{ij} = 0$ (recall that we compose from bottom to top). To show that its image is 0, we start with the following topological fact



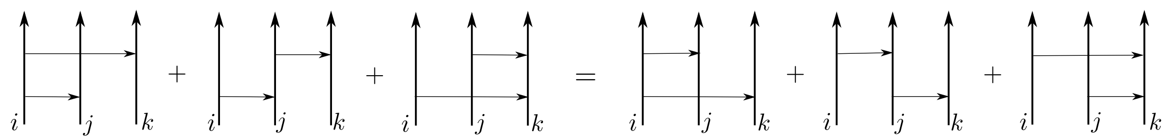
which follows from the OC relations. In terms of equations we obtain

$$R^+_{i,j}R^+_{i,k} - R^+_{i,k}R^+_{i,j} = 0.$$

Then we have

$$\pi(a_{ij}a_{ik} - a_{ik}a_{ij}) = (R^+_{i,j} - 1)(R^+_{i,k} - 1) - (R^+_{i,k} - 1)(R^+_{i,j} - 1) = R^+_{i,j}R^+_{i,k} - R^+_{i,k}R^+_{i,j} = 0,$$

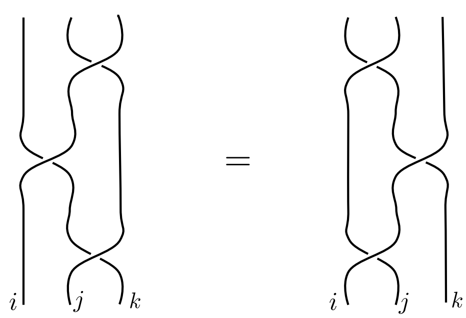
as required. Similarly for the $6T$ relations we first rewrite it in a vertical form



or in equation

$$[a_{i,j}, a_{i,k}] + [a_{i,j}, a_{j,k}] + [a_{i,k}, a_{j,k}] = 0. \tag{4.3}$$

To show that its image is 0 we consider the Reidemeister 3 relation



or in equation $R^+_{j,k}R^+_{i,k}R^+_{i,j} - R^+_{i,j}R^+_{i,k}R^+_{j,k} = 0$. It follows that

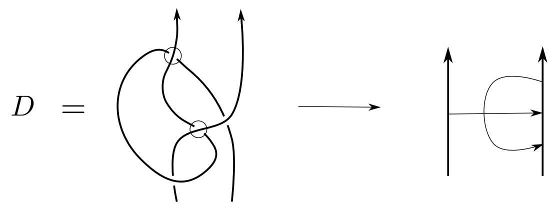
$$(\pi(a_{jk}) + 1)(\pi(a_{ik}) + 1)(\pi(a_{ij}) + 1) - (\pi(a_{ij}) + 1)(\pi(a_{ik}) + 1)(\pi(a_{jk}) + 1) = 0.$$

Note that the terms $\pi(a_{jk}a_{ik}a_{ij})$ and $\pi(a_{ij}a_{ik}a_{jk})$ vanish because we mod out by $\mathcal{I}^3$. Therefore the above equation reduces to the equation

$$\pi([a_{i,j}, a_{i,k}] + [a_{i,j}, a_{j,k}] + [a_{i,k}, a_{j,k}]) = 0,$$

as required.

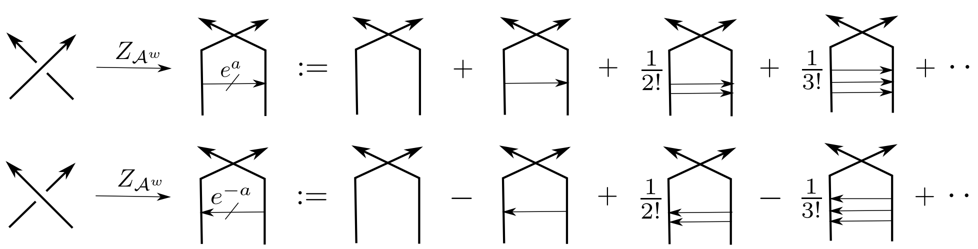
To see that π is surjective, consider an element of $\mathcal{I}^m/\mathcal{I}^{m+1}$, i.e. a w-tangle D with m semi-virtual crossings modulo w-tangles with $m+1$ semi-virtual crossings. We can associate with D an arrow diagram, a.k.a. $\pi^{-1}(D)$ as follows. We go along the skeleton of D, and mark the positions of the semi-virtual crossings, ignoring the usual crossings. Then we replace each semi-virtual crossings by an arrow that goes from the overstrand to the understrand. Concretely let us look at an example, but the argument works for the general case.



Now to see that the arrow diagram is indeed $\pi^{-1}(D)$ we need to show that the following two w-tangles

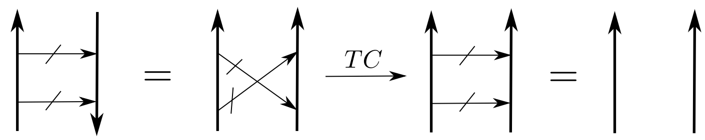
represents the same element in $\mathcal{I}^m/\mathcal{I}^{m+1}$. This follows because one can turn crossings to virtual crossings and vice versa at the cost of w-tangles with $m+1$ semi-virtual crossings, which vanish since we quotient out by $\mathcal{I}^{m+1}$.

Now we define the expansion $Z_{\mathcal{A}^w} : \mathcal{W} \to \mathcal{A}^w$ by sending the crossings to



Here we use the notation e^a to denote an exponential of arrows, as described above. We then extend $Z_{\mathcal{A}^w}$ to an arbitrary w-tangle using homomorphicity. Therefore $Z_{\mathcal{A}^w}$ is homomorphic by construction. To show that $Z_{\mathcal{A}^w}$ is well-defined, we need to check that $Z_{\mathcal{A}^w}$ satisfies the $R2$ relation, the $R3$ relation and the OC relations. For the $R2$ relation we consider two cases depending on the orientations of the strands

The first R2 move is clearly satisfied since $e^a e^{-a} = 1$. For the second $R2$ move the image of the left hand side under $Z_{\mathcal{A}^w}$ can be written as



Here again an arrow with a "/" denotes an exponential of arrows. The TC relation allows us to switch the tails of the arrows, then $e^a e^{-a} = 1$, as required.

Let us look at the left hand side of the $R3$ relation under $Z_{\mathcal{A}^w}$

$$\begin{aligned} Z_{\mathcal{A}^w}(R^+_{jk}R^+_{ik}R^+_{ij}) &= e^{a_{jk}}e^{a_{ik}}e^{a_{ij}} \\ &= e^{a_{jk}}e^{a_{ik}+a_{ij}} \quad \text{(because of the TC relation: } [a_{ij}, a_{ik}] = 0) \\ &= e^{a_{ik}+a_{ij}+a_{jk}} \quad \text{(because of the } \overrightarrow{4T} \text{ relation: } [a_{ij} + a_{ik}, a_{jk}] = 0). \end{aligned}$$

Similarly the right hand side of $R3$ is given by

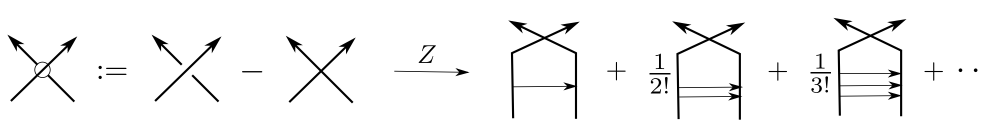
$$\begin{aligned} Z_{\mathcal{A}^w}(R^+_{ij}R^+_{ik}R^+_{jk}) &= e^{a_{ij}}e^{a_{ik}}e^{a_{jk}} \\ &= e^{a_{ij}+a_{ik}}e^{a_{jk}} \quad \text{(because of the TC relation: } [a_{ij}, a_{ik}] = 0) \\ &= e^{a_{ik}+a_{ij}+a_{jk}} \quad \text{(because of the } \overrightarrow{4T} \text{ relation: } [a_{ij} + a_{ik}, a_{jk}] = 0), \end{aligned}$$

as required. For the OC relation, its image under $Z_{\mathcal{A}^w}$ is

The two sides are then the same due to the TC relation. Finally to see that $Z_{\mathcal{A}^w}$ is an expansion we need to verify that

$$Z_{\mathcal{A}^w}(\pi(a_n)) = a_n + \text{ higher order terms,}$$

where $a_n \in \mathcal{A}^w_n$, i.e. an arrow diagram with n arrows. By construction it suffices to verify for the case $n = 1$. We have that



Then for a general w-tangle we obtain the identity by homomorphicity of $Z_{\mathcal{A}^w}$. Identifying $\mathcal{A}^w$ with $\mathrm{gr}\mathcal{W}$ and Z with $\pi \circ Z_{\mathcal{A}^w}$ we obtain a homomorphic expansion. □

Chapter 5

Relations with Lie Algebras

5.1 From $\mathcal{A}^w$ to Lie algebras

In this section we aim to elucidate the connection between $\mathcal{A}^w$ and Lie algebras. First let us recall the semidirect product of two Lie algebras. Let $\mathfrak{g}$ and $\mathfrak{h}$ be finite-dimensional Lie algebras and suppose that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations, this means that

$$x \cdot [\phi, \psi] = [x \cdot \phi, \psi] + [\phi, x \cdot \psi], \quad x \in \mathfrak{g},\ \phi, \psi \in \mathfrak{h}.$$

Then the *semidirect product* of $\mathfrak{g}$ and $\mathfrak{h}$, denoted by $\mathfrak{h} \rtimes \mathfrak{g}$, is $\mathfrak{h} \oplus \mathfrak{g}$ equipped with the following Lie bracket:

$$[(\phi_1, x_1), (\phi_2, x_2)] = ([\phi_1, \phi_2] + x_1 \cdot \phi_2 - x_2 \cdot \phi_1, [x_1, x_2]),$$

where $\phi_1, \phi_2 \in \mathfrak{h}$ and $x_1, x_2 \in \mathfrak{g}$. We leave it to the readers to check that the above is indeed a Lie bracket. When $\mathfrak{h}$ is $\mathfrak{g}^*$ with the trivial Lie bracket we define

$$I\mathfrak{g} := \mathfrak{g}^* \rtimes \mathfrak{g}.$$

Here $\mathfrak{g}$ acts on $\mathfrak{g}^*$ by the *coadjoint action*:

$$(x \cdot \phi)(y) = \phi([y, x]), \quad x, y \in \mathfrak{g},\ \phi \in \mathfrak{g}^*.$$

The Lie algebra $I\mathfrak{g}$ is a special case of what is known as a double (see [CP94]).

Now given a finite dimensional Lie algebra $\mathfrak{g}$ we can define a meta-monoid $\mathbb{U}(I\mathfrak{g})$ as follows. Let $U(I\mathfrak{g})$ be the universal enveloping algebra of $I\mathfrak{g}$. For a finite set of labels X, we let

$$\mathbb{U}(I\mathfrak{g})^X = U(I\mathfrak{g})^{\widehat{\otimes} X}.$$

Here each factor in the tensor product is labeled by an element of X and $\widehat{\otimes}$ is the *completed* tensor product, i.e. we allow series instead of just finite summations. For the completion, we define the degree of $\mathfrak{g}^*$ to be 1 and the degree of $\mathfrak{g}$ to be 0. So for instance the element $\phi_1\phi_2 \otimes \phi_1 x_1 \otimes x_2^2$, where $\phi_1, \phi_2 \in \mathfrak{g}^*$ and $x_1, x_2 \in \mathfrak{g}$, has degree $2 + 1 + 0 = 3$. In general we should also specify the labels of the components of the tensor product, but we suppress the labels when they do not play a role or if no ambiguity is ensued.

The operations in a meta-monoid is defined in a straightforward manner: *disjoint union* corresponds to tensor product, for example:

$$(x_1\phi_2 \otimes x_2\phi_1) \sqcup (x_1 \otimes \phi_2) = x_1\phi_2 \otimes x_2\phi_1 \otimes x_1 \otimes \phi_2,$$

stitching corresponds to multiplication of tensor factors, for example:

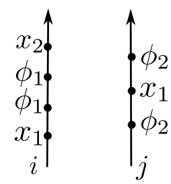
$$(\underbrace{x_1\phi_2}_{1} \otimes \underbrace{x_2^2\phi_1}_{2} \otimes \underbrace{x_1^2x_2}_{3}) \mathbin{/\!/} m_2^{3,2} = \underbrace{x_1\phi_2}_{1} \otimes \underbrace{x_1^2x_2^3\phi_1}_{2},$$

where the underbraces indicate the labels, *deletion* is obtained from the map $U(I\mathfrak{g}) \to \mathbb{Q}$ which is the identity on $\mathbb{Q}$ and zero otherwise, for example

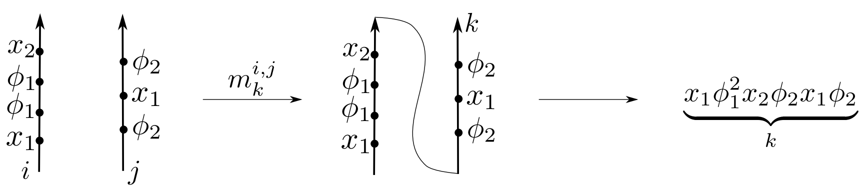
$$(\underbrace{x_1}_{1} \otimes \underbrace{x_1\phi_2}_{2} \otimes \underbrace{1}_{3}) \mathbin{/\!/} \eta_3 = x_1 \otimes x_1\phi_2, \quad \text{but} \quad (\underbrace{x_1}_{1} \otimes \underbrace{x_1\phi_2}_{2} \otimes \underbrace{1}_{3}) \mathbin{/\!/} \eta_1 = 0.$$

We leave it as an exercise to verify that these operations satisfy the axioms of a meta-monoid. One can visualize an element of $U(I\mathfrak{g})^{\widehat{\otimes} X}$ as "beads on strands" as follows. We think of each tensor factor of $U(I\mathfrak{g})^{\widehat{\otimes} X}$ as a directed strand and the generators as beads on a strand. For example,

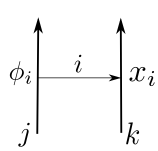
$$\underbrace{x_1\phi_1^2x_2}_{i} \otimes \underbrace{\phi_2x_1\phi_2}_{j} \longrightarrow$$



Then one can interpret the meta-monoid operations visually. For instance the stitching operation is given by

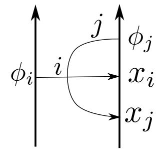


There is a meta-monoid homomorphism $T_{\mathfrak{g}} : \mathcal{A}^w \to \mathbb{U}(I\mathfrak{g})$ given as follows. Since $\mathcal{A}^w$ is generated as a meta-monoid by arrow diagrams with a single arrow, it suffices to define $T_{\mathfrak{g}}$ on these generators. Specifically, choose a basis $\{x_i\}_{i=1}^n$ of $\mathfrak{g}$ with corresponding dual basis $\{\phi_i\}_{i=1}^n$ of $\mathfrak{g}^*$, i.e. $\phi_i(x_j) = \delta_{i,j}$, the Kronecker δ function. For an arrow, we label it with an index $i \in \{1, \dots, n\}$, place ϕ_i at the tail of the arrow and x_i at the head of the arrow and then sum over i:



$$\xrightarrow{T_{\mathfrak{g}}} \sum_{i=1}^{n} \phi_i \otimes x_i$$

Here the image lies in $U(I\mathfrak{g})^{\otimes\{j,k\}}$, where j, k are the labels of the strands. As another example, we have



$$\xrightarrow{T_{\mathfrak{g}}} \quad \sum_{i,j=1}^{n} \phi_i \otimes x_j x_i \phi_j$$

Note that we read the elements along the orientation of the skeleton.

Proposition 5.1. *The map $T_{\mathfrak{g}} : \mathcal{A}^w \to \mathbb{U}(I\mathfrak{g})$ is well-defined, i.e. it does not depend on a choice of basis and satisfies the $\overrightarrow{4T}$ and TC relations.*

Proof. Let us first show that the map $T_{\mathfrak{g}}$ does not depend on a choice of basis. Given two bases $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ of $\mathfrak{g}$ with corresponding dual bases $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ of $\mathfrak{g}^*$ and suppose that

$$y_j = \sum_{i=1}^{n} a_{ij} x_i, \quad \psi_j = \sum_{i=1}^{n} b_{ij} \phi_i, \quad j = 1, 2, \ldots, n.$$

Let $A = (a_{ij})_{1 \le i,j \le n}$ and $B = (b_{ij})_{1 \le i,j \le n}$. We leave it as an exercise in linear algebra to show that $B = (A^{-1})^t$. Then it suffices to show that the term

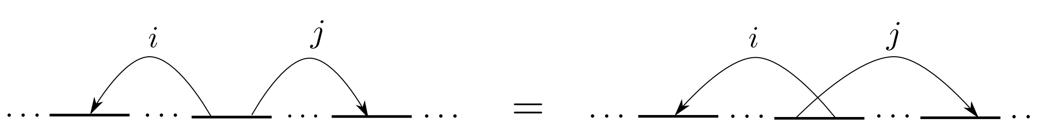
$$\sum_{i=1}^{n} \phi_i \otimes x_i \in U(I\mathfrak{g})^{\otimes 2},$$

which corresponds to a single arrow, does not depend on a choice of basis. Indeed, we have

$$\begin{aligned}
\sum_{j=1}^{n} \psi_j \otimes y_j &= \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} \phi_i \otimes \sum_{k=1}^{n} a_{kj} x_k \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{kj} \phi_i \otimes x_k \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} (BA^t)_{ik} \phi_i \otimes x_k \\
&= \sum_{k=1}^{n} \sum_{i=1}^{n} \delta_{ik} \phi_i \otimes x_k \quad \text{(since } BA^t = I\text{)} \\
&= \sum_{i=1}^{n} \phi_i \otimes x_i,
\end{aligned}$$

as required.

Next let us prove the TC relations



Under $T_{\mathfrak{g}}$, the left hand side is given by

$$\sum_{i,j=1}^{n} \ldots x_i \ldots \phi_j \phi_i \ldots x_j \ldots$$

and the right hand side is given by

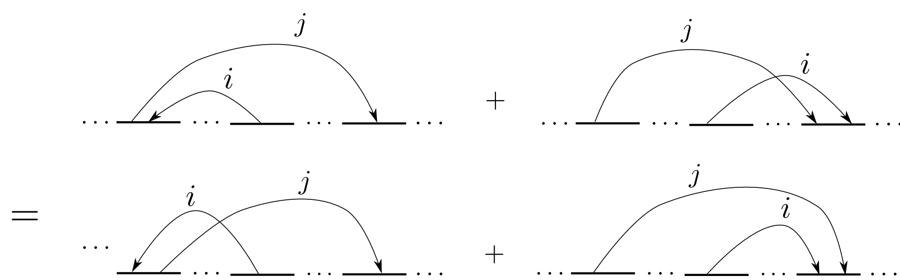
$$\sum_{i,j=1}^{n} \dots x_i \dots \phi_i \phi_j \dots x_j \dots$$

Here again ...'s denote other elements of $U(I\mathfrak{g})$, which stay the same on both sides. Then

$$\sum_{i,j=1}^{n} \dots x_i \dots \phi_j \phi_i \dots x_j \dots \; - \sum_{i,j=1}^{n} \dots x_i \dots \phi_i \phi_j \dots x_j \dots \; = \sum_{i,j=1}^{n} \dots x_i \dots [\phi_j, \phi_i] \dots x_j \dots \; = 0,$$

since the Lie algebra $\mathfrak{g}^*$ is commutative.

Finally let us proceed to show the $\overrightarrow{4T}$ relation



For that we first let c_{ijk} be the structure constants of $\mathfrak{g}$, i.e.

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ijk} x_k, \quad 1 \le i, j \le n.$$

Note that $[\phi_j, x_i] = -x_i \cdot \phi_j$. It is a simple exercise in linear algebra to show that

$$-x_i \cdot \phi_j = \sum_{k=1}^{n} c_{ikj} \phi_k, \quad 1 \le i, j \le n.$$

Under $T_{\mathfrak{g}}$, the left hand side of $\overrightarrow{4T}$ is

$$\sum_{i,j=1}^{n} \dots \phi_j x_i \dots \phi_i \dots x_j \dots \; + \dots \phi_j \dots \phi_i \dots x_j x_i \dots$$

and the right hand side is

$$\sum_{i,j=1}^{n} \dots x_i \phi_j \dots \phi_i \dots x_j \dots \; + \dots \phi_j \dots \phi_i \dots x_i x_j \dots$$

Taking the difference of both sides we obtain

$$\begin{aligned} &\sum_{i,j=1}^{n} \dots [\phi_j, x_i] \dots \phi_i \dots x_j \dots \; - \dots \phi_j \dots \phi_i \dots [x_i, x_j] \dots \\ &= \sum_{i,j,k=1}^{n} \dots c_{ikj} \phi_k \dots \phi_i \dots x_j \dots \; - \sum_{i,j,k=1}^{n} \dots \phi_j \dots \phi_i \dots c_{ijk} x_k \dots \end{aligned}$$

$$= \sum_{i,j,k=1}^{n} \ldots c_{ijk}\phi_j \ldots \phi_i \ldots x_k \ldots \; - \sum_{i,j,k=1}^{n} \ldots \phi_j \ldots \phi_i \ldots c_{ijk}x_k \ldots \; = 0,$$

as required. □

5.2 The Lie Algebra $\mathfrak{g}_0$

In this section let us specialize to the simplest non-trivial case, namely when $\mathfrak{g}$ is the non-abelian 2 dimensional Lie algebra. Specifically $\mathfrak{g}$ has the following presentation

$$\mathfrak{g} = \mathbb{Q}\langle c, w\rangle / [w, c] = w.$$

Then the dual Lie algebra $\mathfrak{g}^*$ has the following presentation

$$\mathfrak{g}^* = \mathbb{Q}\langle b = c^*, u = w^*\rangle / [b, u] = 0.$$

Let us compute the brackets, for instance,

$$[u, w] = -w \cdot u.$$

Now by the definition of the coadjoint action we have

$$(w \cdot u)(c) = u([c, w]) = -u(w) = -w^*(w) = -1.$$

Thus we get $[u, w] = c^* = b$. Similarly we obtain $[u, c] = -u$, and $[b, \cdot] = 0$. In other words, b is central. In this case we let $\mathfrak{g}_0 := I\mathfrak{g}$. So the Lie algebra $\mathfrak{g}_0$ is the four-dimensional vector space

$$\mathfrak{g}_0 = \mathbb{Q}\langle b, c, u, w\rangle$$

equipped with the Lie brackets

$$[b, \cdot] = 0, \quad [c, u] = u, \quad [c, w] = -w, \quad [u, w] = b. \tag{5.1}$$

From our convention the degrees of b and u are 1 and the degrees of c and w are 0. Then one can check that the Lie bracket preserves the degree and hence $\mathfrak{g}_0$ is a graded Lie algebra. In practice, it is useful to have a matrix representation of $\mathfrak{g}_0$

Proposition 5.2. *The Lie algebra $\mathfrak{g}_0$ has the following faithful representation*

$$b \mapsto \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; u \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; c \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; w \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. The four matrices are clearly linearly independent and it is easy to check that they also satisfy the commutation relations given in (5.1). □

5.3 The meta-monoid $\mathbb{G}_0$

In this section we analyze the meta-monoid $\mathbb{U}(\mathfrak{g}_0)$. Let X be a finite set of labels, and consider

$$\mathbb{U}(\mathfrak{g}_0)^X = U(\mathfrak{g}_0)^{\widehat{\otimes} X} = \widehat{U}\left(\bigoplus_{j\in X} \mathfrak{g}_{0,j}\right),$$

where each $\mathfrak{g}_{0,j}$ is a copy of $\mathfrak{g}_0$ and $\widehat{U}$ denotes the degree-completed universal enveloping algebra. The Lie algebra $\bigoplus_{j\in X} \mathfrak{g}_{0,j}$ can be given a description in terms of generators and relations as follows

$$\bigoplus_{j\in X} \mathfrak{g}_{0,j} = \mathbb{Q}\left\langle b_j, c_j, u_j, w_j : j\in X\right\rangle$$

subject to the relations: generators of different indices commute, and

$$[b_j,\cdot] = 0, \quad [c_j,u_j] = u_j, \quad [c_j,w_j] = -w_j, \quad [u_j,w_j] = b_j$$

for all $j\in X$. In other words, we can index the generators of $\mathfrak{g}_0$ by the labels in X to specify which factors of the tensor product they belong to. This results in a more streamlined notations. Concretely, we can write

$$\underbrace{buw}_{i}\otimes\underbrace{wu^2}_{j}\otimes\underbrace{c^2w^3}_{k} \longrightarrow b_iu_iw_iw_ju_j^2c_k^2w_k^3, \quad i,j,k\in X.$$

Since the b_j's are central, we can absorb them into the ground field, and so we think of an element of $\widehat{U}\left(\bigoplus_{j\in X}\mathfrak{g}_{0,j}\right)$ as a power series in c_i, u_j, w_k with coefficients rational functions in b_j's. For the completion recall our convention that the degrees of b_i and u_j are 1 and the degrees of c_i and w_k are 0.

By the PBW theorem [AK08] we can write each element of the universal enveloping algebra in terms of monomials in some particular order of the basis elements which can be fixed in advance. For that purpose let us introduce the *ordering operators* $\mathbb{O}(\cdot|specs)$, which are linear operators

$$\mathbb{O}(\cdot|specs) : \mathbb{Q}[\![b_j,c_j,u_j,w_j : j\in X]\!] \to \widehat{U}\left(\bigoplus_{j\in X}\mathfrak{g}_{0,j}\right)$$

Here $\mathbb{Q}[\![b_j,c_j,u_j,w_j : j\in X]\!]$ is the algebra of power series in the commuting variables b_j,c_j,u_j,w_j where $\deg b_j = \deg u_j = 1$, $\deg c_j = \deg w_j = 0$, for $j\in X$, and $specs$ specifies how we should order the variables. Since b_j's are central, we only need to order c_j,u_j,w_j (compare with *normal ordering* in the physics literature [VMMC06, PHP$^+$07]). For instance,

$$\mathbb{O}(e^{b_1}u_1e^{c_1}e^{u_2}w_2|c_1u_1,w_2u_2) = e^{b_1}e^{c_1}u_1w_2e^{u_2} = w_2e^{u_2}e^{b_1}e^{c_1}u_1,$$

where the second equality follows because generators of different indices commute. As another example, note that

$$\mathbb{O}(e^{u_1w_1}|w_1u_1) = \sum_{k=0}^{\infty}\frac{1}{k!}w_1^ku_1^k \neq e^{u_1w_1} = \sum_{k=0}^{\infty}\frac{1}{k!}(u_1w_1)^k.$$

However

$$\mathbb{O}(e^{u_1w_2}|w_2u_1) = \sum_{k=0}^{\infty}\frac{1}{k!}w_2^ku_1^k = e^{u_1w_2} = \sum_{k=0}^{\infty}\frac{1}{k!}u_1^kw_2^k$$

because u_1 and w_2 commute.

Now we are ready to describe our *meta-monoid* $\mathbb{G}_0$. For a finite set X of labels, let $\mathbb{G}_0^X$ be the collection of *commutative* series of the form

$$f = \omega \exp\left(\sum_{i,j\in X} l_{i,j} b_i c_j + q_{i,j} u_i w_j\right), \tag{5.2}$$

where each $l_{i,j}$ is an integer and ω and $q_{i,j}$ are power series in b_k for $k \in X$. The element f is characterized by a triple consisting of the "scalar" ω and two labeled matrices L and Q

$$L = \left(\begin{array}{c|c} \omega & X \\ \hline X & l_{ij} \end{array}\right), \qquad Q = \left(\begin{array}{c|c} \omega & X \\ \hline X & q_{ij} \end{array}\right).$$

For aesthetic purpose we stick ω to the empty corners of L and Q. The scalar ω is required to satisfied the following condition

(a) ω is a function of e^{b_i}, $i \in X$, and with the substitution $e^{b_i} \to t_i$ we have $\omega|_{t_i\to 1} = 1$.

The matrix Q is also required to satisfy certain conditions. First we need to introduce some notations. Let D be a diagonal matrix labeled by X whose (j,j)-diagonal entry is b_j for $j \in X$. For column vectors $\boldsymbol{t} = (t_j : j \in X)^T$ and $\boldsymbol{a} = (a_j : j \in X)^T$ we define

$$\boldsymbol{t}^{\boldsymbol{a}} := \prod_{j\in X} t_j^{a_j}.$$

Then for a matrix $A = (\boldsymbol{a}_j : j \in X)$, where $\boldsymbol{a}_j = (a_{ij} : i \in X)^T$, we define $\boldsymbol{t}^A$ to be the diagonal matrix whose (j,j)-entry is given by

$$\boldsymbol{t}^{\boldsymbol{a}_j} = \prod_{i\in X} t_i^{a_{ij}}.$$

Now we require the matrix Q to satisfy the following two conditions:

(i) each entry of DQ is a rational function in e^{b_i}'s, so we can make the change of variables $e^{b_i} \to t_i$ for $i \in X$,

(ii) with the substitution $e^{b_i} \to t_i$ for $i \in X$ we have $DQ\big|_{t_i\to 1} = \mathbf{0}$.

where $\mathbf{0}$ is the $n \times n$ matrix consisting of 0's. There is a map from $\mathbb{G}_0^X$ to $\mathbb{U}(\mathfrak{g}_0)^X$ given by

$$f \mapsto \mathbb{O}(f|c_i u_i w_i \text{ for } i \in X).$$

For each index the order is cuw, which we call cuw*-order* for short.

In order to perform stitching, we need to understand how to reorder the generators. For that we introduce the following *switching operators*

$$N^{uc} : \mathbb{Q}[\![b,u,c,w]\!] \to \mathbb{Q}[\![b,u,c,w]\!]$$

defined as follows. For $f \in \mathbb{Q}[\![b,u,c,w]\!]$, we have

$$\mathbb{O}(f|uc) = \mathbb{O}((f \mathbin{/\!\!/} N^{uc})|cu).$$

Notice that we only switch two consecutive variables in the ordering and the order of the remaining variables remains intact. As a simple example, in $\mathfrak{g}_0$ we have $[c, u] = u$, or $cu - uc = u$, so $uc = cu - u = (c-1)u$. It follows that $uc^2 = (c-1)^2u$. Therefore

$$uc^2 \mathbin{/\!/} N^{uc} = (c-1)^2 u.$$

In a similar fashion we can define the switching operators N^{wc} that switches the order wc to the order cw and N^{wu} that switches the order wu to uw. To understand these switching operators the following proposition will be useful.

Proposition 5.3. *In $U(\mathfrak{g}_0)$ we have the following identities*

1. $u^m c^n = (c-m)^n u^m$,
2. $w^m c^n = (c+m)^n w^m$,
3. $w^m u^n = \sum_{j=0}^{\min\{m,n\}} \binom{m}{j}\binom{n}{j} j!\,(-b)^j u^{n-j} w^{m-j} = \sum_{j=0}^{\min\{m,n\}} \frac{m!\,n!\,(-b)^j u^{n-j} w^{m-j}}{j!\,(m-j)!\,(n-j)!}$,

where m and n are non-negative integers.

Proof. The first two identities follow from $[c, u] = u$ and $[c, w] = -w$. For the third identity, using $[u, w] = b$, one can use induction. Alternatively, one can observe that to obtain the term $u^{n-j}w^{m-j}$, we have to choose j elements from u's, j elements from w's, there are $j!$ ways for them to interact, and each interaction will annihilate u and w and return an element $-b$. □

Proposition 5.4. *We have*

$$e^{\beta u+\gamma c} \mathbin{/\!/} N^{uc} = e^{\gamma c + e^{-\gamma}\beta u},$$

$$e^{\alpha w+\gamma c} \mathbin{/\!/} N^{wc} = e^{\gamma c + e^{\gamma}\alpha w},$$

where α, β, γ are scalars.

Proof. Let us show the first identity. The second identity is analogous and we leave it as an exercise. We have

$$\begin{aligned}
\mathbb{O}(e^{\beta u+\gamma c}|uc) = e^{\beta u}e^{\gamma c} &= \sum_{r,s=0}^{\infty} \frac{\beta^r\gamma^s}{r!\,s!} u^r c^s = \sum_{r,s=0}^{\infty} \frac{\beta^r\gamma^s}{r!\,s!}(c-r)^s u^r \\
&= \sum_{r=0}^{\infty} \frac{\beta^r}{r!}\left(\sum_{s=0}^{\infty}\frac{\gamma^s}{s!}(c-r)^s\right) u^r = \sum_{r=0}^{\infty}\frac{\beta^r}{r!} e^{\gamma(c-r)} u^r \\
&= e^{\gamma c}\sum_{r=0}^{\infty}\frac{(e^{-\gamma}\beta u)^r}{r!} = e^{\gamma c} e^{e^{-\gamma}\beta u} = \mathbb{O}(e^{\gamma c + e^{-\gamma}\beta u}|cu),
\end{aligned}$$

as required. □

Example 5.1. As a simple example, we have

$$e^{bc+uw} \mathbin{/\!/} N^{uc} = e^{bc + e^{-b}uw}.$$

To get the corresponding identity in the universal enveloping algebra we need to apply the ordering operator:

$$\mathbb{O}(e^{bc+uw}|wuc) = \mathbb{O}(e^{bc+e^{-b}uw}|wcu).$$

Expanding both sides we obtain the following identity in the universal enveloping algebra

$$\sum_{m,n=0}^{\infty} \frac{b^m}{m!\,n!} w^n u^n c^m = \sum_{m,n=0}^{\infty} \frac{b^m e^{-nb}}{m!\,n!} w^n c^m u^n = \sum_{m,n,p=0}^{\infty} \frac{(-n)^p b^{m+p}}{p!\,m!\,n!} w^n c^m u^n,$$

where equality is interpreted degree by degree (recall that these generators do not have the same degrees). ♣

Proposition 5.5. *We have the following identities*

$$\mathbb{O}(e^{\alpha w+\beta u}|wu) = \mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw),$$
$$e^{\beta u+\alpha w+\gamma uw} /\!\!/ N^{wu} = \nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\gamma uw},$$

where $\nu = (1+b\gamma)^{-1}$, and α, β, γ are scalars.

Proof. The first identity is the familiar Weyl commutation relation [ref]. For completeness we present here a combinatorial proof. The left hand side is

$$\begin{aligned}
\mathbb{O}(e^{\alpha w+\beta u}|wu) = e^{\alpha w}e^{\beta u} &= \sum_{m,n=0}^{\infty} \frac{\alpha^m\beta^n}{m!\,n!} w^m u^n \\
&= \sum_{m,n=0}^{\infty} \frac{\alpha^m\beta^n}{m!\,n!} \left(\sum_{r=0}^{\min\{m,n\}} \frac{m!\,n!\,(-b)^r}{r!\,(m-r)!\,(n-r)!} u^{n-r} w^{m-r} \right) \\
&= \sum_{m,n=0}^{\infty} \sum_{r=0}^{\min\{m,n\}} \frac{\alpha^m\beta^n(-b)^r}{r!\,(m-r)!\,(n-r)!} u^{n-r} w^{m-r} \\
&= \sum_{m,n=0}^{\infty} \sum_{r=0}^{\min\{m,n\}} \frac{(-b\alpha\beta)^r}{r!} \frac{(\beta u)^{n-r}}{(n-r)!} \frac{(\alpha w)^{m-r}}{(m-r)!} \\
&= e^{-b\alpha\beta} e^{\beta u} e^{\alpha w} = \mathbb{O}(e^{-b\alpha\beta+\beta u+\alpha w}|uw),
\end{aligned}$$

as required.

The case that involves the quadratic term uw is a bit more complicated. First let us recall a familiar trick: for a series $p(x)$ we have

$$p(x)e^{\alpha x} = p(\partial_\alpha)e^{\alpha x},$$

where ∂_α denotes the formal partial derivative with respect to α. We can then rewrite the left hand side as follows.

$$\begin{aligned}
\mathbb{O}(e^{\beta u+\alpha w+\gamma uw}|wu) = \mathbb{O}(e^{\gamma uw}e^{\beta u+\alpha w}|wu) = \mathbb{O}(e^{\gamma\partial_\beta\partial_\alpha}e^{\beta u+\alpha w}|wu) &= e^{\gamma\partial_\beta\partial_\alpha}\mathbb{O}(e^{\beta u+\alpha w}|wu) \\
&= e^{\gamma\partial_\alpha\partial_\beta}\mathbb{O}(e^{-b\alpha\beta+\alpha w+\beta u}|uw) \quad \text{(by the first identity)} \\
&= \mathbb{O}(e^{\gamma\partial_\alpha\partial_\beta}e^{-b\alpha\beta+\alpha w+\beta u}|uw)
\end{aligned}$$

Now we let

$$\psi(\alpha,\beta,\gamma) = e^{\gamma\partial_\alpha\partial_\beta} e^{-b\alpha\beta+\alpha w+\beta u}$$

as a formal power series in α, β, γ. Then ψ satisfies

$$\begin{cases} \psi(\alpha,\beta,0) = e^{-b\alpha\beta+\beta u+\alpha w} \\ \partial_\gamma\psi = \partial_\alpha\partial_\beta\psi. \end{cases}$$

Observe that there exists a unique series that satisfies the above initial value problem (IVP) since we can express the coefficient of a term of a certain degree in terms of the coefficients of lower degree terms. All that remains is to show that the series in the right hand side

$$\nu e^{-b\nu\alpha\beta+\nu\alpha w+\nu\beta u+\nu\gamma uw}$$

also satisfies the IVP. Clearly the initial condition is satisfied. To check that the series also satisfies the PDE is an exercise in multivariable calculus and we leave the details to the readers. □

Example 5.2. For a simple example we have

$$\mathbb{O}(e^{uw}|wu) = \mathbb{O}\left(\frac{1}{1+b}e^{\left(\frac{1}{1+b}\right)uw}\middle| uw\right).$$

Expanding both sides we obtain

$$\sum_{m=0}^{\infty}\frac{1}{m!}u^m w^m = \sum_{m=0}^{\infty}\frac{1}{m!}\left(\sum_{s=0}^{\infty}(-b)^s\right)^{m+1} u^m w^m.$$

Again equality is interpreted degree by degree. ♣

Now we are ready to define the stitching operation. First we extend the switching operators to allow indices. For instance we define $N_k^{u_i c_j}$ to be

$$f \mathbin{/\!\!/} N_k^{u_i c_j} := (f \mathbin{/\!\!/} N^{u_i c_j})|_{u_i\to u_k, c_j\to c_k}$$

or in the universal enveloping algebra

$$\mathbb{O}((f|_{u_i\to u_k, c_j\to c_k})\big|\ldots u_k c_k\ldots) = \mathbb{O}(f \mathbin{/\!\!/} N_k^{u_i c_j}|\ldots c_k u_k\ldots).$$

We define the other switching operators similarly. We define the stitching operation $m_k^{i,j}$ in $\mathbb{G}_0$ by pulling back the stitching operation in $\mathbb{U}(\mathfrak{g}_0)$. Namely, for $f \in \mathbb{Q}[\![\ldots, c_i, u_i, w_i, c_j, u_j, w_j\ldots]\!]$ the stitching operation $m_k^{i,j}$ is characterized by

$$\mathbb{O}(f|\ldots c_i u_i w_i c_j u_j w_j\ldots) \mathbin{/\!\!/} m_k^{i,j} = \mathbb{O}(f \mathbin{/\!\!/} m_k^{i,j}|\ldots c_k u_k w_k\ldots).$$

Note that $m_k^{i,j}$ on the left hand side is the stitching operation in $\mathbb{U}(\mathfrak{g}_0)$. Concretely we first put the two orderings $c_i u_i w_i$ and $c_j u_j w_j$ next to each other on the strand labeled k

$$c_i u_i w_i c_j u_j w_j$$

and then use the switching operators to turn the above to the cuw order. Namely

$$c_i u_i (w_i c_j) u_j w_j \xrightarrow{N_k^{w_i c_j}} c_i (u_i c_k) w_k u_j w_j \xrightarrow{N_k^{u_i c_k}} c_i c_k u_k (w_k u_j) w_j \xrightarrow{N_k^{w_k u_j}} c_i c_k u_k u_k w_k w_j.$$

Finally we relabel c_i to c_k and w_j to w_k. From the construction of the stitching operation we see that meta-associativity is automatically satisfied because it is the pullback of the stitching operation in $\mathbb{U}(\mathfrak{g}_0)$. However, we need to check that $m_k^{i,j}$ is well-defined, i.e. after stitching the scalar ω satisfies condition (a), the matrix L consists of integer entries, and the matrix Q satisfies conditions (i) and (ii). For this purpose let us consider an arbitrary element ζ of $\mathbb{G}_0^{\{i,j\}\cup S}$ given in matrix form by

$$L = \left(\begin{array}{c|ccc} \omega & i & j & S \\ \hline i & l_{ii} & l_{ij} & \boldsymbol{l}_{iS} \\ j & l_{ji} & l_{jj} & \boldsymbol{l}_{jS} \\ S & \boldsymbol{l}_{Si} & \boldsymbol{l}_{Sj} & \boldsymbol{l}_{SS} \end{array}\right), \qquad Q = \left(\begin{array}{c|ccc} \omega & i & j & S \\ \hline i & q_{ii} & q_{ij} & \boldsymbol{q}_{iS} \\ j & q_{ji} & q_{jj} & \boldsymbol{q}_{jS} \\ S & \boldsymbol{q}_{Si} & \boldsymbol{q}_{Sj} & \boldsymbol{q}_{SS} \end{array}\right),$$

and the matrix D in this case is

$$D = \left(\begin{array}{c|ccc} & i & j & S \\ \hline i & b_i & 0 & \mathbf{0} \\ j & 0 & b_j & \mathbf{0} \\ S & \mathbf{0} & \mathbf{0} & \boldsymbol{b}_{SS} \end{array}\right).$$

Then it is a computation exercise to verify that for $\zeta /\!\!/ m_k^{i,j}$ the matrix L is given by

$$\left(\begin{array}{c|cc} & k & S \\ \hline k & l_{ii}+l_{ij}+l_{ji}+l_{jj} & \boldsymbol{l}_{iS}+\boldsymbol{l}_{jS} \\ S & \boldsymbol{l}_{Si}+\boldsymbol{l}_{Sj} & \boldsymbol{l}_{SS} \end{array}\right),$$

which consists of integer entries. The scalar part is given by, where we set $e^{b_x} \to t_x$,

$$\frac{\omega}{1 + t_k^{l_{ij}} t_k^{l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji}}.$$

Observe that $b_k q_{ji}$ is a function of e^{b_x} by assumption and when we set $t_x \to 1$ the term $b_k q_{ji}$ vanishes. Thus the scalar part satisfies condition (a). Finally the matrix part $DQ|_{e^{b_x} \to t_x}$ is given by

$$\left(\begin{array}{c|cc} & k & S \\ \hline k & \frac{t_k^{-l_{ij}-l_{jj}} \boldsymbol{t}_S^{-\boldsymbol{l}_{Sj}} b_k q_{ij} + t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji} + b_k(q_{ii}+q_{jj}) + b_k^2(q_{ij}q_{ji}-q_{ii}q_{jj})}{1+t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji}} & \frac{t_k^{-l_{ij}-l_{jj}} \boldsymbol{t}_S^{-\boldsymbol{l}_{Sj}} b_k \boldsymbol{q}_{iS} + b_k \boldsymbol{q}_{jS} + b_k^2(q_{ji}\boldsymbol{q}_{iS}-q_{ii}\boldsymbol{q}_{jS})}{1+t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji}} \\ S & \frac{t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} (\boldsymbol{b}_{SS}\boldsymbol{q}_{Si} - b_k \boldsymbol{b}_{SS}(q_{jj}\boldsymbol{q}_{Si} - q_{ji}\boldsymbol{q}_{Sj})) + \boldsymbol{b}_{SS}\boldsymbol{q}_{Sj}}{1+t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji}} & \frac{\boldsymbol{b}_{SS}\boldsymbol{q}_{SS} - t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k \boldsymbol{b}_{SS}(\boldsymbol{q}_{Si}\boldsymbol{q}_{jS} - q_{ji}\boldsymbol{q}_{SS})}{1+t_k^{l_{ij}+l_{jj}} \boldsymbol{t}_S^{\boldsymbol{l}_{Sj}} b_k q_{ji}} \end{array}\right).$$

In this form we can check readily from the assumption $\zeta \in \mathbb{G}_0^{\{i,j\}\cup S}$ that the two conditions (i) and (ii) are still satisfied. Condition (i) follows because in each term the powers of b's agree with the powers of q's. Condition (ii) is true because when all t_x are set to 1 we have $b_k q_{ij} = b_k q_{ji} = 0$, $b_k \boldsymbol{q}_{iS} = b_k \boldsymbol{q}_{jS} = \mathbf{0}$, $\boldsymbol{b}_{SS}\boldsymbol{q}_{Si} = \boldsymbol{b}_{SS}\boldsymbol{q}_{Sj} = \mathbf{0}$, and $\boldsymbol{b}_k q_{ii} = \boldsymbol{b}_k q_{jj} = 1$, $\boldsymbol{b}_{SS}\boldsymbol{q}_{SS} = I$. Therefore $\mathbb{G}_0$ is a meta-monoid. We summarize the above discussion in the following proposition.

Proposition 5.6. *We have a meta-monoid homomorphism* $\iota : \mathbb{G}_0 \to \mathbb{U}(\mathfrak{g}_0)$ *given by*

$$f \mapsto \mathbb{O}(f|c_j u_j w_j : j \in X),$$

where $f \in \mathbb{G}_0^X$.

Proposition 5.7. *There is a meta-monoid homomorphism* ψ *from the meta-monoid of w-tangles* $\mathcal{W}$ *to the meta-monoid* $\mathbb{G}_0$ *given by*

$$R_{i,j}^{\pm} \mapsto \exp\left(\pm b_i c_j + \frac{e^{\pm b_i} - 1}{b_i} u_i w_j\right)$$

Mathematica®. Before presenting the proof let us describe our implementation of $\mathbb{G}_0$ in Mathematica. A reader with Mathematica can get the notebook here. First we write a subroutine CF to simplify the expressions

```
CF[expr_] := expr // Simplify;
𝔼 /: CF[𝔼[ω_, λ_]] := 𝔼[CF[ω], CF[λ]];
𝔼 /: 𝔼[ω1_, λ1_] 𝔼[ω2_, λ2_] := CF@𝔼[ω1 ω2, λ1 + λ2];
𝔼[ω1_, λ1_] ≡ 𝔼[ω2_, λ2_] := CF[ω1 == ω2 ∧ λ1 == λ2];
```

Notice that here the input has the form $\mathbb{E}[\omega, \lambda]$, where ω is the scalar part and λ is the bilinear form in $b_i c_j$ and $u_k w_l$. So for instance we can input an arbitrary element of $\mathbb{G}_0^{\{i,j\}}$ as

```
𝔼[ω, Sum[l_{x,y} b_x c_y + q_{x,y} u_x w_y, {x, {i, j}}, {y, {i, j}}]]
```

and the output is

```
𝔼[ω, b_i c_i l_{i,i} + b_i c_j l_{i,j} + b_j c_i l_{j,i} + b_j c_j l_{j,j} + u_i w_i q_{i,i} + u_i w_j q_{i,j} + u_j w_i q_{j,i} + u_j w_j q_{j,j}]
```

Notice also that we use the notation $\equiv$ to compare two elements of the form $\mathbb{E}[\omega_1, \lambda_1]$ and $\mathbb{E}[\omega_2, \lambda_2]$. Now we program the switching operators $N_k^{u_i c_j}$, $N_k^{w_i c_j}$, $N_k^{w_i u_j}$:

```
N_{u_i_ c_j_→k_}[𝔼[ω_, λ_]] := CF[
   𝔼[ω, e^{-γ} β u_k + γ c_k + (λ /. c_j | u_i → 0)] /. {γ → ∂_{c_j} λ, β → ∂_{u_i} λ}];
N_{w_i_ c_j_→k_}[𝔼[ω_, λ_]] := CF[
   𝔼[ω, e^γ α w_k + γ c_k + (λ /. c_j | w_i → 0)] /. {γ → ∂_{c_j} λ, α → ∂_{w_i} λ}];
N_{w_i_ u_j_→k_}[𝔼[ω_, λ_]] := CF[
   𝔼[ν ω, -b_k ν α β + ν β u_k + ν δ u_k w_k + ν α w_k + (λ /. w_i | u_j → 0)] /. ν → (1 + b_k δ)^{-1}
    /. {α → ∂_{w_i} λ /. u_j → 0, β → ∂_{u_j} λ /. w_i → 0, δ → ∂_{w_i,u_j} λ}];
```

and the stitching operation

```
gm_{i_,j_→k_}[𝔼[ω_, λ_]] := CF[Module[{x},
   (𝔼[ω, λ] // N_{w_i c_j→x} // N_{u_i c_x→x} // N_{w_x u_j→x}) /. {c_i → c_k, w_j → w_k, y_x ⧴ y_k, b_{i|j} → b_k}]]
```

Note that here we use the notation $gm_k^{i,j}$ to distinguish it from the stitching operations in other meta-monoids.

Proof. Again we just need to check the $R2$ moves, $R3$ moves and OC moves. One can check them directly by hand, but it is faster to use Mathematica. First we define the crossings

```
gR⁺i_,j_ = 𝔼[1, bi cj + bi^-1 (e^bi - 1) ui wj]; gR⁻i_,j_ = 𝔼[1, -bi cj + bi^-1 (e^-bi - 1) ui wj];
```

Again we use the notation $gR^+_{i,j}$ to distinguish it from the crossings in other meta-monoids. For the $R2$ move we consider

```
gR⁺i,j gR⁻k,l // gmi,k→i // gmj,l→j
```

The output is

```
𝔼[1, 0]
```

as expected. For the $R3$ move we test the following equality

```
(gR⁺1,4 gR⁺2,5 gR⁻6,3 // gm1,6→1 // gm2,4→2 // gm3,5→3) ≡
  (gR⁻1,4 gR⁺5,2 gR⁺6,3 // gm1,5→1 // gm2,6→2 // gm3,4→3)
```

and for the OC move we test the following equality

```
(gR⁺4,2 gR⁺1,3 // gm1,4→1) ≡ (gR⁺4,3 gR⁺1,2 // gm1,4→1)
```

They both return True, as required. □

Proposition 5.8. *There is a meta-monoid homomorphism $\eta : \mathbb{G}_0 \to \widetilde{\Gamma}$ given as follows. For a finite set X of labels we send*

$$\omega \exp\left(\sum_{i,j\in X} l_{ij}b_ic_j + q_{ij}u_iw_j\right) \mapsto \left(\left(\begin{array}{c|c} \omega^{-1} & X \\ \hline X & \boldsymbol{t}^L(I - DQ|_{e^{b_i}\to t_i}) \end{array}\right), \sum_{j\in X} \boldsymbol{t}^{\boldsymbol{l}_j}\boldsymbol{v}_j\right),$$

where $L = (l_{ij})_{i,j\in X}$, $Q = (q_{ij})_{i,j\in X}$, $\boldsymbol{l}_j$ is the jth column of L, and D is the diagonal matrix whose (i,i)-entry is b_i for all $i \in X$.

Mathematica®. Again let us implement the above map in Mathematica:

```
G0toΓ[e_] := Module[{A, λ, L, ω, Q, n, II, DD, T, M, σ, i, j},
  ω = e⟦1⟧ /. e^x_ :→ e^Simplify[x /. b_i_ :→ Log[t_i]] // Simplify;
  λ = e⟦2⟧;
  A = Union@Cases[λ, (b | c)_a_ :→ a, ∞];
  L = Outer[Factor[∂_b#1 ∂_c#2 λ] &, A, A];
  Q = Outer[Factor[∂_u#1 w#2 λ] &, A, A];
  n = Length[A];
  II = IdentityMatrix[n];
  DD = DiagonalMatrix[Table[b_i, {i, A}]];
  T = DiagonalMatrix[Table[Product[t_A⟦i⟧^L⟦i,j⟧, {i, 1, n}], {j, 1, n}]];
  σ = Sum[Product[t_A⟦i⟧^L⟦i,j⟧, {i, 1, n}] v_A⟦j⟧, {j, 1, n}];
  M = T.(II - DD.Q) /. e^x_ :→ e^Simplify[x /. b_i_ :→ Log[t_i]] // Simplify;
  eΓ[ω^-1, Table[y_i, {i, A}].M.Table[x_i, {i, A}], σ]];
```

The subroutine G0toΓ takes an element e in $\mathbb{G}_0^X$ and convert it to the corresponding element in $\widetilde{\Gamma}^X$.

Proof. Notice first that condition (a) ensures that we can divide by ω. From condition (a) of ω and conditions (i) and (ii) of Q we see that the image of η is indeed contained in $\widetilde{\Gamma}$. Suppose that $X = \{i,j\} \cup S$, where $\{i,j\} \cap S = \emptyset$. To show that η is a meta-monoid homomorphism we only need to check that

$$\zeta \mathbin{/\!/} m_k^{i,j} \mathbin{/\!/} \eta = \zeta \mathbin{/\!/} \eta \mathbin{/\!/} m_k^{i,j}, \quad \zeta \in \mathbb{G}_0^X. \tag{5.3}$$

This is a matter of computation. First we need to fix some notations. Suppose that ζ is given by

$$L = \left(\begin{array}{c|ccc} \omega & i & j & S \\ \hline i & l_{ii} & l_{ij} & \boldsymbol{l}_{iS} \\ j & l_{ji} & l_{jj} & \boldsymbol{l}_{jS} \\ S & \boldsymbol{l}_{Si} & \boldsymbol{l}_{Sj} & \boldsymbol{l}_{SS} \end{array}\right), \qquad Q = \left(\begin{array}{c|ccc} \omega & i & j & S \\ \hline i & q_{ii} & q_{ij} & \boldsymbol{q}_{iS} \\ j & q_{ji} & q_{jj} & \boldsymbol{q}_{jS} \\ S & \boldsymbol{q}_{Si} & \boldsymbol{q}_{Sj} & \boldsymbol{q}_{SS} \end{array}\right),$$

and the diagonal matrix D in this case is

$$D = \left(\begin{array}{c|ccc} & i & j & S \\ \hline i & b_i & 0 & \mathbf{0} \\ j & 0 & b_j & \mathbf{0} \\ S & \mathbf{0} & \mathbf{0} & \boldsymbol{b}_{SS} \end{array}\right).$$

We can check the equation (5.3) directly in Mathematica as follows. First we input ζ using the command

```
ζ = 𝔼[ω, Sum[l_{x,y} b_x c_y + q_{x,y} u_x w_y, {x, {i, j, S}}, {y, {i, j, S}}]]
```

And then we check (5.3) using the command

```
(ζ // G0toΓ // em_{i,j→k}) ≡ (ζ // gm_{i,j→k} // G0toΓ)
```

Mathematica then returns True, as required. ☐

Remark 5.1. The above proof is purely computational. Let us present a more abstract proof of why equation (5.3) should be true, which will also explains where the map η comes from. Again let

$$f = \exp\left(\sum_{i,j\in X} l_{ij} b_i c_j + q_{ij} u_i w_j\right).$$

To find a matrix representation of f we will define a representation of f on the vector space $\operatorname{span}\{u_i : i \in X\}$ given by

$$f \cdot u_k = \iota(f) u_k \iota(f)^{-1},$$

where ι is the inclusion map defined in Proposition 5.6. We claim that it is indeed a representation, i.e.

$$\iota(f) u_k \iota(f)^{-1} = \sum_{i\in X} \gamma_{ik} u_i.$$

To find the matrix $M = (\gamma_{ij})_{i,j\in X}$ our strategy is to "push" u_k past $\iota(f)$. For that, observe that the following identity can be proven easily by induction

$$w^n u = u w^n - n b w^{n-1}, \quad n \in \mathbb{Z}_{\geq 0}.$$

Then we have

$$\begin{aligned}\exp(q_{ik}u_iw_k)u_k &= \left(\sum_{n=0}^{\infty}\frac{1}{n!}q_{ik}^nu_i^nw_k^n\right)u_k\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}q_{ik}^nu_i^nu_kw_k^n - \sum_{n=1}^{\infty}\frac{b_k}{(n-1)!}q_{ik}^nu_i^nw_k^{n-1}\\ &= (u_k - b_kq_{ik}u_i)\exp(q_{ik}u_iw_k).\end{aligned}$$

Similarly using the identity

$$c^nu = u(c+1)^n,$$

we obtain

$$\begin{aligned}\exp(l_{ik}b_ic_k)u_k &= \sum_{n=0}^{\infty}\frac{1}{n!}l_{ik}^nb_i^nc_k^nu_k = \sum_{n=0}^{\infty}\frac{1}{n!}l_{ik}^nb_i^nu_k(c_k+1)^n\\ &= u_k\exp(l_{ik}b_i(c_k+1)) = \exp(l_{ik}b_i)u_k\exp(l_{ik}b_ic_k).\end{aligned}$$

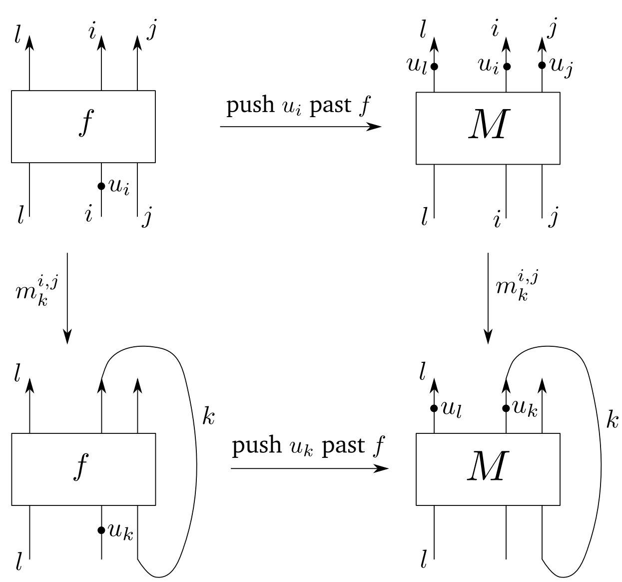
Therefore it follows that

$$\gamma_{ij} = \begin{cases}\prod_{k\in X}\exp(l_{kj}b_k)(1-b_jq_{jj}), & i=j,\\ -\prod_{k\in X}\exp(l_{ki}b_k)b_jq_{ij}, & i\neq j.\end{cases}$$

Then we see that

$$f /\!/ \eta = DMD^{-1},$$

where D again denotes the diagonal matrix whose (j,j)-entry is b_j for $j\in X$. In the "beads on strands" interpretation as in Section 5.1, each term of f can be visualized as a bead diagram. We then obtain a matrix representation of f by putting u_k at the bottom of strand k and then push it past the whole diagram. Together with the interpretation of stitching as connecting output to input (see section [ref]) we see that equation (5.3) is true. Schematically, we can visualize it as follows:



The reason is this, for the left hand side, we first connect the output to the input and then push u_k past the diagram; for the right hand side, we first push u_i past the diagram and then connect the output to the input. Geometrically the two ways should give the same final result, as expected.

To prove the next proposition let us introduce a useful construction known as the *Euler operator* [WKO1]. For a completed graded algebra with unit, in which all degrees are non-negative (think of $\mathbb{U}(\mathfrak{g}_0)$ in our case) the *Euler operator* is the operator $E : A \to A$ given by $Ea = (\deg a)a$ for a homogeneous element $a \in A$. If $f \in A$ is a series that starts with 1, we define the operator $\widetilde{E} : A \to A$ by

$$\widetilde{E}f = f^{-1}Ef.$$

Note that f is invertible because it starts with 1. We call $\widetilde{Z}$ the *normalized Euler operator*. There are several important properties of the Euler operator that we need and we refer the readers to [WKO1] for more details.

(a) The operator E is a derivation, i.e.

$$E(\phi_1\phi_2) = (E\phi_1)\phi_2 + \phi_1 E\phi_2, \quad \phi_1, \phi_2 \in A.$$

(b) The operator $\widetilde{E}$ is one-to-one.

(c) For a series $\phi \in A$,

$$E(e^\phi) = e^\phi \left(\frac{1 - e^{-\operatorname{ad}\phi}}{\operatorname{ad}\phi}\right)(E\phi).$$

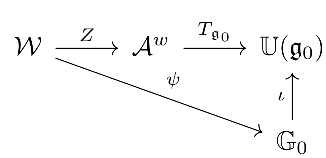
Here $(\operatorname{ad}\phi)(x) = [\phi, x]$ for $x \in A$. In particular when a is an element of degree 1 we have

$$E(e^a) = ae^a \quad \Longrightarrow \quad \widetilde{E}(e^a) = a. \tag{5.4}$$

Thus we see that $\widetilde{E}$ plays a role similar to the logarithm. More generally,

$$\text{if} \quad [\phi, E\phi] = 0, \quad \text{then} \quad \widetilde{E}(e^\phi) = E\phi. \tag{5.5}$$

Proposition 5.9. *We have a commutative diagram*



Proof. Since all the maps are meta-monoid homomorphisms, we just need to check the diagram for the generators of $\mathcal{W}$, namely the positive crossings and negative crossings. Specifically, we want to show that

$$e^{\pm(b_i c_j + u_i w_j)} = \mathbb{O}\left(\exp\left(\pm b_i c_j + \frac{e^{\pm b_i} - 1}{b_i} u_i w_j\right) \middle| u_i, c_j w_j\right).$$

Let us prove the positive case, the negative case can be proven analogously. Note that the exponential on the left hand side is an element of the universal enveloping algebra so it is not a commutative power series. We can expand it explicitly as

$$e^{b_i c_j + u_i w_j} = \sum_{k=0}^{\infty} \frac{(b_i c_j + u_i w_j)^k}{k!}.$$

And the right hand side can be written as

$$\exp(b_i c_j) \exp\left(\frac{e^{b_i} - 1}{b_i} u_i w_j\right)$$

according to the specified order of generators. To show that the two sides are the same we apply $\widetilde{E}$ to both sides. Since $b_i c_j + u_i w_j$ has degree 1 we have

$$\widetilde{E}(e^{b_i c_j + u_i w_j}) = b_i c_j + u_i w_j$$

by (5.4). The image of the right hand side under $\widetilde{E}$, using the derivation property of E, is given by

$$\begin{aligned} & e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} e^{-b_i c_j} \left(E(e^{b_i c_j}) e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} + e^{b_i c_j} E\left(e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} \right) \right) \\ &= e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} e^{-b_i c_j} b_i c_j e^{b_i c_j} e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} + e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} E\left(e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} \right). \end{aligned}$$

For the first term we have

$$e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} e^{-b_i c_j} b_i c_j e^{b_i c_j} e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} = e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} b_i c_j e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j}.$$

We can move c_j past $e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j}$ as follows:

$$\begin{aligned} e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} c_j &= \sum_{k=0}^{\infty} (-1)^k \left(\frac{e^{b_i}-1}{b_i}\right)^k \frac{u_i^k w_j^k}{k!} c_j \\ &= \sum_{k=0}^{\infty} (-1)^k (c_j + k) \left(\frac{e^{b_i}-1}{b_i}\right)^k \frac{u_i^k w_j^k}{k!} \quad \text{(by Proposition 5.3)} \\ &= \sum_{k=0}^{\infty} (-1)^k c_j \left(\frac{e^{b_i}-1}{b_i}\right)^k \frac{u_i^k w_j^k}{k!} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{e^{b_i}-1}{b_i}\right)^k \frac{u_i^k w_j^k}{(k-1)!} \\ &= \left(c_j - \left(\frac{e^{b_i}-1}{b_i}\right) u_i w_j\right) e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j}. \end{aligned}$$

It then follows that

$$e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} e^{-b_i c_j} b_i c_j e^{b_i c_j} e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} = b_i c_j - (e^{b_i} - 1) u_i w_j. \tag{5.6}$$

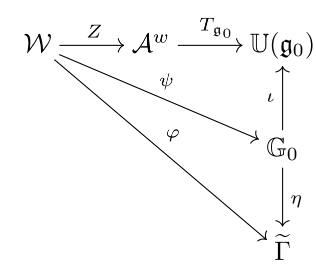
Now let us look at the term

$$e^{-\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} E\left(e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} \right) = \widetilde{E}\left(e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_i w_j} \right).$$

Observe that
$$E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j\right)=E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_i\right)w_j$$
because $\deg w_j = 0$. Then
$$E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_i\right)=E\left(\sum_{k=0}^{\infty}\frac{b_i^k}{(k+1)!}u_i\right)=\sum_{k=0}^{\infty}\frac{b_i^k}{k!}u_i=e^{b_i}u_i$$
since $E(b_i^k u_i) = (k+1)b_i^k u_i$. So
$$E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j\right)=e^{b_i}u_iw_j.$$
In particular
$$\left[E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j\right),\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j\right]=0.$$
Therefore
$$\widetilde{E}\left(e^{\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j}\right)=E\left(\left(\frac{e^{b_i}-1}{b_i}\right)u_iw_j\right)=e^{b_i}u_iw_j \tag{5.7}$$
by (5.5). From (5.6) and (5.7) we see that the image of the right hand side under $\widetilde{E}$ is
$$b_ic_j+u_iw_j,$$
as required. □

Finally we summarize our discussion in the following proposition.

Proposition 5.10. *We have the following commutative diagram*

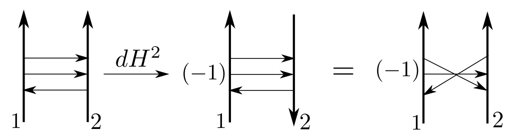


Proof. We have established that all the maps in the diagram are meta-monoid homomorphisms. Therefore it suffices to verify the diagram for the positive crossings and the negative crossings. Notice that the upper half of the diagram is already commutative, thus the remaining part to check is the lower half of the diagram. Namely we just need to show that
$$R^{\pm}_{i,j} /\!\!/ \varphi = R^{\pm}_{i,j} /\!\!/ \psi /\!\!/ \eta.$$
Again the above equations can be verified easily by hand, but we can just use Mathematica via the commands

```
eR⁺i,j ≡ (gR⁺i,j // G0toΓ)
eR⁻i,j ≡ (gR⁻i,j // G0toΓ)
```

Recall that in Mathematica we denote $R^{\pm}_{i,j} \,/\!\!/\, \varphi$ by $eR^{\pm}_{i,j}$; $R^{\pm}_{i,j} \,/\!\!/\, \psi$ by $gR^{\pm}_{i,j}$, and the map η by G0toΓ. The output is True and that establishes the commutativity of the diagram. □

Remark 5.2. Let us investigate the compatibility of the above diagram with the operations orientation reversal and strand doubling, see [WKO1] and [WKO4] for more details. We first consider the operation H^a of reversing the orientation of strand a. In $\mathcal{A}^w$ it is the operation of "flipping" over strand a and multiplying with -1 for each arrow head or tail that connects to strand a. For instance



Correspondingly in $\mathbb{U}(\mathfrak{g}_0)$ it is the *antipode map* H, i.e. the antihomomorphism given by

$$H(x) = -x, \quad x \in \mathfrak{g}_0.$$

So for example

$$H(c^2u^3w^2) = (-w)^2(-u)^3(-c)^2 = -w^2u^3c^2.$$

To obtain the corresponding image in $\mathbb{G}_0$ we would need to apply the switching operators to turn the order wuc to the order cuw. We can implement the antipode operation in Mathematica as follows, where we use $gH[a]$ to denote the subroutine that applies the antipode operation on strand a:

```
gH[a_][e_𝔼] := (e /. {c_a → -c_a, w_a → -w_a, b_a → -b_a, u_a → -u_a}) // N_{u_a c_a→a} // N_{w_a c_a→a} // N_{w_a u_a→a}
```

To show that the diagram is compatible with orientation reversal we have to show that the following diagram

$$\begin{array}{ccc} \mathbb{G}_0^{\{a\}\cup S} & \xrightarrow{dH^a} & \mathbb{G}_0^{\{a\}\cup S} \\ \downarrow \eta & & \downarrow \eta \\ \widetilde{\Gamma}^{\{a\}\cup S} & \xrightarrow{gH^a} & \widetilde{\Gamma}^{\{a\}\cup S} \end{array}$$

is commutative. We can verify the diagram using Mathematica as follows. First we input an element ξ of $\mathbb{G}_0^{\{a\}\cup S}$ in Mathematica using the command

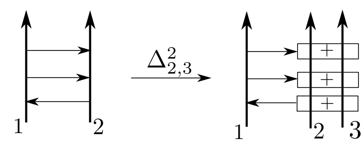
```
ξ = 𝔼[ω, Sum[l_{x,y} b_x c_y + q_{x,y} u_x w_y, {x, {a, S}}, {y, {a, S}}]]
```

Then we check the commutativity of the diagram via the command

```
(ξ // gH[a] // G0toΓ) ≡ (ξ // G0toΓ // dH[a])
```

Mathematica then returns True, as expected.

Next let us look at the strand doubling operation $\Delta^i_{j,k}$ which replaces strand a by two of its parallel copies labeled by j and k. In $\mathcal{A}^w$ it is the operation of replacing the skeleton strand i by two skeleton strands j and k and summing over all ways of connecting arrow heads or arrow tails to strand j or strand k. For instance



Correspondingly in $\mathbb{U}(\mathfrak{g}_0)$ it is the doubling map $\Delta^i_{j,k}$, i.e. the homomorphism given by

$$\Delta^i_{j,k}(x_i) = x_j + x_k, \quad x_i \in \mathfrak{g}_{0,i}.$$

We can implement $\Delta^i_{j,k}$ in Mathematica as follows:

```
gΔ[i_, j_, k_][e_E] := (e /. {ci → cj + ck, wi → wj + wk, bi → bj + bk, ui → uj + uk}) // CF
```

where we denote the subroutine by $g\Delta[i, j, k]$. In this case the doubling operation in $\mathcal{W}$ is NOT compatible with the "naive" doubling operation in Lie algebras. Specifically, the following diagram

$$\begin{array}{ccc} \mathbb{G}_0^{\{i\}\cup S} & \xrightarrow{g\Delta^i_{j,k}} & \mathbb{G}_0^{\{j,k\}\cup S} \\ \downarrow \eta & & \downarrow \eta \\ \widetilde{\Gamma}^{\{i\}\cup S} & \xrightarrow{q\Delta^i_{j,k}} & \widetilde{\Gamma}^{\{j,k\}\cup S} \end{array}$$

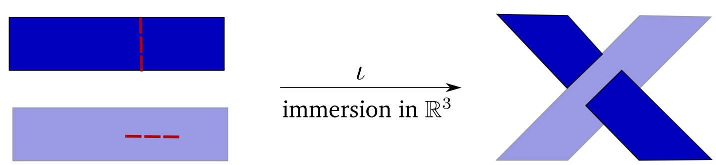
is NOT commutative.

Chapter 6

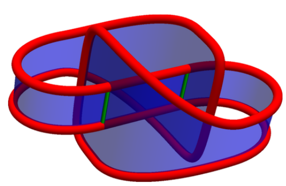
The Fox-Milnor Condition

6.1 Ribbon Knots

We first recall some basic terminologies and refer the readers to [Kau87] for more details. A knot is called *ribbon* if it can be written as the boundary of a 2-disk that is immersed into the 3-sphere S^3 with *ribbon singularities*. More precisely, if $\iota : D^2 \to S^3$ is the immersion and C is a connected component of the singular set of ι, then $\iota^{-1}(C)$ consists of a pair of closed intervals: one lies entirely in the interior of D^2 and one with endpoints on the boundary of D^2. The following figure describes the situation locally.



Here the dashed lines indicate the preimages of the singularity. For instance the following knot is ribbon.



One sees that it can be written as the boundary of a 2-disk (the shaded part) with only ribbon singularities.

A knot is called *(smoothly) slice* if it is the boundary of a smoothly embedded 2-disk D^2 in the 4-dimensional disk D^4. (Here the boundary of D^4 is the 3-sphere S^3, which contains our knot.) It is clear that ribbon knots are slice because we can push the (ribbon) singularities into D^4, thereby obtaining an embedding of D^2 into D^4. However the reverse direction, known as the *slice-ribbon conjecture*, is one of the most famous open problems in classical knot theory. Our goal in this section is to rederive the *Fox-Milnor condition* using the framework of Γ-calculus.

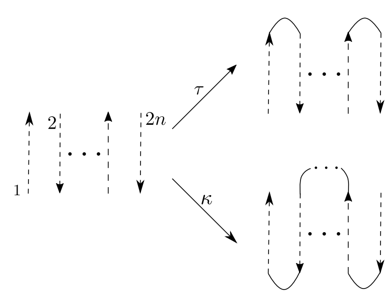
Theorem (Fox-Milnor [Lic97]). *If a knot K is slice, and $\Delta_K(t)$ is the Alexander polynomial of K, then there exists a Laurent polynomial f such that*

$$\Delta_K(t) \doteq f(t)f(t^{-1}), \tag{6.1}$$

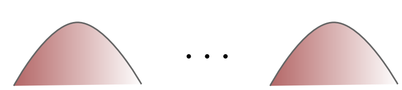
where $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$.

Notice that the Fox-Milnor condition gives us a condition on slice knots, and since the class of slice knots contains ribbon knots, it cannot help resolve the slice-ribbon conjecture. The consensus is that it should be false, and we do have several potential counter-examples with a high number of crossings [GST10]. Our strategy to approach the slice-ribbon conjecture is first to characterize ribbon knots in terms of tangles and operations between them which can be expressed easily in the language of meta-monoids. Then we need an invariant that is polynomial-time computable, so that it can handle tangles with a large number of crossings, and also behaves well with respect to the meta-monoid operations. We argue that Γ-calculus is one example of such an invariant (in fact the simplest of a series of invariants). In the remaining part of the paper we will investigate the ribbon property in Γ-calculus. Although in the end we just obtain the Fox-Milnor condition, our proof uses the characterization for ribbon knots (as opposed to slice knots), thus it has the potential to answer the slice-ribbon conjecture when we generalize it in the context of a stronger invariant, which we are currently developing [BN16a]. The result of this section therefore serves as a preliminary step (or warm-up) in a more exciting project.

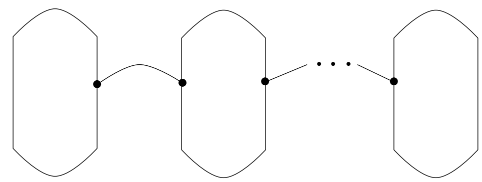
Ribbon knots have the following characterization in terms of tangles. Consider a $2n$-component *pure up-down* tangle. Here *pure* means the permutation induced by the tangle is the identity permutation and *up-down* means that the strands are oriented up and down alternately starting from the first strand, where we label the strands from left to right from 1 to $2n$. There are two special closure operations called *knot closure* and *tangle closure*, denoted by κ and τ, respectively. The τ closure connects strand i to strand $i+1$, where i runs over all odd labels $1, 3, \dots, 2n-1$, which yields an n-component tangle. The κ closure connects strand $i+1$ to strand i, where i runs over the labels $1, 2, \dots, 2n-1$, which yields a long knot.



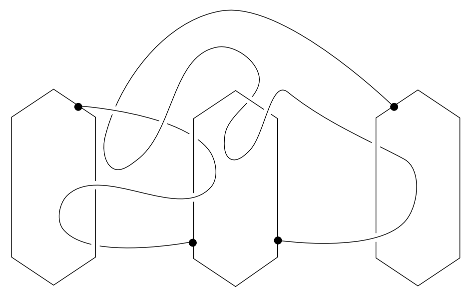
Proposition 6.1. *A long knot K is ribbon if and only if there exists a $2n$-component pure up-down tangle T such that $\kappa(T)$ is the knot K and $\tau(T)$ is the trivial n-component tangle, i.e. it bounds n disjoint embedded half-disks in $\mathbb{R}^2$ as in the following figure.*



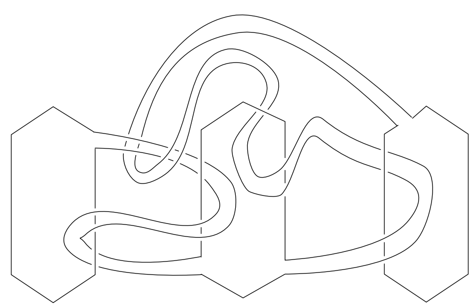
Proof. Let us sketch a proof of the proposition, (see also [Khe17]). For the only if direction, we want to obtain a tangle presentation of a ribbon knot that satisfies the condition of the propositions. First note that a ribbon knot can be presented in a special form, known as a *ribbon presentation* (see [Kaw96]). Namely, every ribbon knot can be obtained from an embedding of a disjoint union of rings and strings between consecutive rings



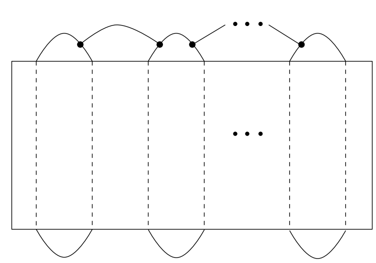
where we require that the rings are embedded trivially, i.e. each bounds a 2-disk, and we require that the ends of the strings, which we denote by dots •, only lie on the boundaries of the disks. For instance a ribbon presentation of a ribbon knot is



To obtain the ribbon knot, we simply "unzip" the strings to obtain

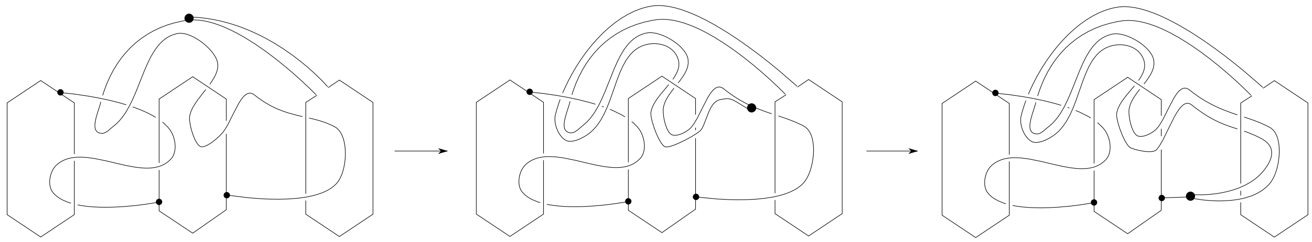


Now given a ribbon presentation of a ribbon knot, observe that if we can deform it into the following form

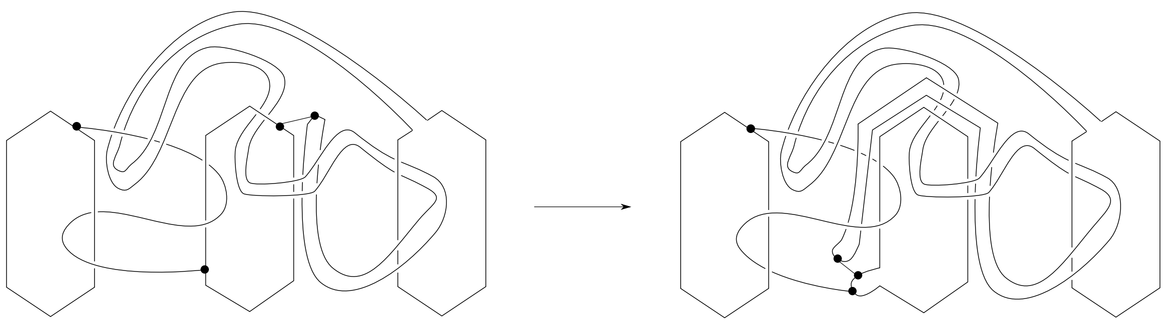


then the tangle inside the rectangle satisfies our requirements (here again dashed lines mean they can be knotted in any manner). To see why, note that the τ closure amounts to removing the strings, which results in a trivial tangle, and the κ closure is equivalent to unzipping to strings, which results in the knot.

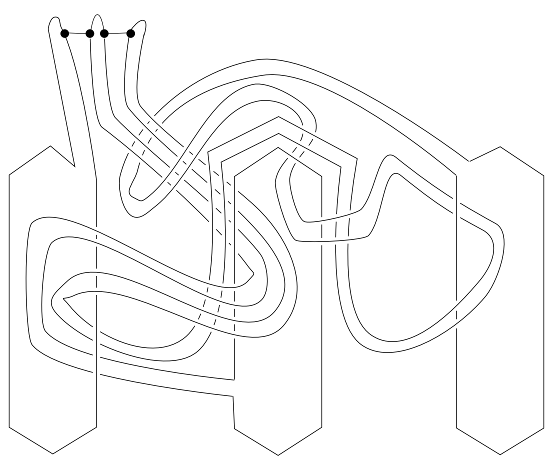
Therefore, it suffices to show that given any ribbon presentation, we can deform it to the above form. For that, we need to make two cuts to the ribbon presentation, the *bottom cut* and the *top cut*. The bottom cut is easy to perform. Namely, for each ring, we can pull the bottom part down below away from interaction with any string simply by choosing a point on the bottom of a ring and perform a "finger move". Then we cut all the bottom parts. For the top cut, we first need to deform the ribbon presentation as follows. We describe the method for a particular example, but it is representative of a general case. Our strategy would be to move the dots along the strings, which will drag parts of the rings along in the process. For our example, we first move the dot from the third ring along the string, which pulls along a part of the third ring



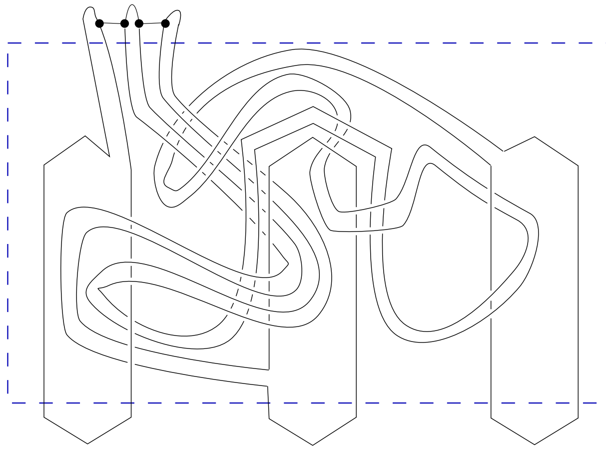
When we get close to the end of the string, we move both dots along the second ring to the other dot on the second ring.



Then we pull all three dots along the remaining strings, which pull along parts of the second and the third rings

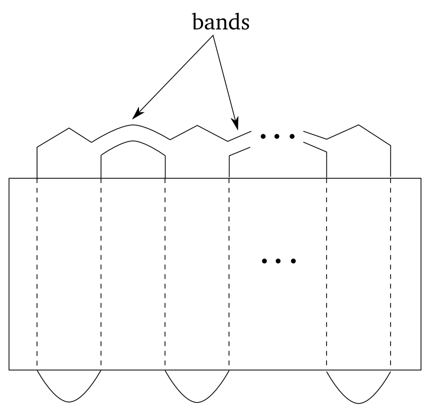


This procedure allows us to pull all the dots and the strings above all rings, then we can easily make the top cut as follows.

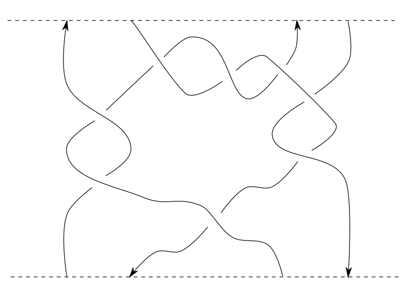


Our required tangle is contained in the dashed rectangle. This completes the only if direction.

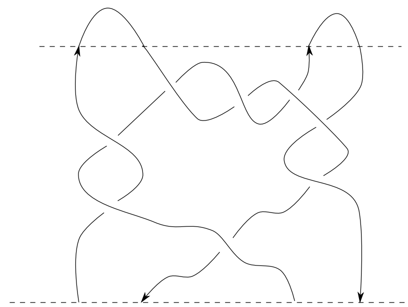
For the if direction, we need to show that if a tangle T satisfies the condition, then its κ closure is ribbon. By assumption, when we take the τ closure, we can deform the link to a trivial position. In the process, we can make sure that the bands in the κ closure intersect the interior of the disks transversely, i.e. ribbon singularities.



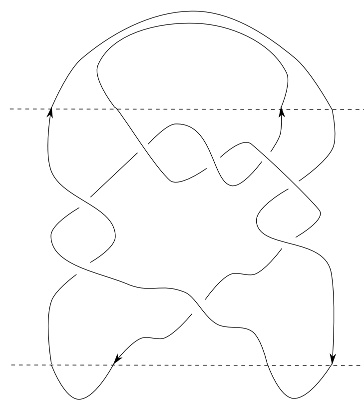
The result is a ribbon presentation and hence the knot is ribbon. Again let us look at a concrete example. Consider the following tangle



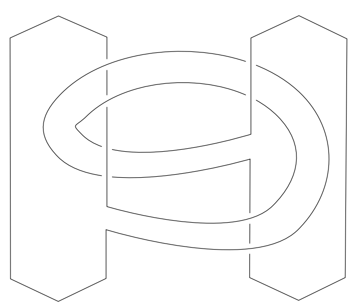
Taking the τ closure we obtain



which one can check to be the trivial tangle. The tangle satisfies the condition of the proposition, therefore it represents a ribbon knot. To see which one it is we look at the κ closure, whereas here we also connect the first and the last strand to obtain a closed knot



which one can deform into the following form



In this form one easily sees that the knot is ribbon. □

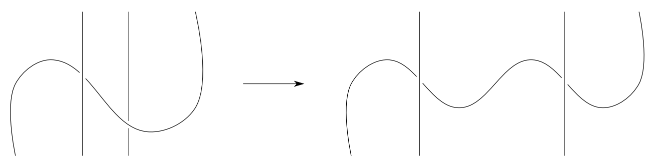
6.2 Unitary Property

To obtain the Fox-Milnor condition (6.1) using the framework of Γ-calculus is not so straightforward. It is not simply a matter of plug-in-and-check in Mathematica like we have done so far because we do not know a formula for the function f. The main difficulty however is that ribbon knots are characterized in terms of (usual) tangles, and the stitching operations in Γ-calculus does not distinguish (usual) tangles from w-tangles. Our first task therefore is to find certain property that can characterizes the image of (usual) tangles in Γ-calculus, which we call a "unitary property". (This is a simplified version of a general problem of characterizing the image of arrow diagrams in chord diagrams, which is quite non-trivial.)

In this section we establish a partial "unitary property" of tangles (which does not hold for w-tangles in general), namely for the case of string links. A key topological fact is given in the following lemma.

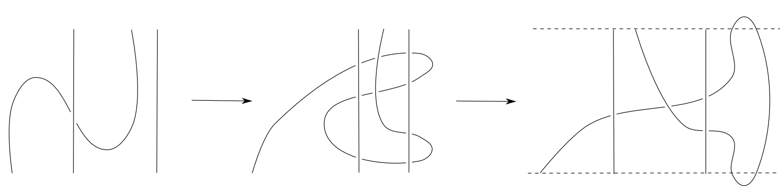
Lemma 6.1. *Every string link can be obtained from a braid by connecting the right-most outgoing strand with the right-most incoming strand successively finitely many times.*

Proof. First we deform the string link to a Morse position. i.e. where we can decompose the string link into elementary pieces consisting of crossings and cups and caps. If the string link contains no downward arcs, then it is a braid and there is nothing to do. Otherwise, because each strand goes from bottom to top, the cups and caps will occur consecutively in pairs, and each downward arc will occur between a pair of consecutive cup and cap. Our strategy will be to transform each downward arc into a closing of the last strand as follows. Look at a particular downward arc which occurs between a pair of cup and cap. There will generally be a number of arcs between them, which go either over or under the downward arc. By introducing new pairs of consecutive cups and caps we can make sure that between a cup and a cap there is only one arc which goes either over or under the downward arc.



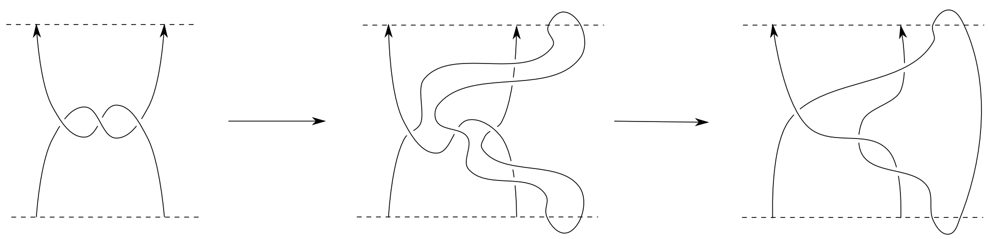
So it suffices to consider the following cases

For the case where the arc goes over the downward arc, we create a "finger" at the cup and and a "finger" at the cap and bring them to the right-most position going under the remaining strands and then pull the downward arc.



This procedure will turn a pair of cup and cap into a stitching (or closing) of the last strand and does not introduce any new downward arc. The case where the arc goes under is similar, we just have to pull the cup and cap to the right going over the remaining strands. We can repeatedly use the procedure to successively move all pairs of consecutive cups and caps to the right-most position, thus obtain our desired form. □

For a simple example we can transform the following string link as follows



Let X be a finite set of labels. For a matrix A with entries rational functions in t_x, $x \in X$, let A^* be $\overline{A^t}$, where A^t is the transpose of A and $\overline{A}$ is the operation sending all variables t_x to t_x^{-1} applied to each entry of A. Recall also that for an $n \times n$ matrix M and a permutation $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$ we let M^ρ be the matrix obtained by permuting the columns of M according to ρ, i.e. the jth column of M^ρ is the ρ_jth column of M (see (3.4)). Now we are ready to state the unitary property:

Theorem 6.1 (Unitary Property). *Let β be a string link and $X = \{a_1, \ldots, a_n\}$ be a finite set of labels. Let ρ be the induced permutation. Then the bottom endpoints of T are labeled by $(a_1, a_2, \ldots, a_n)$ and the top endpoints of T are labeled by $(a_1\rho, \ldots, a_n\rho)$ and*

$$\varphi(\beta) = \left(\begin{array}{c|ccc} \omega & a_1 & \cdots & a_n \\ \hline a_1 & & & \\ \vdots & & M & \\ a_n & & & \end{array}\right).$$

Then we have

$$(M^\rho)^* \Omega M^\rho = \Omega(\rho),$$

and

$$\overline{\omega} \doteq \omega \det(M^\rho),$$

where the matrix Ω is given by

$$\Omega = \begin{pmatrix} (1-t_{a_1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{a_2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_{a_n})^{-1} \end{pmatrix}$$

and $\Omega(\rho)$ is the action of ρ on Ω by permuting the diagonal entries, i.e.

$$\Omega(\rho) = \begin{pmatrix} (1-t_{a_1\rho})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{a_2\rho})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_{a_n\rho})^{-1} \end{pmatrix}.$$

Remark 6.1. Before presenting the proof let us explain the name "unitary property". In the case where ρ is the identity matrix, i.e. pure string links, or when we identify all the variables t_x, i.e. the Burau representation, we obtain

$$M^* \Omega M = \Omega.$$

Taking the conjugate transpose of both sides we obtain

$$M^*\Omega^*M = \Omega^*.$$

Therefore if we let $\Psi = i\Omega - i\Omega^*$, then

$$M^*\Psi M = \Psi.$$

Note that the matrix Ψ is Hermitian since

$$\Psi^* = (i\Omega - i\Omega^*)^* = i\Omega - i\Omega^* = \Psi,$$

hence the matrix M is unitary with respect to the Hermitian form Ψ.

Proof. The general strategy of the proof is as follows. By Lemma 6.1 we just need to show that the unitary property holds for braids and is invariant under stitching the right-most outgoing strand with the right-most incoming strand. To show the property for braids, we verify that it is true for crossings and is preserved under composition. The bulk of the proof is to show invariance under stitching. To that end, we decompose the stitching operation into a sequence of row operations and then it boils down to simple computations in matrix algebra. To streamline the proof, we separate the matrix part and the scalar part.

The matrix part: Let us first check the crossings. Indeed one can verify easily that for $R^+_{a,b}$ we have

$$\begin{pmatrix} 1-t_a^{-1} & t_a^{-1} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (1-t_a)^{-1} & 0 \\ 1 & (1-t_b)^{-1} \end{pmatrix} \begin{pmatrix} 1-t_a & 1 \\ t_a & 0 \end{pmatrix} = \begin{pmatrix} (1-t_b)^{-1} & 0 \\ 1 & (1-t_a)^{-1} \end{pmatrix},$$

and for $R^-_{a,b}$ we have

$$\begin{pmatrix} 0 & 1 \\ t_a & 1-t_a \end{pmatrix} \begin{pmatrix} (1-t_b)^{-1} & 0 \\ 1 & (1-t_a)^{-1} \end{pmatrix} \begin{pmatrix} 0 & t_a^{-1} \\ 1 & 1-t_a^{-1} \end{pmatrix} = \begin{pmatrix} (1-t_a)^{-1} & 0 \\ 1 & (1-t_b)^{-1} \end{pmatrix}.$$

Let us also remark here that the above property does not hold for w-string links, simply because it does not hold for a virtual crossing (recall that in Γ-calculus a virtual crossing is sent to the identity matrix, so in a sense it is "not even there"):

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (1-t_a)^{-1} & 0 \\ 1 & (1-t_b)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} (1-t_b)^{-1} & 0 \\ 1 & (1-t_a)^{-1} \end{pmatrix}.$$

The computation clearly extends to generators of the braid groups (extend by block identity matrix). Next observe that the unitary property is invariant under composition of string links (or braids in particular). Indeed, consider two string links β_1 and β_2 with induced permutations ρ_1 and ρ_2, respectively:

$$\varphi(\beta_1) = \left(\begin{array}{c|c} \omega_1 & \boldsymbol{a}\rho_1 \\ \hline \boldsymbol{a} & M_1^{\rho_1} \end{array}\right) \quad \text{and} \quad \varphi(\beta_2) = \left(\begin{array}{c|c} \omega_2 & \boldsymbol{b}\rho_2 \\ \hline \boldsymbol{b} & M_2^{\rho_2} \end{array}\right)$$

and suppose that we have

$$(M_1^{\rho_1})^*\Omega(\boldsymbol{a})M_1^{\rho_1} = \Omega(\boldsymbol{a}\rho_1) \quad \text{and} \quad (M_2^{\rho_2})^*\Omega(\boldsymbol{b})M_2^{\rho_2} = \Omega(\boldsymbol{b}\rho_2).$$

Recall that the result of composing β_1 and β_2 is

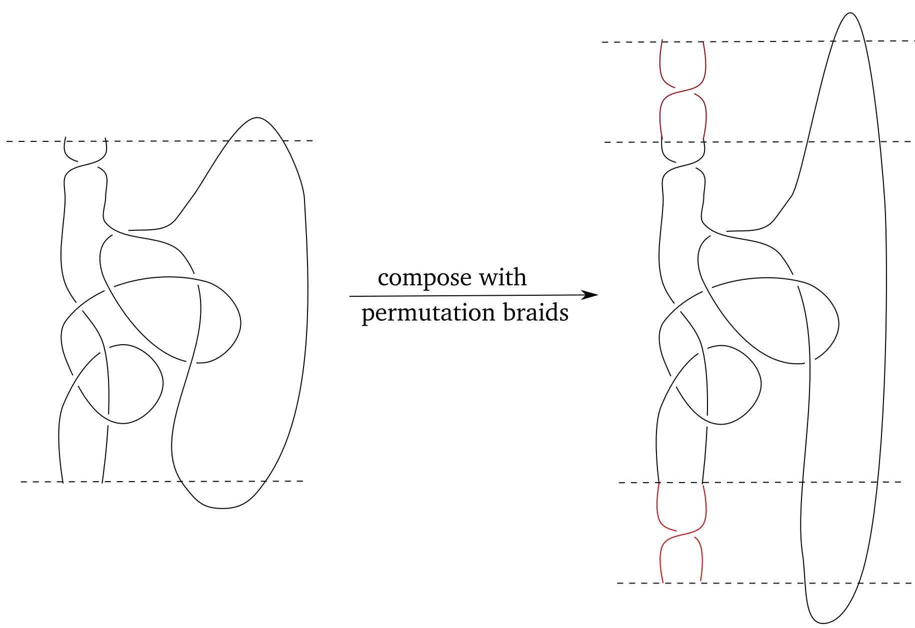
$$\varphi(\beta_1 \cdot \beta_2) = \left(\begin{array}{c|c} \omega_1\omega_2 & x_{\boldsymbol{a}\rho_1\rho_2} \\ \hline y_{\boldsymbol{a}} & M_1^{\rho_1} M_2^{\rho_2} \end{array} \right)_{t_{\boldsymbol{b}} \to t_{\boldsymbol{a}\tau}}.$$

Thus with $t_{\boldsymbol{b}} \to t_{\boldsymbol{a}\tau}$ we have

$$\begin{aligned}
(M_1^{\rho_1} M_2^{\rho_2})^* \Omega(\boldsymbol{a}) (M_1^{\rho_1} M_2^{\rho_2}) &= (M_2^{\rho_2})^* (M_1^{\rho_1})^* \Omega(\boldsymbol{a}) M_1^{\rho_1} M_2^{\rho_2} \\
&= (M_2^{\rho_2})^* \Omega(\boldsymbol{a}\rho_1) M_2^{\rho_2} \\
&= (M_2^{\rho_2})^* \Omega(\boldsymbol{b}) M_2^{\rho_2} \\
&= \Omega(\boldsymbol{b}\rho_2) \\
&= \Omega(\boldsymbol{a}\rho_1\rho_2),
\end{aligned}$$

as required. So the property holds for the case of braids (compare with [BN14]).

Now given a string link β with induced permutation $\rho = (a_1\rho, a_2\rho, \ldots, a_n\rho)$ such that $a_n\rho \neq a_n$ and suppose we want to stitch the right-most outgoing strand to the right-most incoming strand. Note that by composing the top and bottom of β with appropriate permutation braids we can bring β to a standard form where the induced permutation is $(a_1, a_2, \ldots, a_{n-2}, a_n, a_{n-1})$, i.e. the transposition (a_{n-1}, a_n) and we stitch strand a_{n-1} to strand a_n and label the resulting strand a_{n-1}. For example,



Since we have shown unitarity for braids and composition, it suffices to consider the string link β with the induced permutation $\rho = (a_1, a_2, \ldots, a_{n-1}, a_n)$. Let

$$\varphi(\beta) = \left(\begin{array}{c|ccc} \omega & a_{n-1} & a_n & S \\ \hline a_{n-1} & \alpha & \beta & \theta \\ a_n & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array} \right) \xrightarrow{m^{a_{n-1},a_n}_{a_{n-1}}} \left(\begin{array}{c|cc} (1-\gamma)\omega & a_{n-1} & S \\ \hline a_{n-1} & \beta + \frac{\alpha\delta}{1-\gamma} & \theta + \frac{\alpha\epsilon}{1-\gamma} \\ S & \psi + \frac{\delta\phi}{1-\gamma} & \Xi + \frac{\phi\epsilon}{1-\gamma} \end{array} \right)_{t_{a_n} \to t_{a_{n-1}}},$$

where $S = X \setminus \{a_{n-1}, a_n\}$. Assume β satisfies the unitary property, for that we need to rearrange the matrix part as follows

$$\left(\begin{array}{c|ccc} \omega & S & a_n & a_{n-1} \\ \hline S & \Xi & \psi & \phi \\ a_{n-1} & \theta & \beta & \alpha \\ a_n & \epsilon & \delta & \gamma \end{array}\right).$$

Let us denote

$$M = \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix}.$$

Then the unitary statement is

$$M^*\Omega M = \Omega(\rho), \tag{6.2}$$

where to simplify notation we put

$$\Omega = \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \mathbf{1} & 1 & (1-t_{a_n})^{-1} \end{pmatrix},$$

where

$$\Omega_{n-2} = \begin{pmatrix} (1-t_{a_1})^{-1} & 0 & \cdots & 0 \\ 1 & (1-t_{a_2})^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & (1-t_{a_{n-2}})^{-1} \end{pmatrix},$$

and

$$\Omega(\rho) = \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_n})^{-1} & 0 \\ \mathbf{1} & 1 & (1-t_{a_{n-1}})^{-1} \end{pmatrix}.$$

Here **1** denotes either a row or a column or a square matrix (the size of which depends on the context) consists entirely of 1's and similarly for **0**. Now to show that the unitary property is invariant under stitching we first need to decompose the stitching operation into a sequence of elementary operations as follows:

$$\begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} \to \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma - 1 \end{pmatrix} \to \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \frac{\epsilon}{\gamma-1} & \frac{\delta}{\gamma-1} & 1 \end{pmatrix} \to \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} & \mathbf{0} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} & 0 \\ \frac{\epsilon}{\gamma-1} & \frac{\delta}{\gamma-1} & 1 \end{pmatrix}.$$

Note that except for the first one, all the operations are simply elementary row operations. Now under stitching, we identify $t_{a_{n-1}}$ and t_{a_n}. In what follows, we set t_{a_n} to be $t_{a_{n-1}}$. Then $\Omega|_{t_{a_n} \to t_{a_{n-1}}} = \Omega(\rho)|_{t_{a_n} \to t_{a_{n-1}}}$ and again to avoid cumbersome notations we will denote both of them by Ω. We then write (6.2) as

$$\left[\begin{pmatrix}\Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^*-1\end{pmatrix}+\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}\right]\Omega\left[\begin{pmatrix}\Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma-1\end{pmatrix}+\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}\right]=\Omega. \tag{6.3}$$

Observe that

$$\begin{pmatrix}\Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^*-1\end{pmatrix}\Omega\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{0} & \frac{\epsilon^*}{1-t_{a_{n-1}}} \\ \mathbf{0} & 0 & \frac{\delta^*}{1-t_{a_{n-1}}} \\ \mathbf{0} & 0 & \frac{\gamma^*-1}{1-t_{a_{n-1}}}\end{pmatrix}$$

and

$$\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}\Omega\begin{pmatrix}\Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma-1\end{pmatrix}$$
$$=\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \theta+\langle\Xi\rangle+\frac{\epsilon}{1-t_{a_{n-1}}} & \beta+\langle\psi\rangle+\frac{\delta}{1-t_{a_{n-1}}} & \alpha+\langle\phi\rangle+\frac{\gamma-1}{1-t_{a_{n-1}}}\end{pmatrix}.$$

We also have

$$\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}\Omega\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1\end{pmatrix}=\begin{pmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & \frac{1}{1-t_{a_{n-1}}}\end{pmatrix}.$$

Therefore (6.3) becomes

$$\begin{pmatrix}\Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^*-1\end{pmatrix}\Omega\begin{pmatrix}\Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma-1\end{pmatrix}=$$
$$\begin{pmatrix}\Omega_{n-2} & \mathbf{0} & \frac{\epsilon^*}{-1+t_{a_{n-1}}} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & \frac{\delta^*}{-1+t_{a_{n-1}}} \\ \mathbf{1}-\theta-\langle\Xi\rangle+\frac{\epsilon}{-1+t_{a_{n-1}}} & 1-\beta-\langle\psi\rangle+\frac{\delta}{-1+t_{a_{n-1}}} & \frac{-2+\alpha+\gamma+\langle\phi\rangle-(\alpha+\langle\phi\rangle)t_{a_{n-1}}+\gamma^*}{-1+t_{a_{n-1}}}\end{pmatrix}. \tag{6.4}$$

By Lemma 3.4 we can rewrite the above as

$$\begin{pmatrix}\Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^*-1\end{pmatrix}\Omega\begin{pmatrix}\Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma-1\end{pmatrix}=\begin{pmatrix}\Omega_{n-2} & \mathbf{0} & \frac{\epsilon^*}{-1+t_{a_{n-1}}} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & \frac{\delta^*}{-1+t_{a_{n-1}}} \\ \frac{t_{a_{n-1}}\epsilon}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}\delta}{-1+t_{a_{n-1}}} & \frac{-1+\gamma^*-(1-\gamma)t_{a_{n-1}}}{-1+t_{a_{n-1}}}\end{pmatrix}. \tag{6.5}$$

Consider the left hand side of the above identity, we can obtain the stitching formula by a sequence of elementary row and column operations. By employing elementary matrices, we can rewrite it as

$$\begin{pmatrix} \Xi^* + \frac{\epsilon^*\phi^*}{1-\gamma^*} & \theta^* + \frac{\alpha^*\epsilon^*}{1-\gamma^*} & \frac{\epsilon^*}{\gamma^*-1} \\ \psi^* + \frac{\delta^*\phi^*}{1-\gamma^*} & \beta^* + \frac{\alpha^*\delta^*}{1-\gamma^*} & \frac{\delta^*}{\gamma^*-1} \\ \mathbf{0} & 0 & 1 \end{pmatrix} \widetilde{\Omega} \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} & \mathbf{0} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} & 0 \\ \frac{\epsilon}{\gamma-1} & \frac{\delta}{\gamma-1} & 1 \end{pmatrix},$$

where

$$\begin{aligned} \widetilde{\Omega} &= \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \phi^* & \alpha^* & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & \gamma^* - 1 \end{pmatrix} \Omega \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & \gamma - 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & \phi \\ \mathbf{0} & 1 & \alpha \\ \mathbf{0} & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \bullet \\ \mathbf{1} & (1 - t_{a_{n-1}})^{-1} & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}. \end{aligned}$$

Here a $\bullet$ denotes an entry we do not care about. Notice that the row and column operations only affect the last row and the last column. Finally, we apply column operations to the right-most matrix and row operations to the left-most matrix to obtain

$$\begin{pmatrix} \Xi^* + \frac{\epsilon^*\phi^*}{1-\gamma^*} & \theta^* + \frac{\alpha^*\epsilon^*}{1-\gamma^*} & \mathbf{0} \\ \psi^* + \frac{\delta^*\phi^*}{1-\gamma^*} & \beta^* + \frac{\alpha^*\delta^*}{1-\gamma^*} & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \widetilde{\Omega} \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} & \mathbf{0} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix}.$$

We can encode these operations as multiplying with the matrix

$$\begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ -\frac{\epsilon}{\gamma-1} & -\frac{\delta}{\gamma-1} & 1 \end{pmatrix}$$

on the right and its conjugate transpose

$$\begin{pmatrix} I & \mathbf{0} & -\frac{\epsilon^*}{\gamma^*-1} \\ \mathbf{0} & 1 & -\frac{\delta^*}{\gamma^*-1} \\ \mathbf{0} & 0 & 1 \end{pmatrix}$$

on the left. Therefore the right hand side of (6.5) becomes

$$\begin{pmatrix} I & \mathbf{0} & -\frac{\epsilon^*}{\gamma^*-1} \\ \mathbf{0} & 1 & -\frac{\delta^*}{\gamma^*-1} \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \frac{\epsilon^*}{-1+t_{a_{n-1}}} \\ \mathbf{1} & (1 - t_{a_{n-1}})^{-1} & \frac{\delta^*}{-1+t_{a_{n-1}}} \\ \frac{t_{a_{n-1}}\epsilon}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}\delta}{-1+t_{a_{n-1}}} & \frac{-1+\gamma^*-(1-\gamma)t_{a_{n-1}}}{-1+t_{a_{n-1}}} \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ -\frac{\epsilon}{\gamma-1} & -\frac{\delta}{\gamma-1} & 1 \end{pmatrix}.$$

For our purpose we only need to look at the first $n-1$ rows and the first $n-1$ columns. We record these changes below

$$\Omega_{n-2} - \frac{\epsilon^*\epsilon}{(-1 + t_{a_{n-1}})(\gamma - 1)} + \frac{(-1 + \gamma^*)\epsilon^*\epsilon}{(-1 + t_{a_{n-1}})(\gamma^* - 1)(-1 + \gamma)} = \Omega_{n-2},$$

$$\mathbf{1} - \frac{\delta^*\epsilon}{(-1+t_{a_{n-1}})(-1+\gamma)} + \frac{(-1+\gamma^*)\delta^*\epsilon}{(-1+t_{a_{n-1}})(\gamma^*-1)(-1+\gamma)} = \mathbf{1},$$
$$\mathbf{0} - \frac{\delta\epsilon^*}{(-1+t_{a_{n-1}})(-1+\gamma)} + \frac{(-1+\gamma^*)\delta\epsilon^*}{(-1+t_{a_{n-1}})(\gamma^*-1)(-1+\gamma)} = \mathbf{0},$$
$$-\frac{-1+\gamma+\delta\delta^*}{(-1+t_{a_{n-1}})(-1+\gamma)} + \frac{(-1+\gamma^*)\delta\delta^*}{(1-t_{a_{n-1}})(\gamma^*-1)(-1+\gamma)} = \frac{1}{1-t_{a_{n-1}}}.$$

Thus we see that the first $n-1$ rows and the first $n-1$ columns stay unchanged. In summary, we obtain the following identity

$$\begin{pmatrix} \Xi^* + \frac{\epsilon^*\phi^*}{1-\gamma^*} & \theta^* + \frac{\alpha^*\epsilon^*}{1-\gamma^*} & \mathbf{0} \\ \psi^* + \frac{\delta^*\phi^*}{1-\gamma^*} & \beta^* + \frac{\alpha^*\delta^*}{1-\gamma^*} & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \bullet \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} & \mathbf{0} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} = \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \bullet \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix}.$$

It then follows that

$$\begin{pmatrix} \Xi^* + \frac{\epsilon^*\phi^*}{1-\gamma^*} & \theta^* + \frac{\alpha^*\epsilon^*}{1-\gamma^*} \\ \psi^* + \frac{\delta^*\phi^*}{1-\gamma^*} & \beta^* + \frac{\alpha^*\delta^*}{1-\gamma^*} \end{pmatrix} \Omega_{n-1} \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} \end{pmatrix} = \Omega_{n-1},$$

which is precisely the unitary statement after stitching, and the unitary property for the matrix part is proved.

The scalar part: Next let us show the unitary property for the scalar part. The initial setup will be exactly the same as in the proof for the matrix part. Again we first verify the crossings. For the positive crossing $R^+_{a,b}$:

$$1 \cdot \det \begin{pmatrix} 1-t_a & 1 \\ t_a & 0 \end{pmatrix} = -t_a \doteq 1,$$

and for the negative crossing $R^-_{a,b}$:

$$1 \cdot \det \begin{pmatrix} 0 & t_a^{-1} \\ 1 & 1-t_a^{-1} \end{pmatrix} = -t_a^{-1} \doteq 1,$$

as required. It is easy to verify that the property is invariant under disjoint union (the determinant of the direct sum of two matrices is the the product of the determinants) and under composition (the determinant of the product of two square matrices is the product of the determinants). So again we only need to check the property under stitching strand a_{n-1} to strand a_n. Using the same notation as in the proof for the matrix part, we let

$$M = \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix},$$

and M' is the matrix part after stitching strand a_{n-1} to strand a_n and labeling the resulting strand a_{n-1}

$$M' = \begin{pmatrix} \Xi + \frac{\phi\epsilon}{1-\gamma} & \psi + \frac{\delta\phi}{1-\gamma} \\ \theta + \frac{\alpha\epsilon}{1-\gamma} & \beta + \frac{\alpha\delta}{1-\gamma} \end{pmatrix}_{t_{a_n} \to t_{a_{n-1}}}.$$

Suppose that we have

$$\overline{\omega} \doteq \omega \det(M). \tag{6.6}$$

After stitching strand a_{n-1} to strand a_n we want to show that

$$(1-\overline{\gamma})\overline{\omega}\Big|_{t_{a_n} \to t_{a_{n-1}}} \doteq (1-\gamma)\omega \det(M')\Big|_{t_{a_n} \to t_{a_{n-1}}}.$$

Again to simplify notation we assume $t_{a_n} \to t_{a_{n-1}}$ from now on. Using (6.6) we can rewrite the above as

$$(1-\overline{\gamma})\omega \det(M) \doteq (1-\gamma)\omega \det(M').$$

Since $\omega \neq 0$ we can divide both sides by ω to get

$$(1-\overline{\gamma}) \det(M) \doteq (1-\gamma) \det(M'). \tag{6.7}$$

Now from the unitary property of M'

$$(M')^* \Omega_{n-1} M' = \Omega_{n-1},$$

taking the determinant of both sides we obtain

$$\overline{\det(M')} \det(M') = 1.$$

Thus (6.7) becomes

$$\overline{\det(M')} \det(M) \doteq \frac{1-\gamma}{1-\overline{\gamma}}.$$

It follows that we just need to prove the above identity. We see that it only involves the matrix part, so we start with the unitary property for the matrix part:

$$M^* \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \mathbf{1} & 1 & (1-t_{a_{n-1}})^{-1} \end{pmatrix} M = \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \mathbf{1} & 1 & (1-t_{a_{n-1}})^{-1} \end{pmatrix}.$$

We can rewrite the above as

$$\left[\begin{pmatrix} \Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^* - 1 \end{pmatrix} + \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \right] \Omega \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} = \Omega. \tag{6.8}$$

We have

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \mathbf{0} & 0 & 1 \end{pmatrix} \Omega \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & 0 \\ \theta + \langle \Xi \rangle + \frac{\epsilon}{1-t_{a_{n-1}}} & \beta + \langle \psi \rangle + \frac{\delta}{1-t_{a_{n-1}}} & \alpha + \langle \phi \rangle + \frac{\gamma}{1-t_{a_{n-1}}} \end{pmatrix}.$$

Then (6.8) becomes

$$\begin{aligned} &\begin{pmatrix} \Xi^* & \theta^* & \epsilon^* \\ \psi^* & \beta^* & \delta^* \\ \phi^* & \alpha^* & \gamma^* - 1 \end{pmatrix} \Omega \begin{pmatrix} \Xi & \psi & \phi \\ \theta & \beta & \alpha \\ \epsilon & \delta & \gamma \end{pmatrix} \\ &= \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \mathbf{1} - \theta - \langle \Xi \rangle + \frac{\epsilon}{-1+t_{a_{n-1}}} & 1 - \beta - \langle \psi \rangle + \frac{\delta}{-1+t_{a_{n-1}}} & -\alpha - \langle \phi \rangle + \frac{1-\gamma}{1-t_{a_{n-1}}} \end{pmatrix} \\ &= \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \frac{t_{a_{n-1}}\epsilon}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}\delta}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}(1-\gamma)}{1-t_{a_{n-1}}} \end{pmatrix}, \end{aligned}$$

where we use lemma 3.4. Now for the left hand side, we can perform column operations via elementary matrices to get

$$\begin{aligned} &\begin{pmatrix} \Xi^* + \frac{\epsilon^*\phi^*}{1-\gamma^*} & \theta^* + \frac{\alpha^*\epsilon^*}{1-\gamma^*} & \frac{\epsilon^*}{\gamma^*-1} \\ \psi^* + \frac{\delta^*\phi^*}{1-\gamma^*} & \beta^* + \frac{\alpha^*\delta^*}{1-\gamma^*} & \frac{\delta^*}{\gamma^*-1} \\ \mathbf{0} & 0 & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \phi^* & \alpha^* & 1 \end{pmatrix} \begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & 0 \\ \mathbf{0} & 0 & \gamma^* - 1 \end{pmatrix} \Omega M \\ &= \begin{pmatrix} \Omega_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{1} & (1-t_{a_{n-1}})^{-1} & 0 \\ \frac{t_{a_{n-1}}\epsilon}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}\delta}{-1+t_{a_{n-1}}} & \frac{t_{a_{n-1}}(1-\gamma)}{1-t_{a_{n-1}}} \end{pmatrix}. \end{aligned}$$

Finally taking the determinant of both sides we obtain

$$\overline{\det(M')}(\overline{\gamma} - 1)\det(M) = t_{n-1}(1-\gamma).$$

Thus

$$\overline{\det(M')}\det(M) \doteq \frac{1-\gamma}{1-\overline{\gamma}},$$

which completes the proof. □

Remark 6.2. As a consequence of the unitary property for the scalar part, for the case of long knots, the matrix part is 1 and we have

$$\overline{\omega} \doteq \omega,$$

which is the usual fact that the Alexander polynomial is palindromic. (This is not true for w-knots, see Example 3.1.)

6.3 The Fox-Milnor Condition

Now we are ready to tackle the Fox-Milnor condition

Theorem. *If a knot K is ribbon, then the Alexander polynomial of K, $\Delta_K(t)$ satisfies*

$$\Delta_K(t) \doteq f(t)f(t^{-1}),$$

where $\doteq$ means equality up to multiplication by $\pm t^n$, $n \in \mathbb{Z}$ and f is a Laurent polynomial.

Proof. Our strategy is to express proposition 6.1 in the language of Γ-calculus. To that end, consider a pure up-down tangle T with strands labeled by $1, 2, \ldots, 2n$, which satisfies the condition of Proposition 6.1. We let ***odd*** denote the vector $(1, 3, \ldots, 2n-1)$ and ***even*** denote the vector $(2, 4, \ldots, 2n)$. For convenience, we write the matrix part of $\varphi(T)$ as

$$\left(\begin{array}{c|cc} \omega & \textbf{\textit{odd}} & \textbf{\textit{even}} \\ \hline \textbf{\textit{even}} & \gamma & \delta \\ \textbf{\textit{odd}} & \alpha & \beta \end{array}\right)$$

where each α, β, γ, δ is an $n \times n$ matrix. For the τ closure, we stitch the odd strands to the even strands and label the resulting strands odd. Then it follows from Proposition 3.1 that

$$\left(\begin{array}{c|cc} \omega & \textbf{\textit{odd}} & \textbf{\textit{even}} \\ \hline \textbf{\textit{even}} & \gamma & \delta \\ \textbf{\textit{odd}} & \alpha & \beta \end{array}\right) \xrightarrow{\tau = m^{\textbf{\textit{odd}},\textbf{\textit{even}}}_{\textbf{\textit{odd}}}} \left(\begin{array}{c|c} \omega\det(I-\gamma) & \textbf{\textit{odd}} \\ \hline \textbf{\textit{odd}} & \beta + \alpha(I-\gamma)^{-1}\delta \end{array}\right)_{t_{\textbf{\textit{even}}} \to t_{\textbf{\textit{odd}}}}.$$

Since the τ closure yields a trivial tangle we have $\omega\det(I-\gamma) = 1$ and $\beta + \alpha(I-\gamma)^{-1}\delta = I$. Now for the κ closure, the stitching instructions are specified by stitching the strands labeled by $(1, 2, \ldots, 2n-1)$ to the strands labeled by $(2, 3, \ldots, 2n)$, in that order. After we perform all the stitchings, the end result is the original knot K. From the stitching formula (Proposition 3.1) the scalar part is given by

$$\omega\det(I-N)\big|_{t_x \to t},$$

where N is the submatrix of the matrix part of $\varphi(T)$ specified by

$$\left(\begin{array}{c|ccc} \bullet & 2 & \cdots & 2n \\ \hline 1 & & & \\ \vdots & & N & \\ 2n-1 & & & \end{array}\right).$$

(Again $\bullet$ denotes an entry we do not care about.) On the other hand from Proposition 3.8 we know that the scalar part is the Alexander polynomial of the knot K (same as its closure $\widetilde{K}$), i.e.

$$\Delta_K(t) = \omega\det(I-N)\big|_{t_x \to t}. \tag{6.9}$$

Now it is a simple exercise in linear algebra that

$$\det(I-N)=\det(P-M), \tag{6.10}$$

where M is the matrix part of $\varphi(T)$

$$\left(\begin{array}{c|ccc} \bullet & 1 & \cdots & 2n \\ \hline 1 & & & \\ \vdots & & M & \\ 2n & & & \end{array}\right),$$

and P is the matrix given by

$$P=\left(\begin{array}{c|c|ccc} \bullet & 1 & 2 & \cdots & 2n \\ \hline 1 & & & & \\ 2 & \mathbf{0} & & I & \\ \vdots & & & & \\ \hline 2n & 0 & & \mathbf{0} & \end{array}\right).$$

To see why (6.10) is true, observe that if we replace the last row of $P-M$ by the the sum of all the rows, which does not change the value of the determinant, then we obtain the row $(-1,0,\dots,0)$ by Lemma 3.4. We then compute the determinant by expansion along the last row and the result follows.

Now it is useful to rearrange the rows and columns of $P-M$ into ***odd*** and ***even***, which only changes the determinant up to ±1, in order to relate to the τ closure:

$$\left(\begin{array}{c|c|c} \bullet & \textbf{\textit{odd}} & \textbf{\textit{even}} \\ \hline \textbf{\textit{odd}} & -\alpha & I-\beta \\ \hline \textbf{\textit{even}} & \begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix} - \gamma & -\delta \end{array}\right).$$

Then by Lemma 3.2 we have

$$\det\left(\begin{array}{c|c} \alpha & \beta-I \\ \hline \gamma-\begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix} & \delta \end{array}\right)=\det\left(\alpha+(I-\beta)\delta^{-1}\left(\gamma-\begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}\right)\right)\det(\delta).$$

From $\beta+\alpha(I-\gamma)^{-1}\delta=I$ we get

$$\alpha(I-\gamma)^{-1}\delta=I-\beta.$$

Therefore

$$\begin{aligned} &\det\left(\alpha+(I-\beta)\delta^{-1}\left(\gamma-\begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}\right)\right)\det(\delta) \\ &\qquad=\det\left(\alpha+\alpha(I-\gamma)^{-1}\left(\gamma-\begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}\right)\right)\det(\delta) \\ &\qquad=\det(\alpha)\det\left(I+(I-\gamma)^{-1}\left(\gamma-\begin{pmatrix} \mathbf{0} & I_{n-1} \\ 0 & \mathbf{0} \end{pmatrix}\right)\right)\det(\delta) \end{aligned}$$

$$
\begin{aligned}
&= \det(\alpha)\det[(I-\gamma)^{-1}]\det\left(I-\begin{pmatrix}\mathbf{0} & I_{n-1}\\ 0 & \mathbf{0}\end{pmatrix}\right)\det(\delta)\\
&= \frac{\det(\alpha)\det(\delta)}{\det(I-\gamma)}\\
&= \omega\det(\alpha)\det(\delta),
\end{aligned}
$$

where we use $\omega\det(I-\gamma)=1$ in the last equality. From (6.9) it follows that

$$\Delta_K(t) \doteq \omega\det(\alpha)\omega\det(\delta)\big|_{t_x\to t}. \tag{6.11}$$

To finish off, we will employ the unitary property of $\varphi(T)$. But since we only have the unitary property for string links, we first need to reverse the orientations of all the even strands of T. The orientation reversal formula (Proposition 3.10) yields

$$\left(\begin{array}{c|cc}\omega & \textbf{\textit{even}} & \textbf{\textit{odd}}\\ \hline \textbf{\textit{even}} & \delta & \gamma\\ \textbf{\textit{odd}} & \beta & \alpha\end{array}\right) \xrightarrow{dH^{even}} \left(\begin{array}{c|cc}\omega\det(\delta) & \textbf{\textit{even}} & \textbf{\textit{odd}}\\ \hline \textbf{\textit{even}} & \delta^{-1} & \delta^{-1}\gamma\\ \textbf{\textit{odd}} & -\beta\delta^{-1} & \alpha-\beta\delta^{-1}\gamma\end{array}\right)_{t_{even}\to t_{even}^{-1}}.$$

Note that the orientation reversal operation takes value in $\widetilde{\Gamma}$. However here we can safely ignore the σ part because we consider ω up to multiplication of monomials in t_x. Now the unitary property of the scalar part tells us that

$$\omega\det(\delta)\Big|_{t_{odd}\to t_{odd}^{-1}} \doteq \omega\det(\delta)\det\begin{pmatrix}\alpha-\beta\delta^{-1}\gamma & -\beta\delta^{-1}\\ \delta^{-1}\gamma & \delta^{-1}\end{pmatrix}\Bigg|_{t_{even}\to t_{even}^{-1}}.$$

Taking $t_{even}\to t_{even}^{-1}$ in both sides we obtain

$$
\begin{aligned}
\overline{\omega\det(\delta)} &\doteq \omega\det(\delta)\det\begin{pmatrix}\alpha-\beta\delta^{-1}\gamma & -\beta\delta^{-1}\\ \delta^{-1}\gamma & \delta^{-1}\end{pmatrix}\\
&= \omega\det(\delta)\det(\alpha-\beta\delta^{-1}\gamma+\beta\delta^{-1}\delta\delta^{-1}\gamma)\det(\delta^{-1})\\
&= \omega\det(\alpha).
\end{aligned}
$$

Again we use Lemma 3.2 in the second equality. Then setting all t_x to t, (6.11) becomes

$$\Delta_K(t) \doteq \omega\det(\delta)\overline{\omega\det(\delta)} \doteq \omega\det(\alpha)\overline{\omega\det(\alpha)},$$

which is precisely the Fox-Milnor condition.

Note that in our proof we can choose the function f to be $\omega\det(\delta)$ or $\omega\det(\alpha)$. In the first case f is the scalar part of the tangle obtained by reversing the orientations of the even strands of T, and in the second case f is the scalar part of the tangle obtained by reversing the orientations of the odd strands of T (with the relevant $t_x\to t_x^{-1}$). By Proposition 3.5 we see that f is a Laurent polynomial. □

Chapter 7

Extension to w-Links

7.1 The Trace Map

In this section we would like to extend our invariant to links. So far our invariant in Γ-calculus only works for tangles and long knots, since we do not allow closed components. Notice that our stitching formula involves division by $1-\gamma$, and it only makes sense when γ is an off-diagonal term. In other words, we can only stitch strands with distinct labels. When we try to stitch strands of the same label, we may encounter division by zero. Nevertheless, the formula for the scalar part ω only requires multiplication by $1-\gamma$ and so we expect to be able to extend it to links, or more precisely *long w-links*, i.e. w-links with only one open component. The matrix part is no longer well-defined for links. For instance, if a tangle contains a trivial open component, then to stitch the component to itself we will have to divide by $1-1=0$.

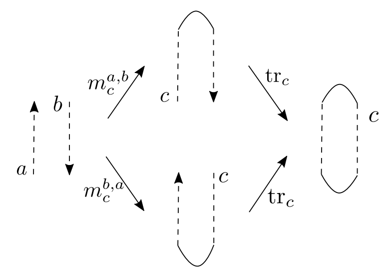
As a first step, we need to describe closed components within the framework of meta-monoids. Let $\mathcal{W}_{cl}^{X\cup\{c\}}$ be the collection of w-tangles whose components are labeled by $X\cup\{c\}$ with exactly one closed component labeled by c. Note that we cannot generate $\mathcal{W}_{cl}^{X\cup\{c\}}$ from crossings using the meta-monoid operations because we cannot stitch the same strand to itself. Let $\mathcal{W}^{X\cup\{c\}}$ be the usual collection of w-tangles (no closed components) whose components are labeled by $X\cup\{c\}$. Then we have a *trace map*

$$\mathrm{tr}_c : \mathcal{W}^{X\cup\{c\}} \to \mathcal{W}_{cl}^{X\cup\{c\}}$$

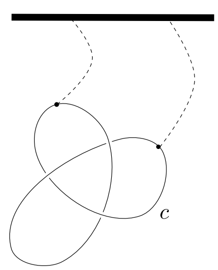
given simply by closing the component c in a trivial way (i.e. only through virtual crossings). We have the following key topological result.

Proposition 7.1. *Two w-tangles T_1 and T_2 have isotopic (same) images in $\mathcal{W}_{cl}^{X\cup\{c\}}$ under the map tr_c if and only if there is a w-tangle $T\in\mathcal{W}^{X\cup\{a,b\}}$ such that $T_1=m_c^{a,b}(T)$ and $T_2=m_c^{b,a}(T)$.*

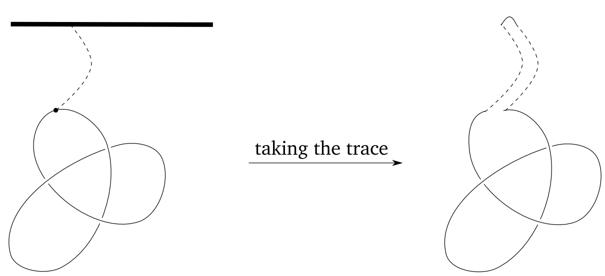
Proof. Note that it suffices to just look at the component labeled c. The if direction is quite clear from the following diagram.



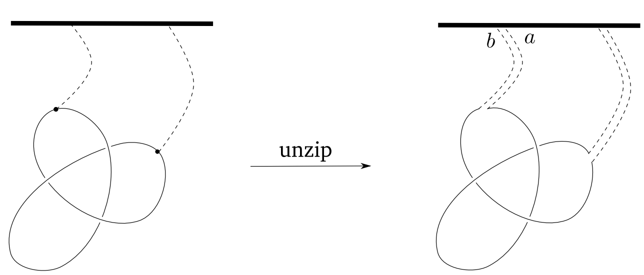
Now for the only if direction, let T_1 and T_2 have isotopic images under the trace map. We can view the image as a closed component c with two beads on it that represent the two positions where we take the trace and two strands that connect the beads to a fixed base as in the following figure (here again dashed line means it can be knotted).



For each position, to take the trace, we unzip the strand, and then cap off the ends.



Then to find a tangle T such that $T_1 = m_c^{a,b}(T)$ and $T_2 = m_c^{b,a}(T)$, we simply unzip the two strands to obtain a tangle T with two components a and b.



It is straightforward to check that T satisfies our requirement. $\square$

Therefore we can think of an element of $\mathcal{W}_{cl}^{X\cup\{c\}}$ as an equivalence class, namely

$$\mathcal{W}_{cl}^{X\cup\{c\}} \simeq \mathcal{W}^{X\cup\{c\}}/(m_c^{a,b}(T) \sim m_c^{b,a}(T)),$$

where $T \in \mathcal{W}^{X\cup\{a,b\}}$. So in particular an invariant Ω on $\mathcal{W}^{X\cup\{c\}}$ will descend to an invariant on $\mathcal{W}_{cl}^{X\cup\{c\}}$ if it satisfies the condition

$$\Omega(m_c^{a,b}(T)) = \Omega(m_c^{b,a}(T))$$

for all w-tangles $T \in \mathcal{W}^{X\cup\{a,b\}}$. In general we would want to include links with more than one closed component, and the above discussion can be generalized in a straightforward manner. For a vector $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ let $W_{cl}^{X\cup\{\boldsymbol{c}\}}$ be the collection of w-tangles whose components labeled by $c_1, \ldots, c_n$ are closed. We also have a *trace map*

$$\mathrm{tr}_{\boldsymbol{c}} : \mathcal{W}^{X\cup\{\boldsymbol{c}\}} \to \mathcal{W}_{cl}^{X\cup\{\boldsymbol{c}\}},$$

obtained by closing the components $c_1, \ldots, c_n$ in a trivial manner and an invariant Ω on $\mathcal{W}^{X\cup\{\boldsymbol{c}\}}$ will descend to an invariant on $\mathcal{W}_{cl}^{X\cup\{\boldsymbol{c}\}}$ if it fulfills the condition

$$\Omega(m_{\boldsymbol{c}}^{\boldsymbol{a},\boldsymbol{b}}(T)) = \Omega(m_{\boldsymbol{c}}^{\boldsymbol{b},\boldsymbol{a}}(T))$$

for two vectors $\boldsymbol{a}, \boldsymbol{b}$ such that $a_i \neq b_j$, and T is a w-tangle in $\mathcal{W}^{X\cup\{\boldsymbol{a},\boldsymbol{b}\}}$.

Proposition 7.2 (The Trace Map)**.** *Let T be a w-tangle in $\mathcal{W}^{X\cup\{\boldsymbol{c}\}}$, then the following composition of maps, which we denote by $\Omega_{\boldsymbol{c}}$*

$$T \xrightarrow{\varphi} \left(\begin{array}{c|cc} \omega & x_{\boldsymbol{c}} & x_S \\ \hline y_{\boldsymbol{c}} & \alpha & \theta \\ y_S & \phi & \Xi \end{array}\right) \xrightarrow{\mathrm{tr}_{\boldsymbol{c}}} \omega \det(I - \alpha)$$

yields an invariant on $\mathcal{W}_{cl}^{X\cup\{\boldsymbol{c}\}}$. For an element $L \in \mathcal{W}_{cl}^{X\cup\{\boldsymbol{c}\}}$, we denote its invariant by ω_L.

Proof. We just have to check that

$$\Omega_{\boldsymbol{c}}(m_{\boldsymbol{c}}^{\boldsymbol{a},\boldsymbol{b}}(T_1)) = \Omega_{\boldsymbol{c}}(m_{\boldsymbol{c}}^{\boldsymbol{b},\boldsymbol{a}}(T_1)),$$

or more specifically

$$\mathrm{tr}_{\boldsymbol{c}}(m_{\boldsymbol{c}}^{\boldsymbol{a},\boldsymbol{b}}(\varphi(T_1))) = \mathrm{tr}_{\boldsymbol{c}}(m_{\boldsymbol{c}}^{\boldsymbol{b},\boldsymbol{a}}(\varphi(T_1)))$$

for all w-tangles $T_1 \in \mathcal{W}^{X\cup\{\boldsymbol{a},\boldsymbol{b}\}}$. Suppose that

$$\varphi(T_1) = \left(\begin{array}{c|ccc} \omega & \boldsymbol{a} & \boldsymbol{b} & S \\ \hline \boldsymbol{a} & \alpha & \beta & \theta \\ \boldsymbol{b} & \gamma & \delta & \epsilon \\ S & \phi & \psi & \Xi \end{array}\right).$$

Then $m_{\boldsymbol{c}}^{\boldsymbol{a},\boldsymbol{b}}$ gives

$$\left(\begin{array}{c|cc} \det(I-\gamma)\omega & \boldsymbol{c} & S \\ \hline \boldsymbol{c} & \beta + \alpha(I-\gamma)^{-1}\delta & \theta + \alpha(I-\gamma)^{-1}\epsilon \\ S & \psi + \phi(I-\gamma)^{-1}\delta & \Xi + \phi(I-\gamma)^{-1}\epsilon \end{array}\right)_{t_{\boldsymbol{a}}, t_{\boldsymbol{b}} \to t_{\boldsymbol{c}}}.$$

Taking $\mathrm{tr}_{\boldsymbol{c}}$ one obtains

$$\det(I - \beta - \alpha(I-\gamma)^{-1}\delta) \det(I-\gamma)\omega\big|_{t_{\boldsymbol{a}}, t_{\boldsymbol{b}} \to t_{\boldsymbol{c}}}.$$

Now the other stitching $m_{\boldsymbol{c}}^{\boldsymbol{b},\boldsymbol{a}}$ yields

$$\left(\begin{array}{c|cc} \det(I-\beta)\omega & \boldsymbol{c} & S \\ \hline \boldsymbol{c} & \gamma+\delta(I-\beta)^{-1}\alpha & \epsilon+\delta(I-\beta)^{-1}\theta \\ S & \phi+\psi(I-\beta)^{-1}\alpha & Xi+\psi(I-\beta)^{-1}\theta \end{array}\right)_{t_{\boldsymbol{a}},t_{\boldsymbol{b}}\to t_{\boldsymbol{c}}}.$$

Taking $\mathrm{tr}_{\boldsymbol{c}}$ one obtains

$$\det(I-\gamma-\delta(I-\beta)^{-1}\alpha)\det(I-\beta)\omega\big|_{t_{\boldsymbol{a}},t_{\boldsymbol{b}}\to t_{\boldsymbol{c}}}.$$

Finally we invoke Lemma 3.2 and obverse that

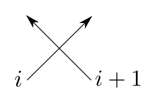
$$\det\begin{pmatrix} I-\beta & \alpha \\ \delta & I-\gamma \end{pmatrix} = \det\begin{pmatrix} I-\gamma & \delta \\ \alpha & I-\beta \end{pmatrix},$$

which completes the proof. □

Remark 7.1. Notice that the trace map agrees with the scalar part of the stitching formula in Proposition 3.1 when we allow $a_i = b_i$. The matrix part is no longer well-defined because the matrix $I-\gamma$ may not always be invertible.

7.2 The Alexander-Conway Skein Relation

In this section we derive the Alexander-Conway skein relation for long w-links. First of all let us recall the notion of *w-braids* (see [BND16] for more details). Let wB_n be the group generated by σ_i, $1 \le i \le n-1$ and s_i, $1 \le i \le n-1$, where σ_i is the usual generator of the braid group B_n and s_i represents the virtual crossing of strand i and strand $i+1$

$s_i \longrightarrow$ 

(we ignore the other strands, which go trivially) subject to the following relations

(a) (permutation relations) $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and if $|i-j|>1$ then $s_i s_j = s_j s_i$,

(b) (braid relations) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, and if $|i-j|>1$ then $\sigma_i\sigma_j = \sigma_j\sigma_i$,

(c) (mixed relations) $s_i\sigma_{i+1}^{\pm1}s_i = s_{i+1}\sigma_i^{\pm1}s_{i+1}$, and if $|i-j|>1$ then $s_i\sigma_j = \sigma_j s_i$.

(d) (OC) $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$.

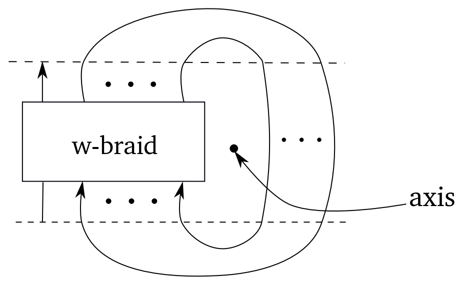
An element of wB_n is called a *w-braid* on n strands. We can extend the Burau representation to wB_n simply as follows

$$\sigma_i \mapsto \begin{pmatrix} 1-t & 1 \\ t & 0 \end{pmatrix}; \quad \sigma_i^{-1} \mapsto \begin{pmatrix} 0 & t^{-1} \\ 1 & 1-t^{-1} \end{pmatrix}; \quad s_i \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

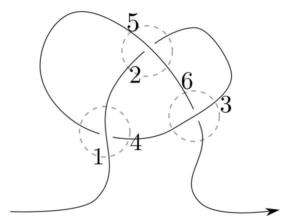
Given a w-braid, its Burau representation agrees with the matrix part of Γ-calculus, up to permutation of the columns, see Proposition 3.7.

We have the following analog of Alexander Theorem (see [KT08]).

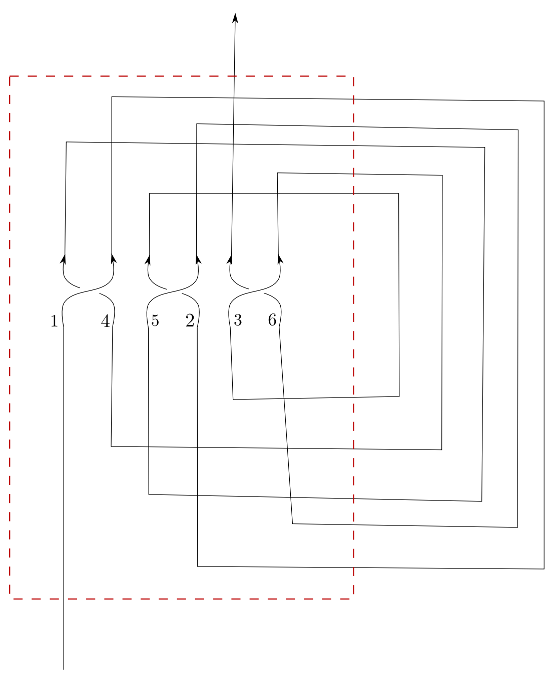
Proposition 7.3. *Every long w-link can be expressed as a partial closure, except the first strand, of a w-braid, i.e. a braid with virtual crossings modulo the OC relation.*



Proof. When we allow virtual crossings, the proof simplifies greatly. Namely we just need to decompose the long w-link into a disjoint union of crossings, put all the crossings in a row and then stitch them. As an example, let us look at the long trefoil.



Putting all the crossings of the long trefoil horizontally and the stitching the strands appropriately we obtain the desired form.



The w-braid is enclosed in the dashed rectangle. □

For a long w-link L, let

$$\Delta_L(t) = t^{-w(L)/2}\omega_L(t),$$

where $\omega_L(t)$ is the invariant as defined in Proposition 7.2 (we identify all the variables t_i to t) and

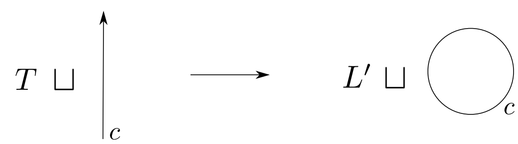
$$w(L) = \sum_{\text{crossings}} \pm 1,$$

with $+1$ for a positive crossing and -1 for a negative crossing. We record here a simple property of Δ_L.

Proposition 7.4. *Let L be a long w-link and suppose L contains a closed trivial component, i.e. bounds an embedded 2-disk that is disjoint from the rest of L. Then*

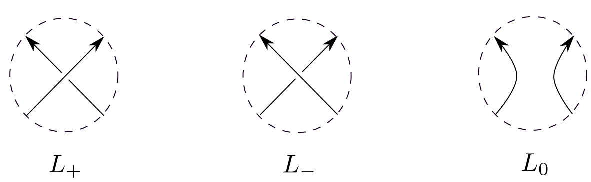
$$\Delta_L(t) = 0.$$

Proof. Suppose the closed component is labeled c. The link L can be obtained by closing a tangle of the form $T \sqcup U_c$, where U_c denotes the trivial strand.



Then the matrix part of $\varphi(T)$ will contain a column of the form $(1, \mathbf{0})^t$, where 1 occurs at position (c, c). From Proposition 7.2 we observe that in this case $\det(I - \alpha) = \det(1 - 1) = 0$. So the invariant vanishes, as required. ☐

Theorem (Alexander-Conway Skein Relation)**.** *Let L_+, L_- and L_0 be three long w-links which are identical except at a neighborhood of a crossing where they are given by,*



then we have

$$\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2})\Delta_{L_0}(t). \tag{7.1}$$

Proof. First of all we prove the following special case $L_+ = \widehat{\beta\sigma_{n-1}}$, $L_- = \widehat{\beta\sigma_{n-1}^{-1}}$, $L_0 = \widehat{\beta}$. Here β is a w-braid and σ_{n-1} is a standard generator of the braid group, n is the number of strands, and $\widehat{\ }$ denotes the partial closure (except the first strand). Observe that

$$w(L_+) = w(L_0) + 1, \quad w(L_-) = w(L_0) - 1.$$

Thus the skein relation becomes

$$t^{-1/2}\omega_{L_+}(t) - t^{1/2}\omega_{L_-}(t) = (t^{-1/2} - t^{1/2})\omega_{L_0}(t). \tag{7.2}$$

From Proposition 7.2 we have

$$\omega_{L_+}(t) = \det([I - \beta\sigma_{n-1}]_1^1), \quad \omega_{L_-}(t) = \det([I - \beta\sigma_{n-1}^{-1}]_1^1), \quad \omega_{L_0} = \det([I - \beta]_1^1),$$

where we identify the braid with its Burau representation by abuse of notations. Let

$$\beta = \left(\begin{array}{c|cc} M_1 & \phi_1 & \psi_1 \\ \hline \theta_1 & a & b \\ \epsilon_1 & c & d \end{array}\right),$$

where M_1 is an $(n-2)\times(n-2)$ matrix, ϕ_1, ψ_1 are column vectors and θ_1, ϵ_1 are row vectors. Then

$$\beta\sigma_{n-1} = \begin{pmatrix} M_1 & \phi_1 & \psi_1 \\ \theta_1 & a & b \\ \epsilon_1 & c & d \end{pmatrix}\begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1-t & 1 \\ \mathbf{0} & t & 0 \end{pmatrix} = \begin{pmatrix} M_1 & (1-t)\phi_1 + t\psi_1 & \phi_1 \\ \theta_1 & (1-t)a + tb & a \\ \epsilon_1 & (1-t)c + td & c \end{pmatrix},$$

and

$$\beta\sigma_{n-1}^{-1} = \begin{pmatrix} M_1 & \phi_1 & \psi_1 \\ \theta_1 & a & b \\ \epsilon_1 & c & d \end{pmatrix}\begin{pmatrix} I & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 0 & t^{-1} \\ \mathbf{0} & 1 & 1-t^{-1} \end{pmatrix} = \begin{pmatrix} M_1 & \psi_1 & t^{-1}\phi_1 + (1-t^{-1})\psi_1 \\ \theta_1 & b & t^{-1}a + (1-t^{-1})b \\ \epsilon_1 & d & t^{-1}c + (1-t^{-1})d \end{pmatrix}.$$

Removing the first column and the first row (correspondingly, we remove the subscript 1 in the notations) we can rewrite (7.2) as

$$\begin{aligned} t^{-1/2}\det\begin{pmatrix} I-M & -(1-t)\phi - t\psi & -\phi \\ -\theta & 1-(1-t)a-tb & -a \\ -\epsilon & -(1-t)c-td & 1-c \end{pmatrix} - t^{1/2}\det\begin{pmatrix} I-M & -\psi & -t^{-1}\phi-(1-t^{-1})\psi \\ -\theta & 1-b & -t^{-1}a-(1-t^{-1})b \\ -\epsilon & -d & 1-t^{-1}c-(1-t^{-1})d \end{pmatrix} \\ = (t^{-1/2}-t^{1/2})\det\begin{pmatrix} I-M & -\phi & -\psi \\ -\theta & 1-a & -b \\ -\epsilon & -c & 1-d \end{pmatrix}. \end{aligned}$$

Now for the first matrix, multiply the third column with $(1-t)$ and subtract it from the second column we obtain

$$\begin{aligned} & t^{-1/2}\det\begin{pmatrix} I-M & -t\psi & -\phi \\ -\theta & 1-tb & -a \\ -\epsilon & -1+t(1-d) & 1-c \end{pmatrix} \\ &= t^{-1/2}\det\begin{pmatrix} I-M & -t\psi & -\phi \\ -\theta & -tb & -a \\ -\epsilon & t(1-d) & 1-c \end{pmatrix} + t^{-1/2}\det\begin{pmatrix} I-M & \mathbf{0} & -\phi \\ -\theta & 1 & -a \\ -\epsilon & -1 & 1-c \end{pmatrix} \\ &= t^{1/2}\det\begin{pmatrix} I-M & -\psi & -\phi \\ -\theta & -b & -a \\ -\epsilon & 1-d & 1-c \end{pmatrix} + t^{-1/2}\det\begin{pmatrix} I-M & \mathbf{0} & -\phi \\ -\theta & 1 & -a \\ -\epsilon & -1 & 1-c \end{pmatrix} \\ &= -t^{1/2}\det\begin{pmatrix} I-M & -\phi & -\psi \\ -\theta & 1-a & -b \\ -\epsilon & -c & 1-d \end{pmatrix} - t^{1/2}\det\begin{pmatrix} I-M & \mathbf{0} & -\psi \\ -\theta & -1 & -b \\ -\epsilon & 1 & 1-d \end{pmatrix} \end{aligned}$$

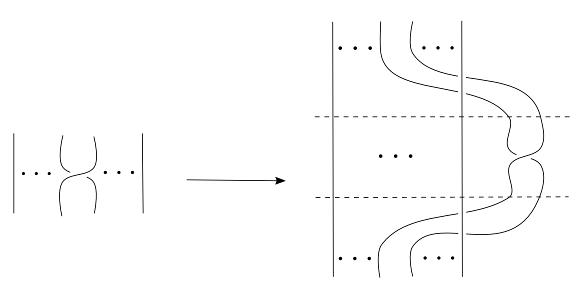
$$+\,t^{-1/2}\det\begin{pmatrix} I-M & \mathbf{0} & -\phi \\ -\theta & 1 & -a \\ -\epsilon & -1 & 1-c \end{pmatrix}.$$

Similarly for the second matrix, multiply the second column with $(1-t^{-1})$ and subtract it from the third column we have

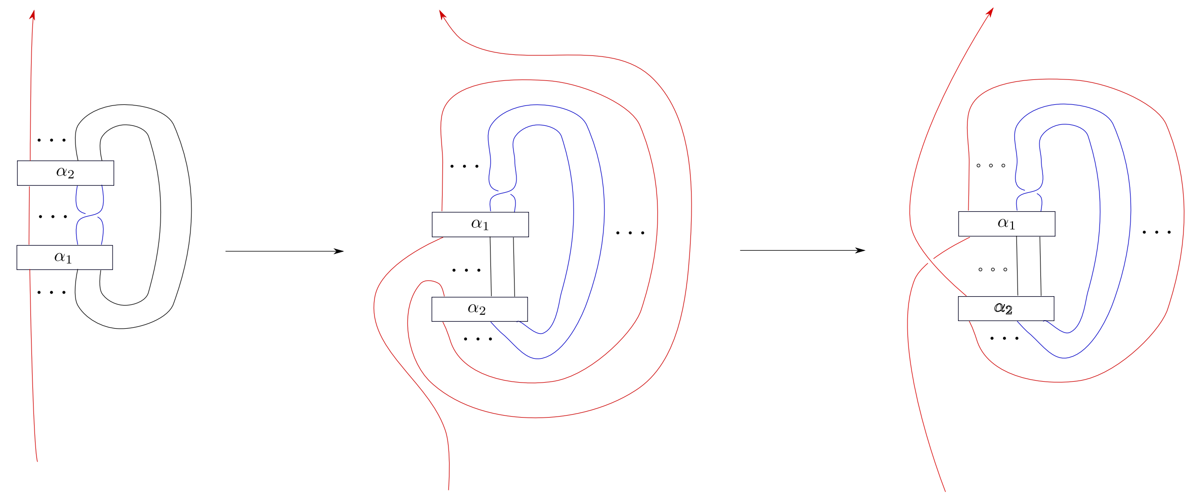
$$\begin{aligned}
&t^{1/2}\det\begin{pmatrix} I-M & -\psi & -t^{-1}\phi \\ -\theta & 1-b & -1+t^{-1}(1-a) \\ -\epsilon & -d & 1-t^{-1}c \end{pmatrix} \\
&= t^{1/2}\det\begin{pmatrix} I-M & -\psi & -t^{-1}\phi \\ -\theta & 1-b & t^{-1}(1-a) \\ -\epsilon & -d & -t^{-1}c \end{pmatrix} + t^{1/2}\det\begin{pmatrix} I-M & -\psi & \mathbf{0} \\ -\theta & 1-b & -1 \\ -\epsilon & -d & 1 \end{pmatrix} \\
&= t^{-1/2}\det\begin{pmatrix} I-M & -\psi & -\phi \\ -\theta & 1-b & 1-a \\ -\epsilon & -d & -c \end{pmatrix} + t^{-1/2}\det\begin{pmatrix} I-M & -\psi & -\mathbf{0} \\ -\theta & 1-b & -1 \\ -\epsilon & -d & 1 \end{pmatrix} \\
&= -t^{-1/2}\det\begin{pmatrix} I-M & -\phi & -\psi \\ -\theta & 1-a & -b \\ -\epsilon & -c & 1-d \end{pmatrix} - t^{-1/2}\det\begin{pmatrix} I-M & -\phi & \mathbf{0} \\ -\theta & 1-a & 1 \\ -\epsilon & -c & -1 \end{pmatrix} \\
&\qquad + t^{1/2}\det\begin{pmatrix} I-M & -\psi & \mathbf{0} \\ -\theta & 1-b & -1 \\ -\epsilon & -d & 1 \end{pmatrix}.
\end{aligned}$$

Subtracting the above identities give us the skein relation.

Finally we show that a general case can be reduced to the special case as follows. Given a long w-link L_+, we first express it as the partial closure of a braid $\beta_1\sigma_i\beta_2$. As a first step, observe that σ_i can be written as a conjugate of σ_{n-1} (simply pulling the crossing σ_i to the right-most position. Thus we can assume that L_+ is the partial closure of $\alpha_1\sigma_{n-1}\alpha_2$.



Now to proceed we can push α_2 along the closure to the bottom of α_1 and then move the open component to the left.



The end result now is the partial closure of a w-braid of the form $\beta\sigma_{n-1}$, as required. □

7.3 Odds and Ends

As we have mentioned our work is just the beginning of a long-term project. There are many potentially interesting directions to be explored. We present a few of these directions below.

- **General "unitary property".** So far we have only proven the unitary property for string links. The general unitary property for tangles is more involved. A unitary property for tangles should characterize the image of (usual) tangles in Γ-calculus. We describe one strategy to accomplish this below, although we suspect that one should be able to arrive at the conclusion by much simpler means. Our underlying technical theorem is the commutativity of the following diagram (see [BND14]):

$$\begin{array}{ccc} sKTG & \xrightarrow{a} & wTF \\ \downarrow{\scriptstyle Z^u} & & \downarrow{\scriptstyle Z^w} \\ \mathcal{A}^u & \xrightarrow{\alpha} & \mathcal{A}^{sw} \end{array} \tag{7.3}$$

To understand the terms and maps in the commutative diagram would require a substantial amount of background. So we can only give a very rough description. The space $sKTG$ consists of *knotted trivalent graphs*, i.e. tangles with trivalent vertices; the space wTF consists of *w-tangled foams*, i.e. w-tangles with trivalent vertices; $\mathcal{A}^u$ is the space of chord diagrams and $\mathcal{A}^{sw}$ is the space of arrow diagrams. The map a includes (usual) tangles into w-tangles; the map α sends a chord diagram to all ways of orienting the chord, namely

$$t_{ij} \mapsto a_{ij} + a_{ji};$$

and Z^u and Z^w are the corresponding homomorphic expansions. It is important to point out that the image of a vertex in $\mathcal{A}^{sw}$ under Z^w gives us a solution to the Kashiwara-Vergne problems.

It is advantageous to work in $\mathbb{G}_0$ because it is simpler than $\mathcal{A}^{sw}$ and all the formulas are readily available. We can solve for a solution of a vertex explicitly in $\mathbb{G}_0$ using Mathematica. Now to state the unitary property we need to introduce an *involution* θ of $\mathfrak{g}_0$ given by

$$b \mapsto -b, \quad c \mapsto -c, \quad u \mapsto w, \quad w \mapsto u.$$

Recall that

$$T_{\mathfrak{g}_0}(a_{ij} + a_{ji}) = b_i c_j + u_i w_j + b_j c_i + u_j w_i.$$

We have

$$\theta(b_i c_j + u_i w_j + b_j c_i + u_j w_i) = b_i c_j + w_i u_j + b_i c_j + u_i w_j.$$

In other words, θ preserves the image of a chord under α. It follows from the commutative diagram (7.3) that for an element $\zeta \in sKTG$ its image $\zeta \not\!/\, a \not\!/\, Z^w \not\!/\, T_{\mathfrak{g}_0}$ is invariant under θ.

Given a (usual) bottom tangle T with $2n$ endpoints, we first split the endpoints in half and designate one half as the bottom and the other half as the top. To convert T to an element of $sKTG$ we pick a canonical "parenthesization" of the bottom endpoints and the top endpoints, i.e. a way of grouping the endpoints together. Then we can compose the bottom and the top of the tangle with binary trees V_b and V_t according to the parenthesization. From the above discussion we have

Theorem 7.1 (General Unitary Property)**.** *The element $V_b T V_t$ is invariant under θ.*

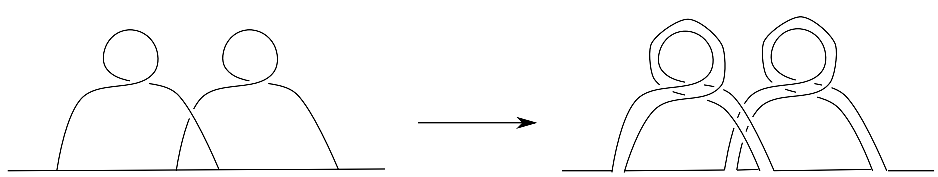
For the case of string links we claim that the general unitary property reduces to Theorem 6.1. It is interesting to express the general unitary property explicitly in the language of Γ-calculus. Then it might be possible to prove the property using more elementary means.

- **Alexander recovery.** Another goal of our work is to convince the readers that the language of Γ-calculus provides an easily accessible way to study the Alexander polynomial, particularly in terms of computer implementation. We have recovered several classical properties of the Alexander polynomial in this thesis but there are more to be explored. Another particularly interesting property on our list is the genus property of the Alexander polynomial, which states that

$$\deg(\Delta_K) \leq 2g(K),$$

where $g(K)$ is the genus of the knot K and $\deg(\Delta_K)$ is the *degree* of Δ_K, i.e. the difference of the smallest and largest exponents of the monomials of Δ_K.

To express the genus in the language of meta-monoids, we use the band presentation of a surface. More specifically, suppose that K is a knot of genus g. We can represent the surface that K bounds as a bottom tangle with $2g$ components. Then to recover the surface, we double each of the strand, reverse the orientation of one side of each band, and then perform stitching. For instance, a knot of genus 1 is given by



In this manner we can use Γ-calculus to investigate the genus since we have already obtained the formulas for strand doubling and orientation reversal.

- **The Lie algebra $\mathfrak{g}_1$.** Our long-term goal would be the generalize this thesis to the case of $\mathfrak{g}_1$ (see [BNV17, BN17, BN16a]). The Lie algebra $\mathfrak{g}_1$ is a deformed version of $\mathfrak{g}_0$. Namely, $\mathfrak{g}_1$ is the 4-dimensional Lie algebra $\mathfrak{g}_1 = \langle b, c, u, w\rangle$ over the ring $R = \mathbb{Q}[\epsilon]/(\epsilon^2 = 0)$, with b central and with the brackets given by

$$[w, c] = w, \quad [c, u] = u, \quad [u, w] = b - 2\epsilon c.$$

Observe that when $\epsilon = 0$ we recover the Lie algebra $\mathfrak{g}_0$. In this case the positive crossing is

$$R^+_{i,j} \mapsto \exp\left((b_i - \epsilon c_i)c_j + u_i w_j\right) \in U(\mathfrak{g}_1)^{\widehat{\otimes}\{i,j\}}$$

Ideally we would like to generalize the Fox-Milnor condition to the case of $\mathfrak{g}_1$, which will hopefully shed some light on the slice-ribbon conjecture. This thesis is the first step in that direction.

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