One step beyond the Alexander polynomial

Dror Bar-Natan and Roland van der Veen

We present a simple, strong knot invariant that is closely related to the Alexander polynomial and seems to share many of its good properties. For example, unlike the commonly used quantum invariants such as the Jones polynomial, our invariant is computable in polynomial time. In the discussion section below we give some hints on how it is related to the process of solvable approximation of Lie groups that may also be of interest outside of knot theory.

Consider a (long) knot K presented as a proper smooth embedding of $[0,1]$ into the closed unit ball such that the projection on the third coordinate is a generic immersion γ in the plane, see for example Figure 1. More specifically, assume that there is an $n \in \mathbb{N}$ such that γ has the following properties. The points $\gamma(\frac{k}{n+1})$ where $k \in \{1,\dots,n\}$ are the union of all double points and all points where γ' is parallel to the positive x-axis. The double points are known as crossings and the latter as cuaps. Close to any crossing we assume γ' has positive y-coordinate. The sign of a crossing is the sign of the x-coordinate of γ' at the overpass. A crossing is denoted $X_{s,i,j}$ where s is the sign and i,j are the labels of the over and under strand. The sign of a cuap is the sign of the y-direction of γ''. A cuap is denoted $u_{s,i}$ where s is the sign and i is its label.

Let E^i_j be the elementary matrix with a single non-zero entry 1 at the (i,j)-th place. Define the matrices

$$Q = \sum_{X_{s,i,j}} st^{\frac{s}{2}}(E^j_j - E^i_j) \quad W = \sum_{i<j} E^i_j \quad c = \sum_{X_{s,i,j},u_{s,i}} t^{-s}$$

$$A = I - (t^{\frac{1}{2}} - t^{-\frac{1}{2}})WQ \quad G = Q\operatorname{adj}(A) \quad H = \operatorname{adj}(A)W$$

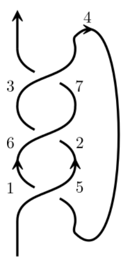
$$Z_G = \sum_{j=2}^{n} \sum_{a,b<j} G^j_a \left(\frac{1}{2} G^j_b + \sum_{c>j} G^c_b \right)$$

$$Z_H = \sum_{X_{s,i,j}} \frac{s}{2}((1-t^s)H^i_j)^2 - \frac{s}{2}((1+t^s)H^j_j)^2 + st^s(H^j_i H^i_j + H^i_i H^j_j) + t^s(1-t)H^j_i((1+s)H^j_j + (1-s)H^i_i)$$
$$+ \det(A) \sum_{u_{s,i}} sH^i_i$$

Theorem 1. *$c^{\frac{1}{2}}\det(A)$ is the Alexander polynomial and $Z_1 = c(Z_G + Z_H)$ is a new knot invariant. Both are elements of $\mathbb{Z}[t,t^{-1}]$ computable in polynomial time.*

The pair Δ, Z_1 distinguishes all knots in the Rolfsen table of prime knots up to ten crossings, see the appendix. That is a better performance than for example the pair Khovanov homology and HOMFLY polynomial. More importantly Z appears to share many of the desirable properties of the Alexander polynomial.

Our invariant Z_1 appears to coincide with the invariant ρ_1 of Rozansky [5] and Overbay [4] through the substitution $\rho_1 = -t/(1-t)^2(Z - t\Delta\frac{d}{dt}\Delta)$. The benefit of working with Z_1 is that it allows for a simpler formula.



$$W = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 & 0 & 0 & -\sqrt{t} & 0 & 0 \\ 0 & \sqrt{t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{t} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{t} & 0 & 0 \\ 0 & -\sqrt{t} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{t} \end{pmatrix} \quad c = t^{-2}$$

Figure 1: A diagram for the trefoil knot 3_1. The double points at the crossings and the right-pointing cuaps are enumerated in order of appearance. The matrices W, Q and the number c necessary for computation of the invariant Z_1 are listed next to it.

Example: Trefoil

We illustrate the computation of Z for the trefoil knot 3_1 shown in Figure 1.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1-t & 0 & 0 \\ 0 & t & 0 & 0 & 1-t & 0 & 0 \\ 0 & t-1 & 1 & 0 & 1-t & 0 & 1-t \\ 0 & t-1 & 0 & 1 & 1-t & 0 & 1-t \\ 0 & t-1 & 0 & 0 & 1 & 0 & 1-t \\ 0 & 0 & 0 & 0 & 0 & 1 & 1-t \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \Delta_{3_1}(t) = c^{\frac{1}{2}}\det(A) = t - 1 + t^{-1}$$

$$G = \begin{pmatrix} 0 & t^{3/2}-t^{\frac{1}{2}} & 0 & 0 & -t^{3/2} & 0 & t^{3/2}-t^{5/2} \\ 0 & t^{\frac{1}{2}} & 0 & 0 & t^{3/2}-t^{\frac{1}{2}} & 0 & t^{5/2}-2t^{3/2}+t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & -t^{5/2}+t^{3/2}-t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & t^{\frac{1}{2}}-t^{3/2} & 0 & 0 & t^{3/2} & 0 & t^{5/2}-t^{3/2} \\ 0 & -t^{\frac{1}{2}} & 0 & 0 & t^{\frac{1}{2}}-t^{3/2} & 0 & -t^{5/2}+2t^{3/2}-t^{\frac{1}{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & t^{5/2}-t^{3/2}+t^{\frac{1}{2}} \end{pmatrix} \qquad Z_G = t^4 - \frac{3t^2}{2} + \frac{1}{2}$$

$$H = \begin{pmatrix} 0 & t^2-t+1 & t & t & t & t^2 & t^2 \\ 0 & 0 & 1 & 1 & 1 & t & t \\ 0 & 0 & t-t^2 & 1 & 1 & t & t \\ 0 & 0 & t-t^2 & t-t^2 & 1 & t & t \\ 0 & 0 & 1-t & 1-t & 1-t & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^2-t+1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad Z_H = t^4 - 3t^3 + \frac{7t^2}{2} - t - \frac{1}{2}$$

It follows that $Z_1(3_1) = c(Z_G + Z_H) = 2 - t^{-1} - 3t + 2t^2$ and its normalization $\rho_1(t) = t + t^{-1}$.

Discussion: Solvable approximation of Lie (bi)algebras

One way to find the invariant Z_1 and many more like it is through solvable approximation of Lie algebras, in the present case $\mathfrak{gl}_2$. Below we give a brief account of Lie bialgebras and some of their solvable approximations. This process is closely related to contraction of Lie algebras [2].
A **Lie bialgebra** $\mathfrak{g}$ over a ring S is an S-module $\mathfrak{g}$ together with module maps $\beta : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying the following, where $\sigma(a \otimes b) = b \otimes a$ and $\tau(a \otimes b \otimes c) = c \otimes a \otimes b$:

$$(\beta)(1+\sigma) = \beta(\beta \otimes \mathrm{id})(1 + \tau + \tau^2) = (1+\sigma)(\delta) = (1 + \tau + \tau^2)\delta(\delta \otimes \mathrm{id}) = 0$$

$$\delta\beta = ((\beta \otimes \mathrm{id})(\mathrm{id} \otimes \delta) + (\mathrm{id} \otimes \beta)(\delta \otimes \mathrm{id}))(1 - \sigma)$$

Example 1: $\mathfrak{gl}_n(S) = \mathrm{End}(S^n)$ with $\delta = 0$ and $\beta(A, B) = AB - BA$. This is the tangent space to $GL(S^n)$. If S has a basis we may write the elementary matrices as e^i_j and $\beta(e^i_j, e^k_l) = e^i_l \delta^k_j - e^k_j \delta^i_l$ (Kronecker delta).
Example 2: $\mathfrak{a}$ with generators X, Y and bracket $\beta(X, Y) = X$ and $\delta(X) = Y \otimes X - X \otimes Y$ and $\delta(Y) = 0$.

The **double** $\mathcal{D}(\mathfrak{g}, \beta, \delta)$ of a Lie bialgebra is the module $\mathfrak{g} \oplus \mathfrak{g}^*$ with $\delta = \delta_{\mathfrak{g}} - \beta^*_g$. We define a bilinear form on $\mathcal{D}$ by $\langle x + X, y + Y\rangle = x(Y) + y(X)$. Now β is defined by $\langle \beta(u, v), w\rangle = \langle u, \beta(v, w)\rangle$.
Example 2 (continued): Over $\mathbb{Q}$ with $\epsilon \neq 0$ we have $\mathcal{D}(\mathfrak{a}, \beta, \epsilon\delta) \cong \mathfrak{gl}_2(\mathbb{Q})$ as Lie algebra with generators X, Y, x, y and relations

$$[X, Y] = X \quad [X, x] = -y - \epsilon Y \quad [X, y] = \epsilon X \quad [Y, x] = x \quad [Y, y] = 0 \quad [x, y] = -\epsilon x$$

The isomorphism ϕ is given by $\phi(X) = e^1_2, \phi(Y) = e^2_2, \phi(x) = e^2_1, \phi(y) = -e^1_1$.

Over the ring $S_k = \mathbb{Q}[\epsilon]/(\epsilon^{k+1})$ the algebra $\mathcal{D}(\mathfrak{g}, \beta, \epsilon\delta)$ is called a **solvable approximation** of $\mathcal{D}(\mathfrak{g}, \beta, \delta)$.

Recall $\mathfrak{g}$ is solvable if $\mathfrak{g}^{(k)} = \{0\}$ for some k where $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{(k+1)} = \beta(\mathfrak{g}^{(k)}, \mathfrak{g}^{(k)})$. Over an algebraically closed field in characteristic 0 all solvable Lie algebras may be presented as upper-triangular matrices.

Example 2 with $k=0$ so $\epsilon=0$ gives $\mathfrak{g}_0 = \mathcal{D}(\mathfrak{a}, \beta, 0)$. It may be represented as matrices $\begin{pmatrix} 0 & a & b \\ 0 & c & d \\ 0 & 0 & 0 \end{pmatrix}$ In other words there is a faithful $3d$ representation ρ of $\mathfrak{g}_0$ given by $\rho(X) = e_2^1, \rho(Y) = e_2^2, \rho(x) = e_3^2, \rho(y) = -e_3^1$.

Coming back to our knot invariant Z_1, it is obtained from the solvable Lie algebra $\mathfrak{g}_1$ by Drinfeld's quantum double construction. The quantum double construction can be applied to the universal enveloping algebra of any Lie bialgebra [1]. Taking the constant term of the universal invariant corresponding to the resulting ribbon Hopf algebra [3] yields Z_1. Applying the same construction without the approximation would yield the usual quantum group $U_q\mathfrak{gl}_2$ and hence invariants such as the colored Jones polynomial. This indicates our invariant is closely related to Rozansky's expansion of the colored Jones polynomial [5].

Question: What problems does solvable approximation solve?

Many of the topics in this conference are related to the Lie algebra $\mathfrak{gl}_2$ in one way or another. For example, hyperbolic geometry is the geometry of (P)GL(2). Is there a geometry based on solvable approximations of this group? Likewise Chern-Simons theory is usually thought of only in the context of semi-simple Lie algebras but surely one can replace those by their solvable approximations. At least at the perturbative level. This is what our Z_1 does. $GL(2)$ also plays an important role in number theory, for example in modular forms and Galois representations. Here it may also be useful to use these approximations.
In conclusion, solvable approximations promise at least two things: First a new way of getting interesting yet tractable examples. Second, a way of expanding out and perhaps solving less tractable problems.

References

[1] P. Etingof and O. Schiffman. *Lectures on quantum groups.* International Press, 1998.

[2] E. Inonu and E. Wigner. On the contraction of groups and their representations. *Proc. Nat. Acad. Sci.*, 39:510524, 1953.

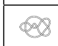
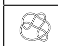
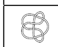
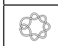
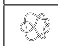
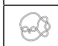
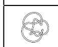
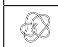
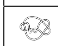
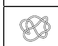
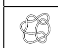
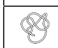
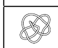
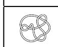
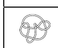
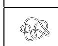
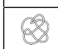
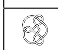
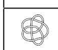
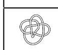
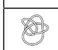
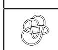
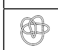
[3] T. Ohtsuki. *Quantum invariants.* World Scientific, 2001.

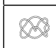
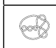
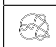
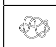
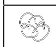
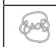
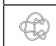
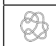
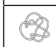
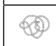
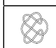
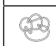
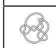
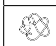
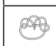
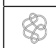
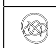
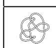
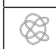
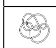
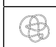
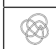
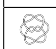
[4] A. Overbay. Perturbative expansion of the colored Jones polynomial. University of North Carolina PhD thesis, 2013.

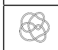
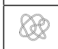
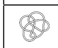
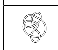
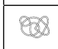
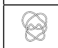
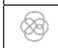
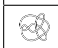
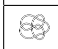
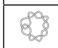
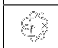
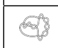
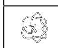
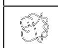
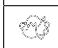
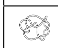
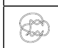
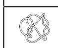
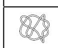
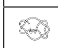
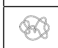
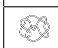
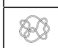
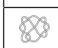
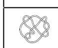
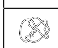
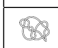
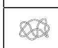
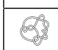
[5] L. Rozansky. The universal r-matrix, Burau representation and the Melvin-Morton expansion of the colored Jones polynomial. *Adv. math.*, 134-1:1–31, 1998.

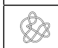
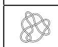
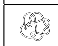
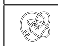
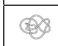
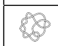
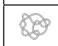
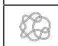
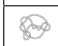
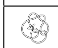
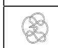
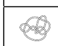
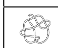
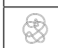
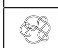
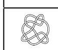
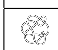
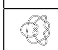
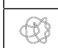
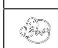
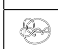
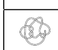
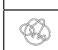
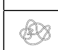
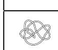
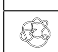
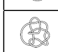
Appendix: Rolfsen table of prime knots up to 10 crossings

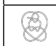
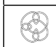
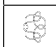
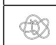
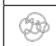
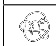
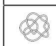
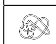
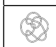
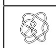
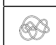
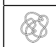
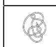
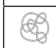
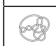
In the table we list the Alexander polynomial $\Delta(t)$ of each knot together with $\rho_1 = -t/(1-t)^2(Z - t\Delta\frac{d}{dt}\Delta)$. Since both polynomials are symmetric Laurent polynomials in t it suffices to list only the non-negative powers of t. Note how *all* knots in the table are distinguished by the pair Δ, Z. Also ρ_1 appears to vanish on all amphicheiral knots [4].

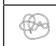
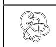
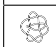
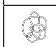
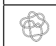
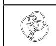
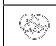
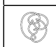
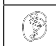
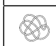
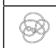
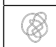
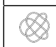
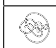
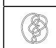
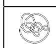
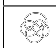
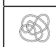
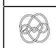
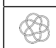
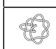
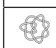
Picture	Name	Alexander	ρ_1-normalization of Z
	3_1	$t-1$	t
	4_1	$3-t$	0
	5_1	t^2-t+1	$2t^3+3t$
	5_2	$2t-3$	$5t-4$
	6_1	$5-2t$	$t-4$
	6_2	$-t^2+3t-3$	t^3-4t^2+4t-4
	6_3	t^2-3t+5	0
	7_1	t^3-t^2+t-1	$3t^5+5t^3+6t$
	7_2	$3t-5$	$14t-16$
	7_3	$2t^2-3t+3$	$-9t^3+8t^2-16t+12$
	7_4	$4t-7$	$32-24t$
	7_5	$2t^2-4t+5$	$9t^3-16t^2+29t-28$
	7_6	$-t^2+5t-7$	$t^3-8t^2+19t-20$
	7_7	t^2-5t+9	$8-3t$
	8_1	$7-3t$	$5t-16$
	8_2	$-t^3+3t^2-3t+3$	$2t^5-8t^4+10t^3-12t^2+13t-12$
	8_3	$9-4t$	0
	8_4	$-2t^2+5t-5$	$3t^3-8t^2+6t-4$
	8_5	$-t^3+3t^2-4t+5$	$-2t^5+8t^4-13t^3+20t^2-22t+24$
	8_6	$-2t^2+6t-7$	$5t^3-20t^2+28t-32$
	8_7	t^3-3t^2+5t-5	$-t^5+4t^4-10t^3+12t^2-13t+12$
	8_8	$2t^2-6t+9$	$-t^3+4t^2-12t+16$
	8_9	$-t^3+3t^2-5t+7$	0
	8_{10}	t^3-3t^2+6t-7	$-t^5+4t^4-11t^3+16t^2-21t+20$
	8_{11}	$-2t^2+7t-9$	$5t^3-24t^2+39t-44$
	8_{12}	$t^2-7t+13$	0
	8_{13}	$2t^2-7t+11$	$-t^3+4t^2-14t+20$
	8_{14}	$-2t^2+8t-11$	$5t^3-28t^2+57t-68$
	8_{15}	$3t^2-8t+11$	$21t^3-64t^2+120t-140$
	8_{16}	t^3-4t^2+8t-9	$t^5-6t^4+17t^3-28t^2+35t-36$
	8_{17}	$-t^3+4t^2-8t+11$	0
	8_{18}	$-t^3+5t^2-10t+13$	0
	8_{19}	t^3-t^2+1	$-3t^5-4t^2-3t$
	8_{20}	t^2-2t+3	$4t-4$
	8_{21}	$-t^2+4t-5$	$t^3-8t^2+16t-20$

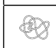
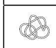
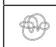
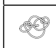
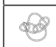
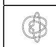
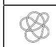
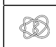
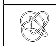
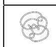
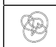
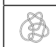
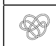
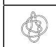
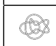
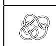
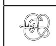
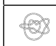
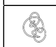
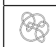
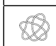
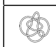
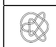
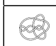
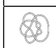
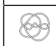
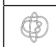
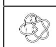
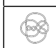
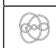
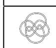
Picture	Name	Alexander	ρ_1-normalization of Z
	9_1	$t^4-t^3+t^2-t+1$	$4t^7+7t^5+9t^3+10t$
	9_2	$4t-7$	$30t-40$
	9_3	$2t^3-3t^2+3t-3$	$-13t^5+12t^4-25t^3+20t^2-32t+24$
	9_4	$3t^2-5t+5$	$23t^3-28t^2+46t-44$
	9_5	$6t-11$	$100-65t$
	9_6	$2t^3-4t^2+5t-5$	$13t^5-24t^4+45t^3-52t^2+68t-64$
	9_7	$3t^2-7t+9$	$23t^3-56t^2+99t-108$
	9_8	$-2t^2+8t-11$	$3t^3-16t^2+29t-28$
	9_9	$2t^3-4t^2+6t-7$	$13t^5-24t^4+55t^3-72t^2+98t-96$
	9_{10}	$4t^2-8t+9$	$-40t^3+72t^2-114t+120$
	9_{11}	$-t^3+5t^2-7t+7$	$-2t^5+16t^4-41t^3+52t^2-66t+64$
	9_{12}	$-2t^2+9t-13$	$5t^3-36t^2+84t-100$
	9_{13}	$4t^2-9t+11$	$-40t^3+92t^2-154t+168$
	9_{14}	$2t^2-9t+15$	$-t^3+8t^2-35t+60$
	9_{15}	$-2t^2+10t-15$	$-5t^3+40t^2-108t+136$
	9_{16}	$2t^3-5t^2+8t-9$	$-13t^5+36t^4-80t^3+120t^2-161t+168$
	9_{17}	t^3-5t^2+9t-9	$t^5-8t^4+23t^3-32t^2+28t-24$
	9_{18}	$4t^2-10t+13$	$40t^3-108t^2+193t-220$
	9_{19}	$2t^2-10t+17$	$t^3-8t^2+20t-24$
	9_{20}	$-t^3+5t^2-9t+11$	$2t^5-16t^4+47t^3-84t^2+117t-124$
	9_{21}	$-2t^2+11t-17$	$-5t^3+44t^2-127t+164$
	9_{22}	$t^3-5t^2+10t-11$	$-t^5+8t^4-24t^3+38t^2-40t+36$
	9_{23}	$4t^2-11t+15$	$40t^3-128t^2+243t-288$
	9_{24}	$-t^3+5t^2-10t+13$	$-4t^2+16t-20$
	9_{25}	$-3t^2+12t-17$	$12t^3-70t^2+153t-188$
	9_{26}	$t^3-5t^2+11t-13$	$-t^5+8t^4-31t^3+64t^2-85t+92$
	9_{27}	$-t^3+5t^2-11t+15$	$t^3-8t^2+24t-32$
	9_{28}	$t^3-5t^2+12t-15$	$t^5-8t^4+30t^3-68t^2+105t-120$
	9_{29}	$t^3-5t^2+12t-15$	$t^5-8t^4+26t^3-48t^2+59t-56$
	9_{30}	$-t^3+5t^2-12t+17$	$2t^3-10t^2+25t-32$
	9_{31}	$t^3-5t^2+13t-17$	$t^5-8t^4+33t^3-80t^2+132t-152$
	9_{32}	$t^3-6t^2+14t-17$	$-t^5+10t^4-42t^3+94t^2-133t+148$
	9_{33}	$-t^3+6t^2-14t+19$	$t^3-10t^2+30t-40$
	9_{34}	$-t^3+6t^2-16t+23$	$3t^3-18t^2+43t-56$
	9_{35}	$7t-13$	$90t-144$

Picture	Name	Alexander	ρ_1-normalization of Z
	9_{36}	$-t^3+5t^2-8t+9$	$-2t^5+16t^4-44t^3+66t^2-87t+88$
	9_{37}	$2t^2-11t+19$	$t^3-8t^2+22t-28$
	9_{38}	$5t^2-14t+19$	$62t^3-204t^2+382t-452$
	9_{39}	$-3t^2+14t-21$	$-12t^3+84t^2-210t+268$
	9_{40}	$t^3-7t^2+18t-23$	$t^5-12t^4+57t^3-144t^2+229t-264$
	9_{41}	$3t^2-12t+19$	$3t^3-20t^2+70t-108$
	9_{42}	$-t^2+2t-1$	$-t^3+2t^2+t-4$
	9_{43}	$-t^3+3t^2-2t+1$	$-2t^5+8t^4-7t^3+2t^2-5t+4$
	9_{44}	t^2-4t+7	$-2t^2+9t-12$
	9_{45}	$-t^2+6t-9$	$t^3-14t^2+47t-60$
	9_{46}	$5-2t$	$3t-12$
	9_{47}	t^3-4t^2+6t-5	$-t^5+6t^4-15t^3+16t^2-10t+12$
	9_{48}	$-t^2+7t-11$	$-t^3+12t^2-42t+52$
	9_{49}	$3t^2-6t+7$	$-21t^3+38t^2-61t+60$
	10_1	$9-4t$	$14t-40$
	10_2	$-t^4+3t^3-3t^2+3t-3$	$3t^7-12t^6+16t^5-20t^4+24t^3-24t^2+27t-24$
	10_3	$13-6t$	$11t-28$
	10_4	$-3t^2+7t-7$	$4t^3-8t^2+t+8$
	10_5	$t^4-3t^3+5t^2-5t+5$	$-2t^7+8t^6-20t^5+28t^4-36t^3+36t^2-39t+36$
	10_6	$-2t^3+6t^2-7t+7$	$9t^5-36t^4+56t^3-72t^2+81t-84$
	10_7	$-3t^2+11t-15$	$14t^3-72t^2+135t-160$
	10_8	$-2t^3+5t^2-5t+5$	$7t^5-20t^4+23t^3-28t^2+26t-24$
	10_9	$-t^4+3t^3-5t^2+7t-7$	$-t^7+4t^6-10t^5+20t^4-25t^3+28t^2-28t+28$
	10_{10}	$3t^2-11t+17$	$-5t^3+24t^2-71t+100$
	10_{11}	$-4t^2+11t-13$	$16t^3-52t^2+68t-72$
	10_{12}	$2t^3-6t^2+10t-11$	$-5t^5+20t^4-50t^3+72t^2-89t+92$
	10_{13}	$2t^2-13t+23$	$t^3-12t^2+51t-84$
	10_{14}	$-2t^3+8t^2-12t+13$	$9t^5-52t^4+119t^3-180t^2+225t-236$
	10_{15}	$2t^3-6t^2+9t-9$	$-3t^5+12t^4-24t^3+24t^2-17t+12$
	10_{16}	$-4t^2+12t-15$	$-16t^3+56t^2-76t+80$
	10_{17}	$t^4-3t^3+5t^2-7t+9$	0
	10_{18}	$-4t^2+14t-19$	$16t^3-68t^2+121t-140$
	10_{19}	$2t^3-7t^2+11t-11$	$3t^5-16t^4+35t^3-40t^2+30t-24$
	10_{20}	$-3t^2+9t-11$	$14t^3-56t^2+88t-104$

Picture	Name	Alexander	ρ_1-normalization of Z
	10_{21}	$-2t^3+7t^2-9t+9$	$9t^5-44t^4+80t^3-104t^2+121t-124$
	10_{22}	$-2t^3+6t^2-10t+13$	$-t^5+4t^4-10t^3+24t^2-37t+44$
	10_{23}	$2t^3-7t^2+13t-15$	$-5t^5+24t^4-67t^3+108t^2-137t+144$
	10_{24}	$-4t^2+14t-19$	$24t^3-116t^2+221t-268$
	10_{25}	$-2t^3+8t^2-14t+17$	$9t^5-52t^4+131t^3-232t^2+314t-344$
	10_{26}	$-2t^3+7t^2-13t+17$	$-t^5+4t^4-10t^3+28t^2-49t+60$
	10_{27}	$2t^3-8t^2+16t-19$	$5t^5-28t^4+87t^3-164t^2+229t-252$
	10_{28}	$4t^2-13t+19$	$-8t^3+36t^2-100t+136$
	10_{29}	$t^3-7t^2+15t-17$	$t^5-12t^4+52t^3-104t^2+124t-128$
	10_{30}	$-4t^2+17t-25$	$24t^3-148t^2+345t-440$
	10_{31}	$4t^2-14t+21$	$-4t^2+9t-12$
	10_{32}	$-2t^3+8t^2-15t+19$	$t^5-4t^4+13t^3-40t^2+78t-96$
	10_{33}	$4t^2-16t+25$	0
	10_{34}	$3t^2-9t+13$	$-5t^3+20t^2-52t+68$
	10_{35}	$2t^2-12t+21$	$-t^3+12t^2-47t+76$
	10_{36}	$-3t^2+13t-19$	$14t^3-88t^2+208t-264$
	10_{37}	$4t^2-13t+19$	0
	10_{38}	$-4t^2+15t-21$	$24t^3-128t^2+270t-336$
	10_{39}	$-2t^3+8t^2-13t+15$	$9t^5-52t^4+125t^3-204t^2+263t-280$
	10_{40}	$2t^3-8t^2+17t-21$	$-5t^5+28t^4-89t^3+176t^2-258t+288$
	10_{41}	$t^3-7t^2+17t-21$	$t^5-12t^4+54t^3-120t^2+157t-164$
	10_{42}	$-t^3+7t^2-19t+27$	$2t^3-8t^2+11t-12$
	10_{43}	$-t^3+7t^2-17t+23$	0
	10_{44}	$t^3-7t^2+19t-25$	$t^5-12t^4+56t^3-140t^2+220t-248$
	10_{45}	$-t^3+7t^2-21t+31$	0
	10_{46}	$-t^4+3t^3-4t^2+5t-5$	$-3t^7+12t^6-21t^5+34t^4-43t^3+52t^2-55t+56$
	10_{47}	$t^4-3t^3+6t^2-7t+7$	$-2t^7+8t^6-23t^5+38t^4-56t^3+60t^2-68t+64$
	10_{48}	$t^4-3t^3+6t^2-9t+11$	$t^5-2t^4+2t^3-3t+4$
	10_{49}	$3t^3-8t^2+12t-13$	$30t^5-94t^4+196t^3-292t^2+372t-392$
	10_{50}	$-2t^3+7t^2-11t+13$	$-9t^5+44t^4-94t^3+150t^2-186t+200$
	10_{51}	$2t^3-7t^2+15t-19$	$-5t^5+24t^4-73t^3+134t^2-194t+212$
	10_{52}	$2t^3-7t^2+13t-15$	$-3t^5+16t^4-37t^3+50t^2-49t+44$
	10_{53}	$6t^2-18t+25$	$93t^3-346t^2+680t-828$
	10_{54}	$2t^3-6t^2+10t-11$	$-3t^5+12t^4-24t^3+26t^2-21t+16$
	10_{55}	$5t^2-15t+21$	$66t^3-246t^2+488t-596$

Picture	Name	Alexander	ρ_1-normalization of Z
	10_{56}	$-2t^3+8t^2-14t+17$	$-9t^5+52t^4-133t^3+234t^2-312t+340$
	10_{57}	$2t^3-8t^2+18t-23$	$-5t^5+28t^4-93t^3+194t^2-300t+340$
	10_{58}	$3t^2-16t+27$	$3t^3-28t^2+94t-140$
	10_{59}	$t^3-7t^2+18t-23$	$-t^5+12t^4-55t^3+128t^2-181t+196$
	10_{60}	$-t^3+7t^2-20t+29$	$5t^3-40t^2+122t-176$
	10_{61}	$-2t^3+5t^2-6t+7$	$-7t^5+20t^4-27t^3+36t^2-35t+36$
	10_{62}	$t^4-3t^3+6t^2-8t+9$	$-2t^7+8t^6-23t^5+40t^4-63t^3+76t^2-89t+88$
	10_{63}	$5t^2-14t+19$	$66t^3-220t^2+416t-496$
	10_{64}	$-t^4+3t^3-6t^2+10t-11$	$-t^7+4t^6-11t^5+24t^4-37t^3+52t^2-60t+64$
	10_{65}	$2t^3-7t^2+14t-17$	$-5t^5+24t^4-71t^3+124t^2-169t+180$
	10_{66}	$3t^3-9t^2+16t-19$	$30t^5-112t^4+279t^3-480t^2+662t-724$
	10_{67}	$-4t^2+16t-23$	$24t^3-140t^2+312t-392$
	10_{68}	$4t^2-14t+21$	$8t^3-40t^2+117t-164$
	10_{69}	$t^3-7t^2+21t-29$	$-t^5+12t^4-68t^3+212t^2-397t+476$
	10_{70}	$t^3-7t^2+16t-19$	$-t^5+12t^4-53t^3+114t^2-146t+152$
	10_{71}	$-t^3+7t^2-18t+25$	t^3-2t^2-t+4
	10_{72}	$-2t^3+9t^2-16t+19$	$-9t^5+60t^4-167t^3+298t^2-410t+448$
	10_{73}	$t^3-7t^2+20t-27$	$t^5-12t^4+65t^3-194t^2+350t-416$
	10_{74}	$-4t^2+16t-23$	$24t^3-136t^2+290t-360$
	10_{75}	$-t^3+7t^2-19t+27$	$-4t^3+36t^2-117t+172$
	10_{76}	$-2t^3+7t^2-12t+15$	$-9t^5+44t^4-104t^3+184t^2-245t+272$
	10_{77}	$2t^3-7t^2+14t-17$	$-5t^5+24t^4-71t^3+132t^2-189t+208$
	10_{78}	$-t^3+7t^2-16t+21$	$2t^5-24t^4+105t^3-244t^2+390t-448$
	10_{79}	$t^4-3t^3+7t^2-12t+15$	0
	10_{80}	$3t^3-9t^2+15t-17$	$30t^5-112t^4+260t^3-426t^2+568t-616$
	10_{81}	$-t^3+8t^2-20t+27$	0
	10_{82}	$-t^4+4t^3-8t^2+12t-13$	$t^7-6t^6+19t^5-42t^4+64t^3-78t^2+84t-84$
	10_{83}	$2t^3-9t^2+19t-23$	$-5t^5+34t^4-110t^3+214t^2-301t+332$
	10_{84}	$2t^3-9t^2+20t-25$	$-5t^5+34t^4-116t^3+246t^2-373t+424$
	10_{85}	$t^4-4t^3+8t^2-10t+11$	$2t^7-12t^6+36t^5-68t^4+101t^3-124t^2+138t-140$
	10_{86}	$-2t^3+9t^2-19t+25$	$-t^5+6t^4-21t^3+58t^2-105t+128$
	10_{87}	$-2t^3+9t^2-18t+23$	$-t^5+6t^4-23t^3+66t^2-125t+152$
	10_{88}	$-t^3+8t^2-24t+35$	0
	10_{89}	$t^3-8t^2+24t-33$	$t^5-14t^4+83t^3-264t^2+495t-596$
	10_{90}	$-2t^3+8t^2-17t+23$	$-t^5+6t^4-21t^3+54t^2-93t+112$

Picture	Name	Alexander	ρ_1-normalization of Z
	10_{91}	$t^4-4t^3+9t^2-14t+17$	$t^5-2t^4+2t^3-3t+4$
	10_{92}	$-2t^3+10t^2-20t+25$	$-9t^5+68t^4-216t^3+428t^2-622t+696$
	10_{93}	$2t^3-8t^2+15t-17$	$3t^5-18t^4+43t^3-58t^2+55t-48$
	10_{94}	$-t^4+4t^3-9t^2+14t-15$	$-t^7+6t^6-20t^5+46t^4-76t^3+102t^2-115t+120$
	10_{95}	$2t^3-9t^2+21t-27$	$-5t^5+32t^4-114t^3+248t^2-384t+436$
	10_{96}	$-t^3+7t^2-22t+33$	$-7t^3+50t^2-147t+212$
	10_{97}	$-5t^2+22t-33$	$-37t^3+242t^2-603t+788$
	10_{98}	$-2t^3+9t^2-18t+23$	$9t^5-60t^4+177t^3-348t^2+501t-564$
	10_{99}	$t^4-4t^3+10t^2-16t+19$	0
	10_{100}	$t^4-4t^3+9t^2-12t+13$	$2t^7-12t^6+39t^5-80t^4+128t^3-164t^2+192t-196$
	10_{101}	$7t^2-21t+29$	$-129t^3+480t^2-942t+1148$
	10_{102}	$-2t^3+8t^2-16t+21$	$-t^5+6t^4-19t^3+50t^2-89t+108$
	10_{103}	$2t^3-8t^2+17t-21$	$5t^5-30t^4+93t^3-178t^2+254t-280$
	10_{104}	$t^4-4t^3+9t^2-15t+19$	$t^5-2t^4+2t^3-3t+4$
	10_{105}	$t^3-8t^2+22t-29$	$-t^5+14t^4-71t^3+184t^2-292t+332$
	10_{106}	$-t^4+4t^3-9t^2+15t-17$	$-t^7+6t^6-20t^5+48t^4-82t^3+114t^2-134t+140$
	10_{107}	$-t^3+8t^2-22t+31$	$2t^3-8t^2+13t-16$
	10_{108}	$2t^3-8t^2+14t-15$	$-3t^5+18t^4-41t^3+50t^2-40t+32$
	10_{109}	$t^4-4t^3+10t^2-17t+21$	0
	10_{110}	$t^3-8t^2+20t-25$	$t^5-14t^4+69t^3-160t^2+219t-236$
	10_{111}	$-2t^3+9t^2-17t+21$	$-9t^5+60t^4-171t^3+316t^2-436t+480$
	10_{112}	$-t^4+5t^3-11t^2+17t-19$	$t^7-8t^6+29t^5-68t^4+115t^3-152t^2+175t-180$
	10_{113}	$2t^3-11t^2+26t-33$	$-5t^5+42t^4-167t^3+394t^2-623t+720$
	10_{114}	$-2t^3+10t^2-21t+27$	$t^5-8t^4+30t^3-78t^2+140t-168$
	10_{115}	$-t^3+9t^2-26t+37$	0
	10_{116}	$-t^4+5t^3-12t^2+19t-21$	$t^7-8t^6+30t^5-74t^4+132t^3-184t^2+217t-228$
	10_{117}	$2t^3-10t^2+24t-31$	$-5t^5+38t^4-144t^3+330t^2-522t+600$
	10_{118}	$t^4-5t^3+12t^2-19t+23$	0
	10_{119}	$-2t^3+10t^2-23t+31$	$-t^5+6t^4-26t^3+86t^2-175t+220$
	10_{120}	$8t^2-26t+37$	$166t^3-692t^2+1433t-1788$
	10_{121}	$2t^3-11t^2+27t-35$	$5t^5-42t^4+167t^3-396t^2+634t-732$
	10_{122}	$-2t^3+11t^2-24t+31$	$-t^5+8t^4-34t^3+104t^2-211t+264$
	10_{123}	$t^4-6t^3+15t^2-24t+29$	0
	10_{124}	t^4-t^3+t-1	$-4t^7-6t^4-4t^2-6t$
	10_{125}	t^3-2t^2+2t-1	$-t^5+2t^4-2t^3+3t-4$

Picture	Name	Alexander	ρ_1-normalization of Z
	10_{126}	t^3-2t^2+4t-5	$t^5-2t^4+10t^3-12t^2+22t-20$
	10_{127}	$-t^3+4t^2-6t+7$	$2t^5-14t^4+32t^3-52t^2+67t-72$
	10_{128}	$2t^3-3t^2+t+1$	$-13t^5+12t^4-3t^3-10t^2-9t+12$
	10_{129}	$2t^2-6t+9$	$-t^3-2t^2+14t-20$
	10_{130}	$2t^2-4t+5$	$t^3-2t^2+19t-24$
	10_{131}	$-2t^2+8t-11$	$5t^3-38t^2+87t-112$
	10_{132}	t^2-t+1	$2t^2+5t-4$
	10_{133}	$-t^2+5t-7$	$t^3-14t^2+37t-48$
	10_{134}	$2t^3-4t^2+4t-3$	$-13t^5+24t^4-33t^3+30t^2-41t+40$
	10_{135}	$3t^2-9t+13$	$t^3-6t^2+18t-24$
	10_{136}	$-t^2+4t-5$	$-t^3+4t^2-2t-4$
	10_{137}	$t^2-6t+11$	$-4t^2+24t-44$
	10_{138}	t^3-5t^2+8t-7	$-t^5+8t^4-22t^3+24t^2-11t+8$
	10_{139}	t^4-t^3+2t-3	$-4t^7-12t^4+5t^3-4t^2-16t+12$
	10_{140}	t^2-2t+3	$8t-8$
	10_{141}	$-t^3+3t^2-4t+5$	$t^3-8t^2+16t-20$
	10_{142}	$2t^3-3t^2+2t-1$	$-13t^5+12t^4-13t^3+4t^2-17t+12$
	10_{143}	t^3-3t^2+6t-7	$t^5-4t^4+15t^3-28t^2+45t-48$
	10_{144}	$-3t^2+10t-13$	$10t^3-44t^2+80t-96$
	10_{145}	t^2+t-3	$2t^3+8t^2+6t-8$
	10_{146}	$2t^2-8t+13$	$t^3-8t^2+21t-28$
	10_{147}	$-2t^2+7t-9$	$-3t^3+12t^2-15t+12$
	10_{148}	t^3-3t^2+7t-9	$t^5-4t^4+18t^3-36t^2+62t-68$
	10_{149}	$-t^3+5t^2-9t+11$	$2t^5-18t^4+55t^3-104t^2+149t-164$
	10_{150}	$-t^3+4t^2-6t+7$	$-2t^5+12t^4-26t^3+38t^2-45t+44$
	10_{151}	$t^3-4t^2+10t-13$	$-t^5+6t^4-21t^3+42t^2-66t+72$
	10_{152}	$t^4-t^3-t^2+4t-5$	$4t^7-7t^5+18t^4-7t^3-12t^2+45t-52$
	10_{153}	t^3-t^2-t+3	$t^5-2t^4+t^3+2t^2-t$
	10_{154}	t^3-4t+7	$-3t^5-6t^4+13t^3-47t+68$
	10_{155}	$-t^3+3t^2-5t+7$	$-2t^3+12t^2-22t+28$
	10_{156}	t^3-4t^2+8t-9	$t^5-6t^4+19t^3-30t^2+33t-32$
	10_{157}	$-t^3+6t^2-11t+13$	$-2t^5+22t^4-78t^3+148t^2-218t+240$
	10_{158}	$-t^3+4t^2-10t+15$	$2t^2-7t+12$
	10_{159}	$t^3-4t^2+9t-11$	$t^5-6t^4+26t^3-60t^2+98t-112$
	10_{160}	$-t^3+4t^2-4t+3$	$-2t^5+12t^4-20t^3+14t^2-16t+12$
	10_{161}	t^3-2t+3	$3t^5+6t^4-3t^3+4t^2+14t-12$
	10_{162}	$-3t^2+9t-11$	$10t^3-38t^2+58t-68$
	10_{163}	$t^3-5t^2+12t-15$	$-t^5+8t^4-30t^3+62t^2-89t+96$
	10_{164}	$3t^2-11t+17$	$t^3-10t^2+29t-40$
	10_{165}	$-2t^2+10t-15$	$-5t^3+50t^2-146t+196$