In[ ]:=
```
PD[epd_EPD] := PD @@ epd /. {X_{i_,j_} :> X[j, i + 1, j + 1, i], X̄_{i_,j_} :> X[j, i, j + 1, i + 1]}
```

In[ ]:=
```
Rot[pd_PD] := Module[{n, xs, x, rots, Xp, Xm, front = {1}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]   PositiveQ@x
                           Xm[x[[2]], x[[1]]]   True        ];
   For[k = 1, k ≤ 2 n, ++k,
    If[FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
            Xp[k, l_] | Xm[l_, k] :> {l + 1, k + 1, -l},
            Xp[l_, k] | Xm[k, l_] :> (++rots[[l]]; {-l, k + 1, l + 1}),
            _Xp | _Xm :> {}
           }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k]];
    ]
   ];
   {xs /. {Xp[i_, j_] :> {+1, i, j}, Xm[i_, j_] :> {-1, i, j}}, rots} ];
Rot[K_] := Rot[PD[K]];
```

```
Once[<< KnotTheory`];
```

In[ ]:=
```
δ_{i_,j_} := If[i === j, 1, 0];
gRules_{s_,i_,j_} := {g_{iβ_} :> δ_{iβ} + T^s g_{i+,β} + (1 - T^s) g_{j+,β}, g_{jβ_} :> δ_{jβ} + g_{j+,β},
   g_{α_,i} :> T^-s (g_{α,i+} - δ_{α,i+}), g_{α_j} :> g_{α,j+} - (1 - T^s) g_{αi} - δ_{α,j+}}
fRules_{s_,i_,j_} := {f_{iβ_} :> δ_{iβ} + F^s f_{i+,β} + (1 - F^s) f_{j+,β}, f_{jβ_} :> δ_{jβ} + f_{j+,β},
   f_{α_,i} :> F^-s (f_{α,i+} - δ_{α,i+}), f_{α_j} :> f_{α,j+} - (1 - F^s) f_{αi} - δ_{α,j+}}
(α_+)+ := α^"++";
```

The ansatz for R, R^{-1} and C

In[ ]:=
```
R_1[1, i_, j_] := Module[{gfs = Join[Flatten@Table[f_{a,b}, {a, {i, j}}, {b, {i, j}}],
      Flatten@Table[g_{a,b}, {a, {i, j}}, {b, {i, j}}]]},
   DeleteDuplicates[Times @@@ Tuples[
        gfs, 2]].Table[r_k, {k, 36}] +
    gfs.Table[r_k, {k, 36 + 1, 36 + 8}] + r_{36+9}]
R_1[-1, i_, j_] := Module[{gfs = Join[Flatten@Table[f_{a,b}, {a, {i, j}}, {b, {i, j}}],
      Flatten@Table[g_{a,b}, {a, {i, j}}, {b, {i, j}}]]},
   DeleteDuplicates[Times @@@ Tuples[
        gfs, 2]].Table[q_k, {k, 36}] +
    gfs.Table[q_k, {k, 36 + 1, 36 + 8}] + q_{36+9}]
(* modified expression for the curl: *)
CC[s_, i_] := s (c_1 f_{i,i} + c_2 g_{i,i} + c_3)
```

The Reidemeister moves as equations

In[ ]:=

```
lhs3 = R_1[1, j, k] + R_1[1, i, k^+] + R_1[1, i^+, j^+] //.
   gRules_{1,j,k} ⋃ gRules_{1,i,k^+} ⋃ gRules_{1,i^+,j^+} ⋃ fRules_{1,j,k} ⋃ fRules_{1,i,k^+} ⋃ fRules_{1,i^+,j^+};
rhs3 = R_1[1, i, j] + R_1[1, i^+, k] + R_1[1, j^+, k^+] //.
   gRules_{1,i,j} ⋃ gRules_{1,i^+,k} ⋃ gRules_{1,j^+,k^+} ⋃ fRules_{1,i,j} ⋃ fRules_{1,i^+,k} ⋃ fRules_{1,j^+,k^+};

lhs2c = R_1[-1, i, j^+] + R_1[1, i^+, j] + CC[1, j^+] //.
   gRules_{-1,i,j^+} ⋃ gRules_{1,i^+,j} ⋃ fRules_{-1,i,j^+} ⋃ fRules_{1,i^+,j};
rhs2c = CC[1, (j^+)^+];

lhs1l = R_1[1, i^+, i] + CC[1, i^+] //. {g_{i^+,β_} :> T^-1 δ_{i^+,β} + g_{i^"++",β}, g_{i,β_} :> δ_{i,β} + g_{i^+,β},
    g_{α_,i^+} :> T^-1 (g_{α,i^"++"} - δ_{α,i^+}), g_{α_,i} :> T g_{α,i^+} - δ_{α,i^+},
    f_{i^+,β_} :> F^-1 δ_{i^+,β} + f_{i^"++",β}, f_{i,β_} :> δ_{i,β} + f_{i^+,β},
    f_{α_,i^+} :> F^-1 (f_{α,i^"++"} - δ_{α,i^+}), f_{α_,i} :> T f_{α,i^+} - δ_{α,i^+}};
lhs1r = R_1[1, i, i^+] + CC[-1, i^+] //. {
    g_{iβ_} :> δ_{iβ} + T g_{i^+,β} + (1 - T) g_{i^++,β}, g_{i^+,β_} :> δ_{i^+,β} + g_{i^++,β},
    g_{α_,i} :> T^-1 (g_{α,i^+} - δ_{α,i^+}), g_{α_,i^+} :> T g_{α,i^++} - (1 - T) δ_{α,i^+} - T δ_{α,i^++},
    f_{iβ_} :> δ_{iβ} + F f_{i^+,β} + (1 - F) f_{i^++,β}, f_{i^+,β_} :> δ_{i^+,β} + f_{i^++,β},
    f_{α_,i} :> F^-1 (f_{α,i^+} - δ_{α,i^+}), f_{α_,i^+} :> T f_{α,i^++} - (1 - F) δ_{α,i^+} - F δ_{α,i^++}};
rhs1l = 0;
rhs1r = 0;
lhssw =
  R_1[1, i, j] + CC[-1, i] + CC[-1, j] + CC[1, i^+] + CC[1, j^+] //. gRules_{1,i,j} ⋃ fRules_{1,i,j};
rhssw = R_1[1, i, j] //. gRules_{1,i,j} ⋃ fRules_{1,i,j};
```

Solve for the coefficients in the Reidemeister equations

In[ ]:=

```
V = Join[Flatten@Table[g_{a,b}, {a, {i^"++", j^"++", k^"++"}}, {b, {i^"++", j^"++", k^"++"}}],
   Flatten@Table[f_{a,b}, {a, {i^"++", j^"++", k^"++"}}, {b, {i^"++", j^"++", k^"++"}}]];
eq3 = Thread[Last /@ CoefficientRules[lhs3 - rhs3, V] == 0];
eq2c = Thread[Last /@ CoefficientRules[lhs2c - rhs2c, V] == 0];
eq1l = Thread[Last /@ CoefficientRules[lhs1l - rhs1l, V] == 0];
eq1r = Thread[Last /@ CoefficientRules[lhs1r - rhs1r, V] == 0];
eqsw = Thread[Last /@ CoefficientRules[lhssw - rhssw, V] == 0];
```

In[∘]:=

```
Soln = First@Solve[Join[eq3, eq2c, eq1l, eq1r, eqsw],
    Join[Table[q_i, {i, 1, 45}], Table[r_i, {i, 1, 45}], Table[c_i, {i, 1, 2}]]] // Simplify
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=

$$\Big\{q_1 \to 0,\ q_2 \to 0,\ q_3 \to \frac{(-1+F)\ r_{17}}{2\ F},\ q_4 \to \frac{r_{17}}{2},\ q_5 \to 0,\ q_6 \to 0,\ q_7 \to \frac{(-1+T)\ r_{21}}{(-1+F)\ T^2},\ q_8 \to -\frac{(-1+T)\ r_{21}}{(-1+F)\ T},$$
$$q_9 \to 0,\ q_{10} \to \frac{r_{17}}{2},\ q_{11} \to 0,\ q_{12} \to 0,\ q_{13} \to 0,\ q_{14} \to 0,\ q_{15} \to 0,\ q_{16} \to -\frac{(-1+F)\ r_{17}}{2\ F},$$
$$q_{17} \to -r_{17},\ q_{18} \to \frac{r_{21}}{F},\ q_{19} \to 0,\ q_{20} \to -\frac{\left(-1+T^2\right)\ r_{21}}{F\ T^2},\ q_{21} \to -\frac{r_{21}}{F\ T},\ q_{22} \to 0,\ q_{23} \to \frac{(-1+T)\ r_{21}}{(-1+F)\ T},$$
$$q_{24} \to 0,\ q_{25} \to -\frac{(-1+T)\ r_{21}}{(-1+F)\ T},\ q_{26} \to 0,\ q_{27} \to 0,\ q_{28} \to 0,\ q_{29} \to \frac{(-3+T)\ r_{29} + 2\ (-1+T)\ r_{35}}{T\ (1+T)},$$
$$q_{30} \to \frac{r_{29}+r_{35}}{1+T},\ q_{31} \to 0,\ q_{32} \to \frac{r_{29}+r_{35}}{1+T},\ q_{33} \to 0,\ q_{34} \to -\frac{(-1+T)\ ((-2+T)\ r_{29} + (-1+2\ T)\ r_{35})}{T^2\ (1+T)},$$
$$q_{35} \to \frac{-2\ (-1+T)\ r_{29} + (1-3\ T)\ r_{35}}{T\ (1+T)},\ q_{36} \to 0,\ q_{37} \to -r_{37},\ q_{38} \to 0,\ q_{39} \to r_{37},\ q_{40} \to 0,$$
$$q_{41} \to -r_{41},\ q_{42} \to 0,\ q_{43} \to \frac{2\ c_3 - r_{37} + (-1+T)\ r_{41}}{T},\ q_{44} \to -2\ c_3 + r_{37} + r_{41},\ q_{45} \to c_3,\ r_1 \to 0,$$
$$r_2 \to 0,\ r_3 \to \frac{1}{2}\ (-1+F)\ r_{17},\ r_4 \to -\frac{r_{17}}{2},\ r_5 \to 0,\ r_6 \to 0,\ r_7 \to -\frac{(-1+T)\ r_{21}}{-1+F},\ r_8 \to \frac{(-1+T)\ r_{21}}{(-1+F)\ T},$$
$$r_9 \to 0,\ r_{10} \to -\frac{r_{17}}{2},\ r_{11} \to 0,\ r_{12} \to 0,\ r_{13} \to 0,\ r_{14} \to 0,\ r_{15} \to 0,\ r_{16} \to -\frac{1}{2}\ (-1+F)\ r_{17},$$
$$r_{18} \to -\frac{r_{21}}{T},\ r_{19} \to 0,\ r_{20} \to -\frac{\left(-1+T^2\right)\ r_{21}}{T},\ r_{22} \to 0,\ r_{23} \to -\frac{(-1+T)\ r_{21}}{(-1+F)\ T},\ r_{24} \to 0,$$
$$r_{25} \to \frac{(-1+T)\ r_{21}}{(-1+F)\ T},\ r_{26} \to 0,\ r_{27} \to 0,\ r_{28} \to 0,\ r_{30} \to -\frac{r_{29}+r_{35}}{1+T},\ r_{31} \to 0,\ r_{32} \to -\frac{r_{29}+r_{35}}{1+T},$$
$$r_{33} \to 0,\ r_{34} \to \frac{(-1+T)\ (r_{29} - T\ r_{35})}{1+T},\ r_{36} \to 0,\ r_{38} \to 0,\ r_{39} \to -r_{37},\ r_{40} \to 0,\ r_{42} \to 0,$$
$$r_{43} \to -2\ T\ c_3 + T\ r_{37} + (-1+T)\ r_{41},\ r_{44} \to 2\ c_3 - r_{37} - r_{41},\ r_{45} \to -c_3,\ c_1 \to -\frac{r_{17}}{2},\ c_2 \to -\frac{r_{29}+r_{35}}{1+T}\Big\}$$

In[∘]:=

```
CCC[s_, i_] := s (CC[1, i] /. Soln)
```

In[ ]:=

```
CC[1, i] /. Soln
R_1[1, i, j] /. Soln
R_1[-1, i, j] /. Soln
```

Out[ ]=

$$c_3 - \frac{1}{2} r_{17} f_{i,i} - \frac{(r_{29} + r_{35}) g_{i,i}}{1 + T}$$

Out[ ]=

$$-c_3 + r_{37} f_{i,i} - r_{37} f_{j,i} + \frac{1}{2} (-1 + F) r_{17} f_{i,i} f_{j,i} - \frac{1}{2} r_{17} f_{i,j} f_{j,i} - \frac{1}{2} (-1 + F) r_{17} f_{j,i}^2 - \frac{1}{2} r_{17} f_{i,i} f_{j,j} + r_{17} f_{j,i} f_{j,j} + r_{41} g_{i,i} - \frac{r_{21} f_{j,i} g_{i,i}}{T} - \frac{(-1 + T) r_{21} f_{j,j} g_{i,i}}{(-1 + F) T} + (-2 T c_3 + T r_{37} + (-1 + T) r_{41}) g_{j,i} - \frac{(-1 + T) r_{21} f_{i,i} g_{j,i}}{-1 + F} - \frac{\left(-1 + T^2\right) r_{21} f_{j,i} g_{j,i}}{T} + \frac{(-1 + T) r_{21} f_{j,j} g_{j,i}}{(-1 + F) T} + r_{29} g_{i,i} g_{j,i} - \frac{(r_{29} + r_{35}) g_{i,j} g_{j,i}}{1 + T} + \frac{(-1 + T) (r_{29} - T r_{35}) g_{j,i}^2}{1 + T} + (2 c_3 - r_{37} - r_{41}) g_{j,j} + \frac{(-1 + T) r_{21} f_{i,i} g_{j,j}}{(-1 + F) T} + r_{21} f_{j,i} g_{j,j} - \frac{(r_{29} + r_{35}) g_{i,i} g_{j,j}}{1 + T} + r_{35} g_{j,i} g_{j,j}$$

Out[ ]=

$$c_3 - r_{37} f_{i,i} + r_{37} f_{j,i} + \frac{(-1 + F) r_{17} f_{i,i} f_{j,i}}{2 F} + \frac{1}{2} r_{17} f_{i,j} f_{j,i} - \frac{(-1 + F) r_{17} f_{j,i}^2}{2 F} + \frac{1}{2} r_{17} f_{i,i} f_{j,j} - r_{17} f_{j,i} f_{j,j} - r_{41} g_{i,i} + \frac{r_{21} f_{j,i} g_{i,i}}{F} + \frac{(-1 + T) r_{21} f_{j,j} g_{i,i}}{(-1 + F) T} + \frac{(2 c_3 - r_{37} + (-1 + T) r_{41}) g_{j,i}}{T} + \frac{(-1 + T) r_{21} f_{i,i} g_{j,i}}{(-1 + F) T^2} - \frac{\left(-1 + T^2\right) r_{21} f_{j,i} g_{j,i}}{F T^2} - \frac{(-1 + T) r_{21} f_{j,j} g_{j,i}}{(-1 + F) T} + \frac{((-3 + T) r_{29} + 2 (-1 + T) r_{35}) g_{i,i} g_{j,i}}{T (1 + T)} + \frac{(r_{29} + r_{35}) g_{i,j} g_{j,i}}{1 + T} - \frac{(-1 + T) ((-2 + T) r_{29} + (-1 + 2 T) r_{35}) g_{j,i}^2}{T^2 (1 + T)} + (-2 c_3 + r_{37} + r_{41}) g_{j,j} - \frac{(-1 + T) r_{21} f_{i,i} g_{j,j}}{(-1 + F) T} - \frac{r_{21} f_{j,i} g_{j,j}}{F T} + \frac{(r_{29} + r_{35}) g_{i,i} g_{j,j}}{1 + T} + \frac{(-2 (-1 + T) r_{29} + (1 - 3 T) r_{35}) g_{j,i} g_{j,j}}{T (1 + T)}$$

In[ ]:=

```
CCC[s_, i_] := s (CC[1, i] /. Soln)
RR₁[1, i_, j_] := R₁[1, i, j] /. Soln
RR₁[-1, i_, j_] := R₁[-1, i, j] /. Soln
ρρ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, FF, ρ1},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :→ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1 ; 0  -1 ))];
  Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
  G = Inverse[A]; FF = G /. T → F;
  ρ1 = Sum[RR₁ @@ Cs⟦k⟧, {k, 1, n}] + Sum[φ⟦k⟧ CCC[1, k], {k, 1, 2 n}];
  Factor@{Δ, (Δ^2 /. T → F) Δ^2 ρ1 /. α_^+ :→ α + 1 /. {g_{α_, β_} :→ G⟦α, β⟧, f_{α_, β_} :→ FF⟦α, β⟧}}];
Rold₁[s_, i_, j_] := s (g_{ji} (g_{j+,j} + g_{j,j+} - g_{ij}) - g_{ii} (g_{j,j+} - 1) - 1 / 2);
Cold[s_, i_] := s (-g_{ii} + 1/2)
ρ[K_] := Module[{Cs, φ, n, A, s, i, j, k, Δ, G, FF, ρ1},
  {Cs, φ} = Rot[K]; n = Length[Cs];
  A = IdentityMatrix[2 n + 1];
  Cases[Cs, {s_, i_, j_} :→ (A⟦{i, j}, {i + 1, j + 1}⟧ += ( -T^s  T^s - 1 ; 0  -1 ))];
  Δ = T^((-Total[φ] - Total[Cs⟦All,1⟧])/2) Det[A];
  G = Inverse[A];
  ρ1 = Sum[((Rold₁ @@ Cs⟦k⟧)), {k, 1, n}] + Sum[φ⟦k⟧ Cold[1, k], {k, 1, 2 n}];
  Factor@{Δ, Δ^2 ρ1 /. α_^+ :→ α + 1 /. {g_{α_, β_} :→ G⟦α, β⟧}}];
```

In[ ]:=

```
ρ[Knot[3, 1]]
Expand[F^2 ρρ[Knot[3, 1]]] /. F → 0
```

Out[ ]=

$$\left\{\frac{1-T+T^2}{T}, \frac{(-1+T)^2\,(1+T^2)}{T^2}\right\}$$

Out[ ]=

$$\left\{0, -\frac{2\,c_3}{1+T}+\frac{2\,c_3}{T^2\,(1+T)}+\frac{2\,T\,c_3}{1+T}-\frac{2\,T^3\,c_3}{1+T}+\frac{2\,r_{29}}{1+T}-\frac{r_{29}}{T\,(1+T)}-\frac{3\,T\,r_{29}}{1+T}+\frac{2\,T^2\,r_{29}}{1+T}+\frac{2\,r_{35}}{1+T}-\frac{r_{35}}{T\,(1+T)}-\frac{3\,T\,r_{35}}{1+T}+\frac{2\,T^2\,r_{35}}{1+T}+\frac{2\,r_{37}}{1+T}-\frac{r_{37}}{T\,(1+T)}-\frac{T^2\,r_{37}}{1+T}+\frac{2\,T^3\,r_{37}}{1+T}\right\}$$

```
(*ρρ does not distinguish the famous mutants*)
ρρ[Knot[11, Knot[11, NonAlternating, 34]]]
ρρ[Knot[11, Knot[11, NonAlternating, 42]]]
```

Out[ ]=

```
{1, 0}
```

Out[ ]=

```
{1, 0}
```

```
(*Yet ρρ does outperform our usual ρ=
 ρ1 a little on all 11 crossing non-alternating knots:*)
```

In[ ]:=

```
ρ11na = ρ /@ Table[PD@Knot[11, NonAlternating, k], {k, 1, 185}];
```

In[ ]:=

```
ρρ11na = ρρ /@ Table[PD@Knot[11, NonAlternating, k], {k, 1, 185}];
```

In[ ]:=

```
Tally[ρ11na] // Length
Tally[ρρ11na] // Length
```

Out[ ]=

```
177
```

Out[ ]=

```
178
```