Pensieve header: The “Speedy” engine.

In[∘]:=
```
Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
PP_ = Identity; $k = 0; γ = 1; ℏ = 1;
```

In[∘]:=
$$\text{Doubler}_{i\_} := \overline{\text{tKink}}_9\, \overline{\text{tKink}}_{10}\, \text{tR}_{1,7}\, \text{tR}_{5,2}\, \text{tC}_4\, \overline{\text{tC}}_6\, \text{t}\Delta_{i\to 3,8} \,//\, \text{tS}_3 \,//\, \text{tm}_{1,2\to 1} \,//\, \text{tm}_{1,3\to 1} \,//\, \text{tm}_{1,4\to 1} \,//\, \text{tm}_{1,5\to 1} \,//\, \text{tm}_{1,6\to 1} \,//\, \text{tm}_{1,7\to 1} \,//\, \text{tm}_{1,8\to 1} \,//\, \text{tm}_{1,9\to 1} \,//\, \text{tm}_{1,10\to i}$$
$$\text{Doubler}_0$$

Out[∘]=
$$\mathbb{E}_{\{0\}\to\{0\}}\left[0,\ \frac{x_0\, y_0\left(1-2\mathcal{A}_0+\mathcal{A}_0^2\right)}{\mathcal{A}_0-\mathcal{A}_0^2+T_0\,\mathcal{A}_0^2}+\frac{y_0\,(1-T_0-\mathcal{A}_0+T_0\,\mathcal{A}_0)\,\eta_0}{1-\mathcal{A}_0+T_0\,\mathcal{A}_0}+\frac{x_0\,(1-T_0-\mathcal{A}_0+T_0\,\mathcal{A}_0)\,\xi_0}{1-\mathcal{A}_0+T_0\,\mathcal{A}_0}+\frac{\left(\mathcal{A}_0-2\,T_0\,\mathcal{A}_0+T_0^2\,\mathcal{A}_0\right)\eta_0\,\xi_0}{1-\mathcal{A}_0+T_0\,\mathcal{A}_0},\ \frac{T_0}{1-\mathcal{A}_0+T_0\,\mathcal{A}_0}+O[\epsilon]^1\right]$$

In[∘]:=
```
$k = 1;
(Z@Knot[3, 1] /. {a → a₀, x → x₀, y → y₀, T → T₀}) // Doubler₀
(Z@Knot[4, 1] /. {a → a₀, x → x₀, y → y₀, T → T₀}) // Doubler₀
(Z@Knot[5, 1] /. {a → a₀, x → x₀, y → y₀, T → T₀}) // Doubler₀
(Z@Knot[5, 2] /. {a → a₀, x → x₀, y → y₀, T → T₀}) // Doubler₀
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ 1+\frac{\left(-4+8\,T_0-4\,T_0^2\right)\epsilon}{T_0}+O[\epsilon]^2\right]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ 1+\frac{\left(4-8\,T_0+4\,T_0^2\right)\epsilon}{T_0}+O[\epsilon]^2\right]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ 1+\frac{\left(-12+24\,T_0-12\,T_0^2\right)\epsilon}{T_0}+O[\epsilon]^2\right]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ 1+\frac{\left(-8+16\,T_0-8\,T_0^2\right)\epsilon}{T_0}+O[\epsilon]^2\right]$$

In[∘]:=
```
Vassiliev[2] /@ AllKnots[{3, 5}]
```

Out[∘]= {1, -1, 3, 2}

In[∘]:=
$$\text{Union}\left[\left(\text{Vassiliev}[2]\,@\#-\text{Together}\left[\left(-\frac{4\,(-1+T_0)^2}{T_0}\right)^{-1}\text{Coefficient}\left[((Z@\#\ /.\ \{a\to a_0,\ x\to x_0,\ y\to y_0,\ T\to T_0\})\ //\ \text{Doubler}_0)[\![3]\!],\ \epsilon\right]\right]\right)\ \&\ /@\ \text{AllKnots}[\{3,7\}]\right]$$

Out[∘]= {0}

The “Speedy” Engine

Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
          Expand[ℰ] //. 𝕖^x_ 𝕖^y_ ⧴ 𝕖^(x+y) /. 𝕖^x_ ⧴ 𝕖^CCF[x]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := Module[
          {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
          Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)]
         ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $\mathbb{e}^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
In[∘]:= U2l = {B_i_^p_. ⧴ 𝕖^(-p ℏ γ b_i), B^p_. ⧴ 𝕖^(-p ℏ γ b), T_i_^p_. ⧴ 𝕖^(-p ℏ t_i), T^p_. ⧴ 𝕖^(-p ℏ t), 𝒜_i_^p_. ⧴ 𝕖^(p γ α_i), 𝒜^p_. ⧴ 𝕖^(p γ α)};
        l2U = {𝕖^(c_. b_i_ + d_.) ⧴ B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) ⧴ B^(-c/(ℏ γ)) 𝕖^d,
          𝕖^(c_. t_i_ + d_.) ⧴ T_i^(-c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) ⧴ T^(-c/ℏ) 𝕖^d,
          𝕖^(c_. α_i_ + d_.) ⧴ 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) ⧴ 𝒜^(c/γ) 𝕖^d,
          𝕖^ℰ_ ⧴ 𝕖^Expand@ℰ};
```

Derivatives in the presence of exponentiated variables:

In[∘]:=
```
D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_{b_i_}[f_] := ∂_{b_i} f - ℏ γ B_i ∂_{B_i} f;
D_t[f_] := ∂_t f - ℏ T ∂_T f; D_{t_i_}[f_] := ∂_{t_i} f - ℏ T_i ∂_{T_i} f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_{α_i_}[f_] := ∂_{α_i} f + γ 𝒜_i ∂_{𝒜_i} f;
D_{v_}[f_] := ∂_v f; D_{{v_,0}}[f_] := f; D_{{}}[f_] := f; D_{{v_,n_Integer}}[f_] := D_v[D_{{v,n-1}}[f]];
D_{{l_List,ls___}}[f_] := D_{{ls}}[D_l[f]];
```

Finite Zips:

In[∘]:=
```
collect[sd_SeriesData, ξ_] := MapAt[collect[#, ξ] &, sd, 3];
collect[ℰ_, ξ_] := Collect[ℰ, ξ];
Zip_{{}}[P_] := P;
Zip_{ξs_}[Ps_List] := Zip_{ξs} /@ Ps;
Zip_{{ξ_,ξs___}}[P_] :=
  (collect[P // Zip_{{ξs}}, ξ] /. f_. ξ^d_. :> (D_{{ξ*,d}}[f])) /. ξ* → 0 /.
   ((ξ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)\mathbb{e}^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|\mathbb{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[∘]:=
```
QZip_{ξs_List}@𝔼[L_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule, out},
    zs = Table[ξ*, {ξ, ξs}];
    c = CF[Q /. Alternatives @@ (ξs ⋃ zs) → 0];
    ys = CF@Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ξrule = Thread[ξs → ξs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ξs}[P /. (zrule ⋃ ξrule)]] ];
MQZip_{ξs_List}@𝔼[L_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule, out},
    zs = Table[ξ*, {ξ, ξs}] // Echo;
    c = CF[Q /. Alternatives @@ (ξs ⋃ zs) → 0];
    ys = CF@Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}] // Echo;
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}] // Echo;
    Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}] // MatrixForm // Echo;
    qt = CF@Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}];
    qt // MatrixForm // Echo;
    zrule = Thread[zs → CF[qt.(zs + ys)]] // Echo;
    ξrule = Thread[ξs → ξs + ηs.qt] // Echo;
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ξs}[P /. (zrule ⋃ ξrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single”P”. Here the z's are b and α and the ζ's are β and a.

```
In[]:= LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule,
     zrule /. r_Rule :> ((U = r⟦1⟧ /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
    Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
       Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ] ];
```

```
In[]:= B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Join@@Table[{β_{n@i}, τ_{n@i}, a_{n@i}}, {i, {is}}]} // QZip_{Join@@Table[{ξ_{n@i}, y_{n@i}}, {i, {is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

```
In[]:= B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{dr_}[L_, Q_, P_]_{$k_} := 𝔼_{dr} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

𝔼[Λ]

In[∘]:=
```
𝔼dr_[Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, ⅇ^(Λ-Λ0)]$k /. l2U]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_nisp,$k_Integer, Block[{i, j, k}, op_isp,$k = ℰ; op_nis,$k]];
      SD[op_isp, op_{is},$k]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Objects

Symmetric Algebra Objects

In[∘]:=
```
sm_i_,j_→k_ := 𝔼_{i,j}→{k}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_i_→j_,k_ := 𝔼_{i}→{j,k}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_i_ := 𝔼_{i}→{i}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_i_ := 𝔼_{}→{i}[0];
sη_i_ := 𝔼_{i}→{}[0];
```

In[∘]:=
```
sσ_i_→j_ := 𝔼_{i}→{j}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_i_→j_,k_,l_,m_ := 𝔼_{i}→{j,k,l,m}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Booting Up QU

In[∘]:=
```
Define[aσ_i→j = 𝔼_{i}→{j}[a_j α_i + x_j ξ_i], bσ_i→j = 𝔼_{i}→{j}[b_j β_i + y_j η_i]]
```

In[∘]:=

```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$. As a map $\mathrm{P}:\mathbb{A}\otimes\mathbb{B}\to Q$.

$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=

```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} ((1 - e^{γϵℏ})^k (ℏ y_i x_j)^k)/(k (1 - e^{kγϵℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
       (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_j α_i / ℏ, η_j ξ_i / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
       (R_{1,2} // ((P_{{i,1},0})_{$k} (P_{{2,j},$k-1})_{$k}))[3]]]]
```

In[∘]:= `R_{1,2} // P_{2,3}`

Out[∘]= $\mathbb{E}_{\{3\}\to\{1\}}\left[b_1\,\beta_3,\ y_1\,\eta_3,\ 1+O[\epsilon]^3\right]$

In[∘]:= `(R_{1,2} // ((P_{{i,1},0})_2 (P_{{2,j},1})_2))[3]`

Out[∘]= $1+\left(-\frac{1}{8}\eta_j^2\,\xi_i^2-\frac{\eta_j^3\,\xi_i^3}{4\,\hbar}-\frac{\eta_j^4\,\xi_i^4}{16\,\hbar^2}\right)\epsilon^2+O[\epsilon]^3$

In[∘]:=

```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{2,1},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
       ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

In[∘]:=

```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{2,1},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{2,1},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{i,3},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{3,i}]
```

In[∘]:=

```
Define[
  dm_{i,j→k} = ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3) (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{1,-3} P_{3,-1} am_{2,-4→k} bm_{4,-2→k})]
```

NB. We use the co-algebra structure B tensor A^{cop}. This has the benefit of making our algebra quasi-triangular in the traditional sense of the word.

Watch out: $\Delta_{i\to j,k}$ means j is to the RIGHT of strand k and dS looks like an S.

In[∘]:=
```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
  dϵ_i = sϵ_i, dη_i = sη_i,
  dS_i = sΥ_{i→1,1,2,2} // (bS_1 aS̄_2) // dm_{2,1→i},
  dS̄_i = sΥ_{i→1,1,2,2} // (bS̄_1 aS_2) // dm_{2,1→i},
  dΔ_{i→j,k} = (bΔ_{i→1,3} aΔ_{i→4,2}) // (dm_{3,4→k} dm_{1,2→j})]
```

In[∘]:=
```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ℏ ϵ a_i/2}]_{$k},
  C̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ℏ ϵ a_i/2}]_{$k},
  c_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/4} e^{-ℏ ϵ a_i/4}]_{$k},
  c̄_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/4} e^{ℏ ϵ a_i/4}]_{$k},
  Kink_i = (R_{1,3} C̄_2) // dm_{1,2→1} // dm_{1,3→i},
  Kink̄_i = (R̄_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i},
  ρ_i = (c_1 c̄_3 dS_i) // dm_{1,i→i} // dm_{i,3→i}] (*ρ reverses a strand*)
```

Note. $t = -\epsilon \text{a} + \gamma \text{b}$ and $b = t/\gamma + \epsilon \text{a}/\gamma$

In[∘]:=
```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i + t_i) / γ + ξ_i x_i + η_i y_i],
  t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (-ϵ a_i + γ b_i) + ξ_i x_i + η_i y_i]]
```

In[∘]:=
```
𝔼_{{}→{1}}[0, 0, x_1] // dΔ_{1→1,2}
𝔼_{{}→{1}}[0, 0, x_1] // dS_1
𝔼_{{}→{1}}[0, 0, y_1] // dS_1
𝔼_{{}→{1}}[0, 0, x_1] // dS̄_1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (x_1 + x_2) - \hbar\, a_2\, x_1\, \epsilon + \frac{1}{2} \hbar^2 a_2^2\, x_1\, \epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 + (\hbar\, x_1 - \hbar\, a_1\, x_1)\, \epsilon + \left(-\frac{1}{2} \hbar^2 x_1 + \hbar^2 a_1\, x_1 - \frac{1}{2} \hbar^2 a_1^2\, x_1\right) \epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -\frac{y_1}{B_1} + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 - \hbar\, a_1\, x_1\, \epsilon - \frac{1}{2} \left(\hbar^2 a_1^2\, x_1\right) \epsilon^2 + O[\epsilon]^3\right]$

In[∘]:=
```
𝔼_{{}→{1}}[0, 0, (1 + ϵ a_1 ℏ) x_1] // dS_1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 + \left(\frac{\hbar^2 x_1}{2} - \hbar^2 a_1\, x_1 + \frac{1}{2} \hbar^2 a_1^2\, x_1\right) \epsilon^2 + O[\epsilon]^3\right]$

In[∘]:=
```
((-1 + ℏ) x_1 + (1 - ℏ) a_1 x_1) // Expand
```

Out[∘]= $-x_1 + \hbar\, x_1 + a_1\, x_1 - \hbar\, a_1\, x_1$

In[∘]:= $\mathbf{t2b_1\, t2b_2 \,//\, P_{2,1}}$

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2\,\tau_1}{\hbar},\ \frac{\eta_1\,\xi_2}{\hbar},\ 1+\left(\frac{\eta_1^2\,\xi_2^2}{4\,\hbar}-\frac{\tau_1\,\tau_2}{\hbar}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0,0,y_1] \,//\, b\Delta_{1\to1,2}$
$\mathbb{E}_{\{\}\to\{1\}}[0,0,y_1] \,//\, d\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (B_2\,y_1+y_2)+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (B_2\,y_1+y_2)+O[\epsilon]^2\right]$

In[∘]:= $(R_{1,2} \,//\, bS_1) \equiv \overline{R}_{1,2}$
$(R_{1,2} \,//\, aS_2) \equiv \overline{R}_{1,2}$

Out[∘]= True

Out[∘]= True

$\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1] \,//\, d\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[0,\ 0,\ (x_1+x_2)-\hbar\,a_2\,x_1\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1] \,//\, a\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (x_1+x_2)-\hbar\,a_1\,x_2\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1] \,//\, (\overline{aS})_1$
$\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1] \,//\, aS_1$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1+(\hbar\,x_1-\hbar\,a_1\,x_1)\,\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1-\hbar\,a_1\,x_1\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1,2\}}[0,0,b_1\,y_2] \,//\, bm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ b_1\,y_1-y_1\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $a\Delta_{i\text{->}1,2} \,//\, aS_1 \,//\, am_{1,2\to1}$
$a\Delta_{i\text{->}1,2} \,//\, aS_2 \,//\, am_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$

In[∘]:= $a\Delta_{1\to1,2}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[a_1\,\alpha_1+a_2\,\alpha_1,\ x_1\,\xi_1+x_2\,\xi_1,\ 1+\left(-\hbar\,a_1\,x_2\,\xi_1+\frac{1}{2}\hbar\,x_1\,x_2\,\xi_1^2\right)\epsilon+O[\epsilon]^2\right]$

Testing

co-associativity

```
In[∘]:= (dΔ1→1,2 // dΔ2→2,3) ≡ (dΔ1→2,3 // dΔ2→1,2)
Out[∘]= True
```

algebra morphism

```
In[∘]:= (dΔi→1,2 dΔj→3,4 // dm1,3→i // dm2,4→j) ≡ (dmi,j→k // dΔk→i,j)
Out[∘]= True
```

associativity

```
In[∘]:= (dm1,2→k // dmk,3→k) ≡ (dm2,3→k // dm1,k→k)
Out[∘]= True
```

antipode

```
In[∘]:= dΔi->1,2 // dS1 // dm1,2→1
        dΔi->1,2 // dS2 // dm1,2→1
Out[∘]= 𝔼{i}→{1}[0, 0, 1 + O[∈]^2]
Out[∘]= 𝔼{i}→{1}[0, 0, 1 + O[∈]^2]
```

quasi-triangular axioms

```
In[∘]:= (R1,3 // dΔ1→1,2) ≡ (R1,3 R2,4 // dm3,4→3)
        (R1,3 // dΔ3→2,3) ≡ (R1,3 R0,2 // dm1,0→1)
        (dΔi→k,j R1,2 // dmj,1→1 // dmk,2→2) ≡ (R1,2 dΔi→j,k // dm1,j→1 // dm2,k→2)
Out[∘]= True
Out[∘]= True
Out[∘]= True

In[∘]:= (R1,2 // aS2) ≡ (R̄1,2)
Out[∘]= True

In[∘]:= (R1,2 // dS1) ≡ (R̄1,2)
        (R1,2 // dS̄2) ≡ (R̄1,2)
Out[∘]= True
Out[∘]= True

In[∘]:= QQs_,r_ := R11,22 R33,44 // dm11,44→s // dm22,33→r
        Q̄Q̄s_,r_ := R̄22,11 R̄44,33 // dm11,44→s // dm22,33→r

In[∘]:= QQ1,2 Q̄Q̄3,4 // dm1,3→1 // dm2,4→2
Out[∘]= 𝔼{}→{1,2}[0, 0, 1 + O[∈]^2]
```

Drinfeld element u

In[∘]:= $u_{i\_}$:= $R_{11,22}$ // dS_{22} // $dm_{22,11\to i}$
$\overline{u}_{i\_}$:= $\overline{R}_{11,22}$ // $\overline{dS}_{22}$ // $dm_{22,11\to i}$
$\overline{uu}_{i\_}$:= $\overline{R}_{11,22}$ // $\overline{dS}_{22}$ // $dm_{11,22\to i}$
$\overline{u2}_{i\_}$:= $\overline{R}_{11,22}$ // dS_{11} // $dm_{11,22\to i}$
$\overline{u3}_{i\_}$:= $R_{11,22}$ // dS_{11} // dS_{11} // $dm_{22,11\to i}$

In[∘]:= $u_i\,\overline{u}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{uu}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{u2}_j$ // $dm_{i,j\to i}$
$u_i\,\overline{u3}_j$ // $dm_{i,j\to i}$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

In[∘]:= (u_1 // dS_1)
$R_{11,22}$ // dS_{22} // $dm_{11,22\to i}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\,a_1\,b_1,\,-\frac{\hbar\,x_1\,y_1}{B_1},\,1+\left(\frac{\hbar^2\,x_1\,y_1}{B_1}-\frac{\hbar^2\,a_1\,x_1\,y_1}{B_1}-\frac{3\,\hbar^3\,x_1^2\,y_1^2}{4\,B_1^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_1\,y_1}{2\,B_1}+\frac{\hbar^3\,a_1\,x_1\,y_1}{B_1}-\frac{\hbar^3\,a_1^2\,x_1\,y_1}{2\,B_1}+\frac{5\,\hbar^4\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{5\,\hbar^4\,a_1\,x_1^2\,y_1^2}{2\,B_1^2}+\frac{\hbar^4\,a_1^2\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{67\,\hbar^5\,x_1^3\,y_1^3}{36\,B_1^3}+\frac{3\,\hbar^5\,a_1\,x_1^3\,y_1^3}{4\,B_1^3}+\frac{9\,\hbar^6\,x_1^4\,y_1^4}{32\,B_1^4}\right)\epsilon^2+O[\epsilon]^3\right]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,b_i,\,-\frac{\hbar\,x_i\,y_i}{B_i},\,1+\left(\frac{\hbar^2\,x_i\,y_i}{B_i}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_i\,y_i}{2\,B_i}+\frac{\hbar^3\,a_i\,x_i\,y_i}{B_i}-\frac{\hbar^3\,a_i^2\,x_i\,y_i}{2\,B_i}+\frac{5\,\hbar^4\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{5\,\hbar^4\,a_i\,x_i^2\,y_i^2}{2\,B_i^2}+\frac{\hbar^4\,a_i^2\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{67\,\hbar^5\,x_i^3\,y_i^3}{36\,B_i^3}+\frac{3\,\hbar^5\,a_i\,x_i^3\,y_i^3}{4\,B_i^3}+\frac{9\,\hbar^6\,x_i^4\,y_i^4}{32\,B_i^4}\right)\epsilon^2+O[\epsilon]^3\right]$$

In[∘]:=
$$\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,B_1^{-1}\left(1+\epsilon\,a_1\,\hbar+\frac{\epsilon^2}{2}\,a_1^2\,\hbar^2\right)\right]u_2\;//\;dm_{1,2\to 1}\right)\equiv(u_1\;//\;dS_1)$$

Out[∘]= True

In[∘]:= u_1

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ B_1 + \left(-\hbar\, a_1\, B_1 - \hbar^2\, x_1\, y_1 - \hbar^2\, a_1\, x_1\, y_1 - \frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1}\right)\epsilon + \left(\frac{1}{2}\hbar^2\, a_1^2\, B_1 - \frac{1}{2}\hbar^3\, x_1\, y_1 + \frac{1}{2}\hbar^3\, a_1^2\, x_1\, y_1 - \frac{\hbar^4\, x_1^2\, y_1^2}{2\, B_1} + \frac{\hbar^4\, a_1\, x_1^2\, y_1^2}{4\, B_1} + \frac{\hbar^4\, a_1^2\, x_1^2\, y_1^2}{2\, B_1} - \frac{13\,\hbar^5\, x_1^3\, y_1^3}{36\, B_1^2} + \frac{3\,\hbar^5\, a_1\, x_1^3\, y_1^3}{4\, B_1^2} + \frac{9\,\hbar^6\, x_1^4\, y_1^4}{32\, B_1^3}\right)\epsilon^2 + O[\epsilon]^3\Big]$

q

In[∘]:= $(u_1\ //\ dS_1)\ \overline{u3}_2\ //\ dm_{1,2\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{1}{B_1} + \frac{\hbar\, a_1\,\epsilon}{B_1} + O[\epsilon]^2\Big]$

In[∘]:= $(u_1\ //\ d\Delta_{1\to 2,1}) \equiv \left(\overline{QQ}_{1,2}\ u_3\ u_4\ //\ dm_{1,3\to 1}\ //\ dm_{2,4\to 2}\right)$

Out[∘]= True

In[∘]:= $\mathbb{E}_{\{\}\to\{i\}}[0,\ 0,\ x_i]\ //\ dS_i$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ -x_i + (\hbar\, x_i - \hbar\, a_i\, x_i)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= Kink_1

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[\hbar\, a_1\, b_1,\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{B_1}} + \left(\frac{\hbar\, a_1}{2\,\sqrt{B_1}} - \frac{\hbar^3\, x_1^2\, y_1^2}{4\,\sqrt{B_1}}\right)\epsilon + O[\epsilon]^2\Big]$

In[∘]:= $(u_1\ //\ dS_1)\ u_2\ //\ dm_{1,2\to 1}$
$(u_1\ //\ dS_1)\ u_2\ //\ dm_{2,1\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[-2\,\hbar\, a_1\, b_1,\ \frac{(-\hbar - \hbar\, B_1)\, x_1\, y_1}{B_1^2},\ B_1 + \left(-\hbar\, a_1\, B_1 + \frac{a_1\left(-2\,\hbar^2 - \hbar^2\, B_1\right) x_1\, y_1}{B_1} + \frac{\left(-7\,\hbar^3 - 8\,\hbar^3\, B_1 - 3\,\hbar^3\, B_1^2\right) x_1^2\, y_1^2}{4\, B_1^3}\right)\epsilon + O[\epsilon]^2\Big]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[-2\,\hbar\, a_1\, b_1,\ \frac{(-\hbar - \hbar\, B_1)\, x_1\, y_1}{B_1^2},\ B_1 + \left(-\hbar\, a_1\, B_1 + \frac{a_1\left(-2\,\hbar^2 - \hbar^2\, B_1\right) x_1\, y_1}{B_1} + \frac{\left(-7\,\hbar^3 - 8\,\hbar^3\, B_1 - 3\,\hbar^3\, B_1^2\right) x_1^2\, y_1^2}{4\, B_1^3}\right)\epsilon + O[\epsilon]^2\Big]$

In[∘]:=
```
(u₁ // dS₁)
u₂
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ 1+\left(\frac{\hbar^2\, x_1\, y_1}{B_1}-\frac{\hbar^2\, a_1\, x_1\, y_1}{B_1}-\frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1^2}\right)\in + O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[-\hbar\, a_2\, b_2,\ -\frac{\hbar\, x_2\, y_2}{B_2},\ B_2+\left(-\hbar\, a_2\, B_2-\hbar^2\, x_2\, y_2-\hbar^2\, a_2\, x_2\, y_2-\frac{3\,\hbar^3\, x_2^2\, y_2^2}{4\, B_2}\right)\in + O[\in]^2\right]$

In[∘]:=
```
R₁,₂
R̄₁,₂
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, b_1,\ \hbar\, x_2\, y_1,\ 1-\frac{1}{4}\left(\hbar^3\, x_2^2\, y_1^2\right)\in + O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[-\hbar\, a_2\, b_1,\ -\frac{\hbar\, x_2\, y_1}{B_1},\ 1+\left(-\frac{\hbar^2\, a_2\, x_2\, y_1}{B_1}-\frac{3\,\hbar^3\, x_2^2\, y_1^2}{4\, B_1^2}\right)\in + O[\in]^2\right]$

In[∘]:=
```
C₁
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \sqrt{B_1}-\frac{1}{2}\left(\hbar\, a_1\,\sqrt{B_1}\right)\in + O[\in]^2\right]$

The Knot Tensors

In[∘]:=
```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
  k̄R̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
  km_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k) /. {t_k → t, T_k → T, τ_{i|j} → 0},
  kC_i = (C_i // b2t_i) /. T_i → T,
  k̄C̄_i = (C̄_i // b2t_i) /. T_i → T,
  kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
  k̄K̄īn̄k̄_i = K̄īn̄k̄_i // b2t_i /. {t_i → t, T_i → T}]
```

In[∘]:=
```
(*Define[
  BS_{i,j→k}=C_3C_4dΔ_{i→r1,l1}dΔ_{j→r2,l2}//d̄S̄_{r1}//dS_{r2}//dm_{l1,3→k}//dm_{k,r2→k}//dm_{k,r1→k}//dm_{k,4→k}//dm_{k,l2→k}]*)
Define[BS_{i,j→k} =
   C_3 C_4 dΔ_{i→r1,l1} dΔ_{j→r2,l2} // d̄S̄_{r1} // dS_{r2} // dm_{l1,r2→k} // dm_{k,3→k} // dm_{k,4→k} // dm_{k,r1→k} // dm_{k,l2→k}]
Define[tBS_{i,j→k} = (t2b_i t2b_j) // BS_{i,j→k} // b2t_k]
Define[tm_{i,j→k} = t2b_i // t2b_j // dm_{i,j→k} // b2t_k]
Define[tΔ_{i→j,k} = t2b_i // dΔ_{i→j,k} // b2t_j // b2t_k]
Define[tS_i = t2b_i // dS_i // b2t_i]
Define[t̄S̄_i = t2b_i // d̄S̄_i // b2t_i]
Define[tR_{i,j} = R_{i,j} // b2t_i // b2t_j, t̄R̄_{i,j} = R̄_{i,j} // b2t_i // b2t_j]
Define[tC_i = C_i // b2t_i , t̄C̄_i = C̄_i // b2t_i ]
Define[tKink_i = Kink_i // b2t_i , t̄K̄īn̄k̄_i = K̄īn̄k̄_i // b2t_i]
```

In[∘]:= $t\Delta_{i\to j,k}$ // tS_k // $tm_{j,k\to k}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{k\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

In[∘]:= G = $\overline{tKink_1}\;\overline{tKink_4}\;\overline{tR}_{2,3}$ // $tm_{1,3\to i}$ // $tm_{2,4\to j}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[-a_i t_i - a_i t_j - a_j t_j,\; -\frac{x_i y_i}{T_i T_j} - \frac{x_i y_j}{T_j} - \frac{x_j y_j}{T_j},$$
$$\sqrt{T_i}\sqrt{T_j} + \Big(-a_i\sqrt{T_i}\sqrt{T_j} - a_i^2\sqrt{T_i}\sqrt{T_j} - a_j\sqrt{T_i}\sqrt{T_j} - a_i a_j\sqrt{T_i}\sqrt{T_j} - a_j^2\sqrt{T_i}\sqrt{T_j} - \frac{2 a_i x_i y_i}{\sqrt{T_i}\sqrt{T_j}} -$$
$$\frac{a_j x_i y_i}{\sqrt{T_i}\sqrt{T_j}} - \frac{3 x_i^2 y_i^2}{4 T_i^{3/2} T_j^{3/2}} - \frac{a_i\sqrt{T_i}\, x_i y_j}{\sqrt{T_j}} - \frac{a_j\sqrt{T_i}\, x_i y_j}{\sqrt{T_j}} - \frac{a_i\sqrt{T_i}\, x_j y_j}{\sqrt{T_j}} - \frac{2 a_j\sqrt{T_i}\, x_j y_j}{\sqrt{T_j}} -$$
$$\frac{x_i^2 y_i y_j}{\sqrt{T_i}\, T_j^{3/2}} - \frac{x_i x_j y_i y_j}{\sqrt{T_i}\, T_j^{3/2}} - \frac{3\sqrt{T_i}\, x_i^2 y_j^2}{4 T_j^{3/2}} - \frac{\sqrt{T_i}\, x_i x_j y_j^2}{T_j^{3/2}} - \frac{3\sqrt{T_i}\, x_j^2 y_j^2}{4 T_j^{3/2}}\Big)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= G // $tBS_{i,j\to k}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{k\}}\Big[0,\,0,\,\frac{T_k}{1-T_k+T_k^2} +$$
$$\Big(\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4} + \frac{-T_k^2+2T_k^3-3T_k^4+2T_k^5}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6} + \frac{\left(-2T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\Big)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= G2 = $tKink_1\, tKink_4\, tR_{3,2}$ // $tm_{1,3\to i}$ // $tm_{2,4\to j}$
G2 // $tBS_{i,j\to k}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[a_i t_i + a_j t_i + a_j t_j,\; x_i y_i + x_j y_i + T_i x_j y_j,$$
$$\frac{1}{\sqrt{T_i}\sqrt{T_j}} + \Big(\frac{a_i}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_i^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_i a_j}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j^2}{\sqrt{T_i}\sqrt{T_j}} - \frac{a_j x_i y_i}{\sqrt{T_i}\sqrt{T_j}} -$$
$$\frac{x_i^2 y_i^2}{4\sqrt{T_i}\sqrt{T_j}} - \frac{x_j^2 y_i^2}{4\sqrt{T_i}\sqrt{T_j}} - \frac{a_i\sqrt{T_i}\, x_j y_j}{\sqrt{T_j}} + \frac{\sqrt{T_i}\, x_i x_j y_i y_j}{\sqrt{T_j}} - \frac{T_i^{3/2} x_j^2 y_j^2}{4\sqrt{T_j}}\Big)\epsilon + O[\epsilon]^2\Big]$$

In[∘]:=
$$\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4} + \frac{-2T_k+3T_k^2-2T_k^3+T_k^4}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6} \text{ // Together}$$

Out[∘]=
$$\frac{-2T_k-2a_kT_k+3T_k^2+2a_kT_k^2-2T_k^3+T_k^4-2a_kT_k^4+2a_kT_k^5}{\left(1-T_k+T_k^2\right)^3}$$

In[∘]:=
$$\frac{-2T_k-2a_kT_k+3T_k^2+2a_kT_k^2-2T_k^3+T_k^4-2a_kT_k^4+2a_kT_k^5}{\left(1-T_k+T_k^2\right)^3} \text{ /. } a_k \to -1/2$$

Out[∘]=
$$\frac{-T_k+2T_k^2-2T_k^3+2T_k^4-T_k^5}{\left(1-T_k+T_k^2\right)^3}$$

In[∘]:= **G3 = (G2 /. {t_ → 0, T_ → 1})**

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\Big[0,\ x_i\, y_i + x_j\, y_i + x_j\, y_j,\ 1 + \Big(a_i + a_i^2 + a_j + a_i\, a_j + a_j^2 - a_j\, x_i\, y_i - \frac{1}{4}\, x_i^2\, y_i^2 - \frac{1}{4}\, x_j^2\, y_i^2 - a_i\, x_j\, y_j + x_i\, x_j\, y_i\, y_j - \frac{1}{4}\, x_j^2\, y_j^2\Big)\, \epsilon + O[\epsilon]^2\Big]$

In[∘]:= **tBS$_{i,j\to k}$ /. ϵ -> 0**

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\,(1-\mathcal{A}_j)\,\eta_i + \frac{y_k\,(-\mathcal{A}_j+\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,(-\mathcal{A}_i+\mathcal{A}_i\,\mathcal{A}_j)\,\xi_i}{\mathcal{A}_j} + (\mathcal{A}_i - T_k\,\mathcal{A}_i - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_i + (-\mathcal{A}_i + T_k\,\mathcal{A}_i - \mathcal{A}_j + T_k\,\mathcal{A}_j + \mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_i + x_k\,(1-\mathcal{A}_i)\,\xi_j + (\mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_j + (\mathcal{A}_j - T_k\,\mathcal{A}_j - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_j,\ T_k\Big]$

In[∘]:= **(tBS$_{i,j\to k}$ /. 𝒜_ → 1 /. x$_k$ → 0 /. y$_k$ → 0 /. a$_k$ → 0)**

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ T_k + \Big((-T_k+T_k^2)\,\eta_i\,\xi_i + \frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_i\,\eta_j\,\xi_i^2 + \frac{1}{4}\,(3\,T_k-4\,T_k^2+T_k^3)\,\eta_j^2\,\xi_i^2 + (-2\,T_k+2\,T_k^2)\,\eta_i\,\xi_j + (-T_k+T_k^2)\,\eta_j\,\xi_j + (3\,T_k-4\,T_k^2+T_k^3)\,\eta_i\,\eta_j\,\xi_i\,\xi_j + \frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_j^2\,\xi_i\,\xi_j + \frac{1}{4}\,(-3\,T_k+4\,T_k^2-T_k^3)\,\eta_i^2\,\xi_j^2\Big)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= **(tBS$_{i,j\to k}$ /. 𝒜_ → 1)**

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ T_k + \Big(-2\,a_k\,T_k - T_k\,y_k\,\eta_i + T_k\,y_k\,\eta_j + T_k\,x_k\,\xi_i + (-T_k+T_k^2)\,\eta_i\,\xi_i - 2\,a_k\,T_k^2\,\eta_j\,\xi_i + T_k\,x_k\,y_k\,\eta_j\,\xi_i - 2\,T_k\,y_k\,\eta_i\,\eta_j\,\xi_i + T_k\,y_k\,\eta_j^2\,\xi_i + T_k\,x_k\,\eta_j\,\xi_i^2 + \frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_i\,\eta_j\,\xi_i^2 + \frac{1}{4}\,(3\,T_k-4\,T_k^2+T_k^3)\,\eta_j^2\,\xi_i^2 - T_k\,x_k\,\xi_j + 2\,a_k\,T_k^2\,\eta_i\,\xi_j + (-2\,T_k+2\,T_k^2)\,\eta_i\,\xi_j - T_k\,x_k\,y_k\,\eta_i\,\xi_j + T_k\,y_k\,\eta_i^2\,\xi_j + (-T_k+T_k^2)\,\eta_j\,\xi_j - 2\,T_k\,x_k\,\eta_j\,\xi_i\,\xi_j + (3\,T_k-4\,T_k^2+T_k^3)\,\eta_i\,\eta_j\,\xi_i\,\xi_j + \frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_j^2\,\xi_i\,\xi_j + T_k\,x_k\,\eta_i\,\xi_j^2 + \frac{1}{4}\,(-3\,T_k+4\,T_k^2-T_k^3)\,\eta_i^2\,\xi_j^2\Big)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= **(tBS$_{i,j\to k}$ /. ϵ → 0)**

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\,(1-\mathcal{A}_j)\,\eta_i + \frac{y_k\,(-\mathcal{A}_j+\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,(-\mathcal{A}_i+\mathcal{A}_i\,\mathcal{A}_j)\,\xi_i}{\mathcal{A}_j} + (\mathcal{A}_i - T_k\,\mathcal{A}_i - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_i + (-\mathcal{A}_i + T_k\,\mathcal{A}_i - \mathcal{A}_j + T_k\,\mathcal{A}_j + \mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_i + x_k\,(1-\mathcal{A}_i)\,\xi_j + (\mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_j + (\mathcal{A}_j - T_k\,\mathcal{A}_j - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_j,\ T_k\Big]$

In[∘]:= **(tBS$_{i,j\to k}$ /. ϵ → 0 /. x$_k$ → 0 /. y$_k$ → 0)**

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (\mathcal{A}_i - T_k\,\mathcal{A}_i - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_i + (-\mathcal{A}_i + T_k\,\mathcal{A}_i - \mathcal{A}_j + T_k\,\mathcal{A}_j + \mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_i + (\mathcal{A}_i\,\mathcal{A}_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_i\,\xi_j + (\mathcal{A}_j - T_k\,\mathcal{A}_j - \mathcal{A}_i\,\mathcal{A}_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j)\,\eta_j\,\xi_j,\ T_k\Big]$

In[∘]:= **Coefficient**$\left[\left(tBS_{i,j\to k}\ /.\ \mathcal{A}\_ \to 1\right)[\![3]\!],\ \epsilon\, a_k\right]$

Out[∘]= $-2\,T_k - 2\,T_k^2\,\eta_j\,\xi_i + 2\,T_k^2\,\eta_i\,\xi_j$

In[∘]:= $\left(tBS_{i,j\to k}\ /.\ \{\mathcal{A}\_ \to 1,\ x_k \to 0,\ y_k \to 0\}\right)$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,$
$T_k + \Big(-2\,a_k\,T_k + \left(-T_k + T_k^2\right)\eta_i\,\xi_i - 2\,a_k\,T_k^2\,\eta_j\,\xi_i + \frac{1}{2}\left(-2\,T_k + 2\,T_k^2\right)\eta_i\,\eta_j\,\xi_i^2 +$
$\frac{1}{4}\left(3\,T_k - 4\,T_k^2 + T_k^3\right)\eta_j^2\,\xi_i^2 + 2\,a_k\,T_k^2\,\eta_i\,\xi_j + \left(-2\,T_k + 2\,T_k^2\right)\eta_i\,\xi_j + \left(-T_k + T_k^2\right)\eta_j\,\xi_j +$
$\left(3\,T_k - 4\,T_k^2 + T_k^3\right)\eta_i\,\eta_j\,\xi_i\,\xi_j + \frac{1}{2}\left(-2\,T_k + 2\,T_k^2\right)\eta_j^2\,\xi_i\,\xi_j + \frac{1}{4}\left(-3\,T_k + 4\,T_k^2 - T_k^3\right)\eta_i^2\,\xi_j^2\Big)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= **(G3 /. ϵ → 0)**

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ x_i\,y_i + x_j\,y_i + x_j\,y_j,\ 1\right]$

In[∘]:= $tBS_{i,j\to k}\ /.\ \{\epsilon \to 0,\ \mathcal{A}\_ \to 1,\ x_k \to 0,\ y_k \to 0\}$
$(G3\ /.\ \epsilon \to 0)\ //\ \left(tBS_{i,j\to k}\ /.\ \{\epsilon \to 0,\ \mathcal{A}\_ \to 1,\ x_k \to 0,\ y_k \to 0\}\right)$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\left[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ T_k\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0,\ 0,\ \frac{T_k}{1 - T_k + T_k^2}\right]$

In[∘]:= $(G3\ /.\ \epsilon \to 0)\ //\ \left(\mathbb{E}_{\{i,j\}\to\{k\}}\left[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ \frac{1}{4}\left(-3\,T_k + 4\,T_k^2 - T_k^3\right)\eta_i^2\,\xi_j^2\right]\right)$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0,\ 0,\ \frac{-3\,T_k^3 + 4\,T_k^4 - T_k^5}{2 - 6\,T_k + 12\,T_k^2 - 14\,T_k^3 + 12\,T_k^4 - 6\,T_k^5 + 2\,T_k^6}\right]$

In[∘]:= **(G3 /. ϵ → 0)**

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ x_i\,y_i + x_j\,y_i + x_j\,y_j,\ 1\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ x_i\,y_i + x_j\,y_i + x_j\,y_j,\ 1\right]\ //\ \Big(\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,$
$(-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ 0\,\eta_i^2\,\xi_j^2\,\epsilon + T_k{}^2\,\eta_i\,\eta_j\,\xi_i\,\xi_j\,\epsilon + \frac{1}{2}\left(-2\,T_k + 2\,T_k^2\right)\eta_j^2\,\xi_i\,\xi_j\,\epsilon\Big]\Big)$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0,\ 0,\ \frac{-2\,\epsilon\,T_k + 5\,\epsilon\,T_k^2 - \epsilon\,T_k^3 - \epsilon\,T_k^4}{1 - 3\,T_k + 6\,T_k^2 - 7\,T_k^3 + 6\,T_k^4 - 3\,T_k^5 + T_k^6}\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ x_i\,y_i + x_j\,y_i + x_j\,y_j,\ 1 + \epsilon\,x_i{}^2\,y_j{}^2\right]\ //$
$\left(\mathbb{E}_{\{i,j\}\to\{k\}}\left[0,\ (-1+T_k)\,\eta_j\,\xi_i + (1-T_k)\,\eta_i\,\xi_j,\ 1 + \eta_i^2\,\xi_j^2\,\epsilon + O[\epsilon]^2\right]\right)$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0,\ 0,\ \frac{1}{1 - T_k + T_k^2} + \frac{\left(4\,T_k^2 - 4\,T_k^3 + 2\,T_k^4\right)\epsilon}{1 - 3\,T_k + 6\,T_k^2 - 7\,T_k^3 + 6\,T_k^4 - 3\,T_k^5 + T_k^6} + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}[0, x_i y_i + x_j y_i + x_j y_j, 1]\ \mathbb{E}[0, (-1 + T_k)\,\eta_j \xi_i + (1 - T_k)\,\eta_i \xi_j,\ \eta_i^2 \xi_j^2 \epsilon]$

Out[∘]= $\mathbb{E}[0, x_i y_i + x_j y_i + x_j y_j + (-1 + T_k)\,\eta_j \xi_i + (1 - T_k)\,\eta_i \xi_j,\ \epsilon\,\eta_i^2 \xi_j^2]$

In[∘]:= $\text{MQZip}_{\{y_i, y_j, \xi_i, \xi_j\}}$@
$\mathbb{E}[0, x_i y_i + x_j y_i + x_j y_j + (-1 + T_k)\,\eta_j \xi_i + (1 - T_k)\,\eta_i \xi_j,\ \epsilon\, y_i^2 x_i^2 + 0\,\epsilon\,\eta_j^2 \xi_i^2 + 0\,\epsilon\,\eta_i^2 \xi_j \xi_i]$

» $\left\{\left\{\frac{1}{1 - T_k + T_k^2}, \frac{-1 + T_k}{1 - T_k + T_k^2}, \frac{1}{1 - T_k + T_k^2}, \frac{T_k}{1 - T_k + T_k^2}\right\}, \left\{\frac{1 - T_k}{1 - T_k + T_k^2}, \frac{T_k}{1 - T_k + T_k^2}, \frac{1 - T_k}{1 - T_k + T_k^2}, \frac{1}{1 - T_k + T_k^2}\right\}, \left\{\frac{-1 + 2 T_k - T_k^2}{1 - T_k + T_k^2}, \frac{-T_k + T_k^2}{1 - T_k + T_k^2}, \frac{T_k}{1 - T_k + T_k^2}, \frac{-1 + T_k}{1 - T_k + T_k^2}\right\}, \left\{\frac{1 - T_k}{1 - T_k + T_k^2}, \frac{-1 + 2 T_k - T_k^2}{1 - T_k + T_k^2}, \frac{1 - T_k}{1 - T_k + T_k^2}, \frac{1}{1 - T_k + T_k^2}\right\}\right\}$

» $\Big\{\eta_i \to \frac{x_i}{1 - T_k + T_k^2} + \frac{T_k x_j}{1 - T_k + T_k^2} + \frac{\eta_i}{1 - T_k + T_k^2} + \frac{(-1 + T_k)\,\eta_j}{1 - T_k + T_k^2},$
$\eta_j \to \frac{(1 - T_k)\, x_i}{1 - T_k + T_k^2} + \frac{x_j}{1 - T_k + T_k^2} + \frac{(1 - T_k)\,\eta_i}{1 - T_k + T_k^2} + \frac{T_k \eta_j}{1 - T_k + T_k^2},$
$x_i \to \frac{T_k x_i}{1 - T_k + T_k^2} + \frac{(-1 + T_k)\, x_j}{1 - T_k + T_k^2} + \frac{(-1 + 2 T_k - T_k^2)\,\eta_i}{1 - T_k + T_k^2} + \frac{(-T_k + T_k^2)\,\eta_j}{1 - T_k + T_k^2},$
$x_j \to \frac{(1 - T_k)\, x_i}{1 - T_k + T_k^2} + \frac{x_j}{1 - T_k + T_k^2} + \frac{(1 - T_k)\,\eta_i}{1 - T_k + T_k^2} + \frac{(-1 + 2 T_k - T_k^2)\,\eta_j}{1 - T_k + T_k^2}\Big\}$

» $\{y_i \to y_i,\ y_j \to y_j,\ \xi_i \to \xi_i,\ \xi_j \to \xi_j\}$

Out[∘]= $\mathbb{E}\left[0, 0, \frac{2 \epsilon - 8 \epsilon T_k + 12 \epsilon T_k^2 - 8 \epsilon T_k^3 + 2 \epsilon T_k^4}{1 - 3 T_k + 6 T_k^2 - 7 T_k^3 + 6 T_k^4 - 3 T_k^5 + T_k^6}\right]$

In[∘]:= (G3 /. $\epsilon \to 0$) // ($\text{tBS}_{i,j\to k}$ /. $\mathcal{A}$_ → 1 /. $x_k \to 0$ /. $y_k \to 0$ /. $a_k \to 0$)

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0, 0, \frac{T_k}{1 - T_k + T_k^2} + \frac{(-3 T_k + 8 T_k^2 - 11 T_k^3 + 4 T_k^4 + 2 T_k^5)\,\epsilon}{2 - 6 T_k + 12 T_k^2 - 14 T_k^3 + 12 T_k^4 - 6 T_k^5 + 2 T_k^6} + O[\epsilon]^2\right]$

In[∘]:= G3 // ($\text{tBS}_{i,j\to k}$)

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\left[0, 0, \frac{T_k}{1 - T_k + T_k^2} + \left(\frac{a_k\,(-2 T_k + 2 T_k^3)}{1 - 2 T_k + 3 T_k^2 - 2 T_k^3 + T_k^4} + \frac{-2 T_k + 3 T_k^2 - 2 T_k^3 + T_k^4}{1 - 3 T_k + 6 T_k^2 - 7 T_k^3 + 6 T_k^4 - 3 T_k^5 + T_k^6} + \frac{(-2 T_k - 2 T_k^2)\, x_k y_k}{1 - 2 T_k + 3 T_k^2 - 2 T_k^3 + T_k^4}\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:=
```
Al = T^-1 - 1 + T; 2 T Al^-2 D[Al, T] // Expand // Together // Expand // Together
Al = -T^-1 + 3 - T; 2 T Al^-2 D[Al, T] // Expand // Together // Expand // Together
Al = (T^2/(1 - T + T^2 - T^3 + T^4))^-1; 2 T Al^-2 D[Al, T] // Expand // Together // Expand // Together
```

Out[∘]= $$\frac{2\left(-T+T^3\right)}{\left(1-T+T^2\right)^2}$$

Out[∘]= $$-\frac{2\left(-T+T^3\right)}{\left(1-3T+T^2\right)^2}$$

Out[∘]= $$\frac{2\left(-2T^2+T^3-T^5+2T^6\right)}{\left(1-T+T^2-T^3+T^4\right)^2}$$

In[∘]:=
```
Z@Knot[5, 1]
```

Out[∘]= $$\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T^2}{1-T+T^2-T^3+T^4}+\left(\frac{a\left(-4T^2+2T^3-2T^5+4T^6\right)}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}+\frac{-4T^2+7T^3-8T^4+8T^5-6T^6+4T^7-2T^8+T^9}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\frac{\left(-4T^2-2T^3-2T^4-4T^5\right)x\,y}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
G3 /. a_ → 0
```

Out[∘]= $$\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\,x_i y_i+x_j y_i+x_j y_j,\,1+\left(-\frac{1}{4}x_i^2y_i^2-\frac{1}{4}x_j^2y_i^2+x_i x_j y_i y_j-\frac{1}{4}x_j^2y_j^2\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
(G3 /. a_ → 0) // (tBS_{i,j→k} /. 𝒜_ → 1)
```

Out[∘]= $$\mathbb{E}_{\{\}\to\{k\}}\left[0,\,0,\,\frac{T_k}{1-T_k+T_k^2}+\left(\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4}+\frac{-2T_k+4T_k^2-4T_k^3+T_k^4+T_k^5}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6}+\frac{\left(T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
G3 // tBS_{i,j→k} /. a_k → -1/2
```

Out[∘]= $$\mathbb{E}_{\{\}\to\{k\}}\left[0,\,0,\,\frac{T_k}{1-T_k+T_k^2}+\left(\frac{-T_k+2T_k^2-2T_k^3+2T_k^4-T_k^5}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6}+\frac{\left(-2T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= **Z@Knot[3, 1]**

Get: ParentDirectory[File] in $Path is not a string.

KnotTheory: Loading precomputed data in PD4Knots`.

$$\text{Out[∘]=}\ \mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{a\left(-2T+2T^3\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T+3T^2-2T^3+T^4}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T-2T^2\right)x\,y}{1-2T+3T^2-2T^3+T^4}\right)\in+O[\in]^2\Big]$$

In[∘]:= $\overline{\mathbf{tR}}_{i,j}$

$$\text{Out[∘]=}\ \mathbb{E}_{\{\}\to\{i,j\}}\Big[-\hbar\, a_j\, t_i,\ -\frac{\hbar\, x_j\, y_i}{T_i},\ 1+\left(-\hbar\, a_i\, a_j-\frac{\hbar^2 a_i\, x_j\, y_i}{T_i}-\frac{\hbar^2 a_j\, x_j\, y_i}{T_i}-\frac{3\hbar^3 x_j^2 y_i^2}{4T_i^2}\right)\in+O[\in]^2\Big]$$

In[∘]:= $\mathbf{tC}_1$

$$\text{Out[∘]=}\ \mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \sqrt{T_1}-\hbar\, a_1\sqrt{T_1}\in+O[\in]^2\Big]$$

In[∘]:= $\mathbf{tR}_{i,j}$

$$\text{Out[∘]=}\ \mathbb{E}_{\{\}\to\{i,j\}}\Big[\hbar\, a_j\, t_i,\ \hbar\, x_j\, y_i,\ 1+\left(\hbar\, a_i\, a_j-\frac{1}{4}\hbar^3 x_j^2 y_i^2\right)\in+O[\in]^2\Big]$$

In[∘]:= $\mathbf{tBS}_{i,j\to k}$

$$\text{Out[∘]=}\ \mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ \frac{y_k\left(1-\mathcal{A}_j\right)\eta_i}{T_k}+\frac{y_k\left(-\mathcal{A}_j+\mathcal{A}_i\mathcal{A}_j\right)\eta_j}{T_k\,\mathcal{A}_i}+\frac{x_k\left(-\mathcal{A}_i+\mathcal{A}_i\mathcal{A}_j\right)\xi_i}{\mathcal{A}_j}+$$
$$\frac{\left(\mathcal{A}_i-T_k\mathcal{A}_i-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\,\xi_i}{\hbar\,T_k}+\frac{\left(-\mathcal{A}_i+T_k\mathcal{A}_i-\mathcal{A}_j+T_k\mathcal{A}_j+\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_j\,\xi_i}{\hbar\,T_k}+$$
$$x_k\left(1-\mathcal{A}_i\right)\xi_j+\frac{\left(\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\,\xi_j}{\hbar\,T_k}+\frac{\left(\mathcal{A}_j-T_k\mathcal{A}_j-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_j\,\xi_j}{\hbar\,T_k},$$
$$T_k+\Bigg(-2\hbar\, a_k\, T_k+a_k\, y_k\left(\hbar-\hbar\mathcal{A}_j\right)\eta_i+\frac{y_k^2\left(\hbar\mathcal{A}_j-\hbar\mathcal{A}_j^2\right)\eta_i^2}{2T_k}+\frac{y_k\left(-\hbar\mathcal{A}_j+\hbar\mathcal{A}_i\mathcal{A}_j\right)\eta_j}{\mathcal{A}_i}+$$
$$\frac{a_k\, y_k\left(-\hbar\mathcal{A}_j+\hbar\mathcal{A}_i\mathcal{A}_j\right)\eta_j}{\mathcal{A}_i}+\frac{y_k^2\left(-\hbar\mathcal{A}_i\mathcal{A}_j-\hbar\mathcal{A}_j^2+2\hbar\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i\,\eta_j}{T_k\,\mathcal{A}_i}+$$
$$\frac{y_k^2\left(-\hbar\mathcal{A}_j^2+3\hbar\mathcal{A}_i\mathcal{A}_j^2-2\hbar\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2}{2T_k\,\mathcal{A}_i^2}+\frac{x_k\left(-\hbar T_k\mathcal{A}_i+\hbar T_k\mathcal{A}_i\mathcal{A}_j\right)\xi_i}{\mathcal{A}_j}+$$
$$\frac{a_k\, x_k\left(-\hbar T_k\mathcal{A}_i+\hbar T_k\mathcal{A}_i\mathcal{A}_j\right)\xi_i}{\mathcal{A}_j}+a_k\left(2\mathcal{A}_i-2\mathcal{A}_i\mathcal{A}_j\right)\eta_i\,\xi_i+$$
$$\left(-\mathcal{A}_i+T_k\mathcal{A}_i+\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\,\xi_i+\frac{x_k\, y_k\left(\hbar\mathcal{A}_i-2\hbar\mathcal{A}_i\mathcal{A}_j+\hbar\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i\,\xi_i}{\mathcal{A}_j}+$$

$$\frac{y_k\left(-\mathcal{A}_i+T_k\mathcal{A}_i+4\mathcal{A}_i\mathcal{A}_j-2T_k\mathcal{A}_i\mathcal{A}_j-3\mathcal{A}_i\mathcal{A}_j^2+T_k\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i^2\xi_i}{2T_k}-\hbar\,x_k\,y_k\,\mathcal{A}_i\mathcal{A}_j\eta_j\xi_i+$$

$$\left(-\mathcal{A}_i+T_k\mathcal{A}_i-\mathcal{A}_j+T_k\mathcal{A}_j\right)\eta_j\xi_i+a_k\left(-2\mathcal{A}_i-2\mathcal{A}_j+2\mathcal{A}_i\mathcal{A}_j\right)\eta_j\xi_i+$$

$$\frac{y_k\left(\mathcal{A}_i-T_k\mathcal{A}_i-5\mathcal{A}_i\mathcal{A}_j+3T_k\mathcal{A}_i\mathcal{A}_j-2\mathcal{A}_j^2+5\mathcal{A}_i\mathcal{A}_j^2-3T_k\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i\eta_j\xi_i}{T_k}+$$

$$\frac{y_k\left(-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j+3\mathcal{A}_i^2\mathcal{A}_j-T_k\mathcal{A}_i^2\mathcal{A}_j-\mathcal{A}_j^2-T_k\mathcal{A}_j^2+6\mathcal{A}_i\mathcal{A}_j^2-2T_k\mathcal{A}_i\mathcal{A}_j^2-5\mathcal{A}_i^2\mathcal{A}_j^2+3T_k\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2\xi_i}{2T_k\mathcal{A}_i}$$

$$+\frac{x_k^2\left(-\hbar T_k\mathcal{A}_i^2+3\hbar T_k\mathcal{A}_i^2\mathcal{A}_j-2\hbar T_k\mathcal{A}_i^2\mathcal{A}_j^2\right)\xi_i^2}{2\mathcal{A}_j^2}+$$

$$\frac{x_k\left(3\mathcal{A}_i^2-T_k\mathcal{A}_i^2-8\mathcal{A}_i^2\mathcal{A}_j+4T_k\mathcal{A}_i^2\mathcal{A}_j+5\mathcal{A}_i^2\mathcal{A}_j^2-3T_k\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i\xi_i^2}{2\mathcal{A}_j}+$$

$$\frac{\left(-3\mathcal{A}_i^2+4T_k\mathcal{A}_i^2-T_k^2\mathcal{A}_i^2+8\mathcal{A}_i^2\mathcal{A}_j-12T_k\mathcal{A}_i^2\mathcal{A}_j+4T_k^2\mathcal{A}_i^2\mathcal{A}_j-5\mathcal{A}_i^2\mathcal{A}_j^2+8T_k\mathcal{A}_i^2\mathcal{A}_j^2-3T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i^2\xi_i^2}{4\hbar T_k}+$$

$$\frac{x_k\left(-\mathcal{A}_i^2-T_k\mathcal{A}_i^2-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j+6\mathcal{A}_i^2\mathcal{A}_j-2T_k\mathcal{A}_i^2\mathcal{A}_j+3\mathcal{A}_i\mathcal{A}_j^2-T_k\mathcal{A}_i\mathcal{A}_j^2-5\mathcal{A}_i^2\mathcal{A}_j^2+3T_k\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j\xi_i^2}{2\mathcal{A}_j}$$

$$+\frac{1}{2\hbar T_k}\left(3\mathcal{A}_i^2-4T_k\mathcal{A}_i^2+T_k^2\mathcal{A}_i^2+\mathcal{A}_i\mathcal{A}_j-2T_k\mathcal{A}_i\mathcal{A}_j+T_k^2\mathcal{A}_i\mathcal{A}_j-10\mathcal{A}_i^2\mathcal{A}_j+16T_k\mathcal{A}_i^2\mathcal{A}_j-\right.$$
$$\left.6T_k^2\mathcal{A}_i^2\mathcal{A}_j-3\mathcal{A}_i\mathcal{A}_j^2+4T_k\mathcal{A}_i\mathcal{A}_j^2-T_k^2\mathcal{A}_i\mathcal{A}_j^2+7\mathcal{A}_i^2\mathcal{A}_j^2-12T_k\mathcal{A}_i^2\mathcal{A}_j^2+5T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i\eta_j\xi_i^2+$$

$$\frac{1}{4\hbar T_k}\left(-\mathcal{A}_i^2+T_k^2\mathcal{A}_i^2-4\mathcal{A}_i\mathcal{A}_j+8T_k\mathcal{A}_i\mathcal{A}_j-4T_k^2\mathcal{A}_i\mathcal{A}_j+8\mathcal{A}_i^2\mathcal{A}_j-12T_k\mathcal{A}_i^2\mathcal{A}_j+4T_k^2\mathcal{A}_i^2\mathcal{A}_j-\right.$$
$$\left.\mathcal{A}_j^2+T_k^2\mathcal{A}_j^2+8\mathcal{A}_i\mathcal{A}_j^2-12T_k\mathcal{A}_i\mathcal{A}_j^2+4T_k^2\mathcal{A}_i\mathcal{A}_j^2-7\mathcal{A}_i^2\mathcal{A}_j^2+12T_k\mathcal{A}_i^2\mathcal{A}_j^2-5T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2\xi_i^2+$$

$$a_k\,x_k\left(\hbar T_k-\hbar T_k\mathcal{A}_i\right)\xi_j+2a_k\mathcal{A}_i\mathcal{A}_j\eta_i\xi_j+x_k\,y_k\left(\hbar\mathcal{A}_i+\hbar\mathcal{A}_j-\hbar\mathcal{A}_i\mathcal{A}_j\right)\eta_i\xi_j+$$

$$\left(-2\mathcal{A}_i\mathcal{A}_j+2T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\xi_j+\frac{y_k\left(-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j+3\mathcal{A}_i\mathcal{A}_j^2-T_k\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i^2\xi_j}{2T_k}+$$

$$a_k\left(2\mathcal{A}_j-2\mathcal{A}_i\mathcal{A}_j\right)\eta_j\xi_j+\left(-\mathcal{A}_j+T_k\mathcal{A}_j+\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_j\xi_j+$$

$$\frac{x_k\,y_k\left(\hbar\mathcal{A}_j-2\hbar\mathcal{A}_i\mathcal{A}_j+\hbar\mathcal{A}_i^2\mathcal{A}_j\right)\eta_j\xi_j}{\mathcal{A}_i}+$$

$$\frac{y_k\left(\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j+3\mathcal{A}_j^2-T_k\mathcal{A}_j^2-5\mathcal{A}_i\mathcal{A}_j^2+3T_k\mathcal{A}_i\mathcal{A}_j^2\right)\eta_i\eta_j\xi_j}{T_k}+$$

$$\frac{y_k\left(3\mathcal{A}_j^2-T_k\mathcal{A}_j^2-8\mathcal{A}_i\mathcal{A}_j^2+4T_k\mathcal{A}_i\mathcal{A}_j^2+5\mathcal{A}_i^2\mathcal{A}_j^2-3T_k\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2\xi_j}{2T_k\mathcal{A}_i}+$$

$$\frac{x_k^2\left(-\hbar T_k\mathcal{A}_i^2-\hbar T_k\mathcal{A}_i\mathcal{A}_j+2\hbar T_k\mathcal{A}_i^2\mathcal{A}_j\right)\xi_i\xi_j}{\mathcal{A}_j}+$$

$$x_k\left(3\mathcal{A}_i^2-T_k\mathcal{A}_i^2+\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j-5\mathcal{A}_i^2\mathcal{A}_j+3T_k\mathcal{A}_i^2\mathcal{A}_j\right)\eta_i\xi_i\xi_j+$$

$$\frac{\left(-3\mathcal{A}_i^2\mathcal{A}_j+4T_k\mathcal{A}_i^2\mathcal{A}_j-T_k^2\mathcal{A}_i^2\mathcal{A}_j+5\mathcal{A}_i^2\mathcal{A}_j^2-8T_k\mathcal{A}_i^2\mathcal{A}_j^2+3T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i^2\xi_i\xi_j}{2\hbar T_k}+$$

$$x_k \left(-2\mathcal{A}_i^2+\mathcal{A}_j-T_k\mathcal{A}_j-5\mathcal{A}_i\mathcal{A}_j+3T_k\mathcal{A}_i\mathcal{A}_j+5\mathcal{A}_i^2\mathcal{A}_j-3T_k\mathcal{A}_i^2\mathcal{A}_j\right)\eta_j\xi_i\xi_j+\frac{1}{\hbar T_k}$$
$$\left(4\mathcal{A}_i^2\mathcal{A}_j-6T_k\mathcal{A}_i^2\mathcal{A}_j+2T_k^2\mathcal{A}_i^2\mathcal{A}_j+4\mathcal{A}_i\mathcal{A}_j^2-6T_k\mathcal{A}_i\mathcal{A}_j^2+2T_k^2\mathcal{A}_i\mathcal{A}_j^2-7\mathcal{A}_i^2\mathcal{A}_j^2+12T_k\mathcal{A}_i^2\mathcal{A}_j^2-5T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)$$
$$\eta_i\eta_j\xi_i\xi_j+\frac{1}{2\hbar T_k}\left(\mathcal{A}_i\mathcal{A}_j-2T_k\mathcal{A}_i\mathcal{A}_j+T_k^2\mathcal{A}_i\mathcal{A}_j-3\mathcal{A}_i^2\mathcal{A}_j+4T_k\mathcal{A}_i^2\mathcal{A}_j-T_k^2\mathcal{A}_i^2\mathcal{A}_j+3\mathcal{A}_j^2-4T_k\mathcal{A}_j^2+\right.$$
$$\left.T_k^2\mathcal{A}_j^2-10\mathcal{A}_i\mathcal{A}_j^2+16T_k\mathcal{A}_i\mathcal{A}_j^2-6T_k^2\mathcal{A}_i\mathcal{A}_j^2+7\mathcal{A}_i^2\mathcal{A}_j^2-12T_k\mathcal{A}_i^2\mathcal{A}_j^2+5T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2\xi_i\xi_j+$$
$$\frac{1}{2}x_k^2\left(\hbar T_k\mathcal{A}_i-\hbar T_k\mathcal{A}_i^2\right)\xi_j^2+\frac{1}{2}x_k\left(-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j+3\mathcal{A}_i^2\mathcal{A}_j-T_k\mathcal{A}_i^2\mathcal{A}_j\right)\eta_i\xi_j^2+$$
$$\frac{\left(-3\mathcal{A}_i^2\mathcal{A}_j^2+4T_k\mathcal{A}_i^2\mathcal{A}_j^2-T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i^2\xi_j^2}{4\hbar T_k}+$$
$$\frac{1}{2}x_k\left(-\mathcal{A}_j+T_k\mathcal{A}_j+4\mathcal{A}_i\mathcal{A}_j-2T_k\mathcal{A}_i\mathcal{A}_j-3\mathcal{A}_i^2\mathcal{A}_j+T_k\mathcal{A}_i^2\mathcal{A}_j\right)\eta_j\xi_j^2+$$
$$\frac{\left(-3\mathcal{A}_i\mathcal{A}_j^2+4T_k\mathcal{A}_i\mathcal{A}_j^2-T_k^2\mathcal{A}_i\mathcal{A}_j^2+5\mathcal{A}_i^2\mathcal{A}_j^2-8T_k\mathcal{A}_i^2\mathcal{A}_j^2+3T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_i\eta_j\xi_j^2}{2\hbar T_k}+$$
$$\left.\frac{\left(-3\mathcal{A}_j^2+4T_k\mathcal{A}_j^2-T_k^2\mathcal{A}_j^2+8\mathcal{A}_i\mathcal{A}_j^2-12T_k\mathcal{A}_i\mathcal{A}_j^2+4T_k^2\mathcal{A}_i\mathcal{A}_j^2-5\mathcal{A}_i^2\mathcal{A}_j^2+8T_k\mathcal{A}_i^2\mathcal{A}_j^2-3T_k^2\mathcal{A}_i^2\mathcal{A}_j^2\right)\eta_j^2\xi_j^2}{4\hbar T_k}\right)$$
$$\left.\epsilon+O[\epsilon]^2\right]$$

In[∘]:= **$R_{1,3}\,R_{2,6}$ // $dm_{3,6\to3}$**
$R_{1,3}$ // $d\Delta_{1\to2,1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar a_3 b_1+\hbar a_3 b_2,\ \hbar B_2 x_3 y_1+\hbar x_3 y_2,\ 1+\left(-\frac{1}{4}\hbar^3 B_2^2 x_3^2 y_1^2-\frac{1}{4}\hbar^3 x_3^2 y_2^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar a_3 b_1+\hbar a_3 b_2,\ \hbar B_2 x_3 y_1+\hbar x_3 y_2,\ 1+\left(-\frac{1}{4}\hbar^3 B_2^2 x_3^2 y_1^2-\frac{1}{4}\hbar^3 x_3^2 y_2^2\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **$tR_{1,2}$ // $\overline{tS}_1$ // $\overline{tS}_1$ // $tm_{1,2\to1}$**
$\left(tR_{1,2} \text{ // } \overline{tS}_1 \text{ // } \overline{tS}_1 \text{ // } tm_{1,2\to1} \text{ // } tS_1\right)\mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2\,(1-2\,\epsilon\,\hbar\,a_1)]$ // $tm_{1,2\to1}$
$\left(tR_{1,2} \text{ // } \overline{tS}_1 \text{ // } \overline{tS}_1 \text{ // } tm_{2,1\to1}\right)\mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2\,(1-2\,\epsilon\,\hbar\,a_1)]$ // $tm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1,\ \hbar x_1 y_1,\ 1+\left(\hbar a_1^2+\hbar^2 x_1 y_1-\frac{1}{4}\hbar^3 x_1^2 y_1^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1,\ \hbar x_1 y_1,\ 1+\left(\hbar a_1^2-\hbar^2 x_1 y_1-\frac{1}{4}\hbar^3 x_1^2 y_1^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar a_1 t_1,\ \hbar x_1 y_1,\ 1+\left(\hbar a_1^2-\hbar^2 x_1 y_1-\frac{1}{4}\hbar^3 x_1^2 y_1^2\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:= **$\mathbb{E}_{\{\}\to\{2\}}[0, 0, x_2]$ // dS_2**

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0,\ 0,\ -x_2-\hbar a_2 x_2\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2]$ // $\overline{\text{dS}}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -\frac{y_2}{B_2} + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2]$ // $\overline{\text{dS}}_2$ // $\overline{\text{dS}}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, y_2 + \hbar\, y_2\, \epsilon + O[\epsilon]^2\right]$

In[∘]:= $\text{tm}_{i,j\to k}$

$\text{tR}_{i,j}$

$\overline{\text{tR}}_{i,j}$

tC_i

$\overline{\text{tC}}_i$

tKink_i

$\overline{\text{tKink}}_i$

$\text{t}\Delta_{i\to j,k}$

tS_i

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i + a_k\,\alpha_j + t_k\,\tau_i + t_k\,\tau_j,\ y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \frac{(1-T_k)\,\eta_j\,\xi_i}{\hbar} + x_k\,\xi_j,\ 1 + \left(2\,a_k\,T_k\,\eta_j\,\xi_i + \frac{\hbar\,x_k\,y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j} + \frac{(1-3\,T_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i} + \frac{(1-3\,T_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j} + \frac{\left(1-4\,T_k+3\,T_k^2\right)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,t_i,\ \hbar\,x_j\,y_i,\ 1 + \left(\hbar\,a_i\,a_j - \frac{1}{4}\,\hbar^3\,x_j^2\,y_i^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,t_i,\ -\frac{\hbar\,x_j\,y_i}{T_i},\ 1 + \left(-\hbar\,a_i\,a_j - \frac{\hbar^2\,a_i\,x_j\,y_i}{T_i} - \frac{\hbar^2\,a_j\,x_j\,y_i}{T_i} - \frac{3\,\hbar^3\,x_j^2\,y_i^2}{4\,T_i^2}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \sqrt{T_i} - \hbar\,a_i\,\sqrt{T_i}\,\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0, 0, \frac{1}{\sqrt{T_i}} + \frac{\hbar\,a_i\,\epsilon}{\sqrt{T_i}} + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\,a_i\,t_i,\ \hbar\,x_i\,y_i,\ \frac{1}{\sqrt{T_i}} + \left(\frac{\hbar\,a_i}{\sqrt{T_i}} + \frac{\hbar\,a_i^2}{\sqrt{T_i}} - \frac{\hbar^3\,x_i^2\,y_i^2}{4\,\sqrt{T_i}}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\,a_i\,t_i,\ -\frac{\hbar\,x_i\,y_i}{T_i},\ \sqrt{T_i} + \left(-\hbar\,a_i\,\sqrt{T_i} - \hbar\,a_i^2\,\sqrt{T_i} - \frac{2\,\hbar^2\,a_i\,x_i\,y_i}{\sqrt{T_i}} - \frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,T_i^{3/2}}\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\,\alpha_i + a_k\,\alpha_i + t_j\,\tau_i + t_k\,\tau_i,\ y_j\,\eta_i + T_j\,y_k\,\eta_i + x_j\,\xi_i + x_k\,\xi_i,\ 1 + \left(-\hbar\,a_j\,T_j\,y_k\,\eta_i + \frac{1}{2}\,\hbar\,T_j\,y_j\,y_k\,\eta_i^2 - \hbar\,a_j\,x_k\,\xi_i + \frac{1}{2}\,\hbar\,x_j\,x_k\,\xi_i^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $$\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i - t_i\,\tau_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{T_i} - x_i\,\mathcal{A}_i\,\xi_i + \frac{(\mathcal{A}_i - T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,T_i},\right.$$
$$1 + \left(\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,a_i\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,T_i^2} - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i + \frac{2\,a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{T_i} - \frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{T_i} + \frac{(-\mathcal{A}_i + T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{T_i} + \frac{y_i\,\left(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2\right)\,\eta_i^2\,\xi_i}{2\,T_i^2} -\right.$$
$$\left.\left.\frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 + \frac{x_i\,\left(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2\right)\,\eta_i\,\xi_i^2}{2\,T_i} + \frac{\left(-3\,\mathcal{A}_i^2 + 4\,T_i\,\mathcal{A}_i^2 - T_i^2\,\mathcal{A}_i^2\right)\,\eta_i^2\,\xi_i^2}{4\,\hbar\,T_i^2}\right)\,\epsilon + O[\epsilon]^2\right]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[a_i\,t_i,\ x_i\,y_i,\ \frac{1}{\sqrt{T_i}} + \left(\frac{a_i}{\sqrt{T_i}} + \frac{a_i^2}{\sqrt{T_i}} - \frac{x_i^2\,y_i^2}{4\,\sqrt{T_i}}\right)\,\epsilon + O[\epsilon]^2\right]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[-a_i\,t_i,\ -\frac{x_i\,y_i}{T_i},\ \sqrt{T_i} + \left(-a_i\,\sqrt{T_i} - a_i^2\,\sqrt{T_i} - \frac{2\,a_i\,x_i\,y_i}{\sqrt{T_i}} - \frac{3\,x_i^2\,y_i^2}{4\,T_i^{3/2}}\right)\,\epsilon + O[\epsilon]^2\right]$$

In[∘]:=
```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X :→ { Xp[x⟦4⟧, x⟦1⟧]  PositiveQ@x
                            Xm[x⟦2⟧, x⟦1⟧]  True       ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
               Xp[k + 1, l_] | Xm[l_, k + 1] :→ {l, k + 1, 1 - l},
               Xp[l_, k + 1] | Xm[k + 1, l_] :→ (++rots⟦l⟧; {1 - l, k + 1, l})
              })],
      Cases[front, k | -k] /. {k, -k} :→ --rots⟦k + 1⟧;
     ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := 𝔼_{}→{i}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC̄_j] // km_{i,j→i}];
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{}→{0}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (kR_{i,j} kKink‾_k) // km_{j,k→j},
     Xm, (kR‾_{i,j} kKink_k) // km_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // km_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // km_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // km_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // km_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // km_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // km_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
 ]
```

In[∘]:=
```
Z@Knot[3, 1]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T}{1-T+T^2}+\left(\frac{a\left(-2T\hbar+2T^3\hbar\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T\hbar+3T^2\hbar-2T^3\hbar+T^4\hbar}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{x\,y\left(-2T\hbar^2-2T^2\hbar^2\right)}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
$$\text{Coefficient}\left[\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T}{1-T+T^2}+\left(\frac{a\left(-2T\hbar+2T^3\hbar\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T\hbar+3T^2\hbar-2T^3\hbar+T^4\hbar}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{xy\left(-2T\hbar^2-2T^2\hbar^2\right)}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right][[3]],\epsilon\right]\text{ /. }\{a\to-1/2,x\to0\}\text{ // Together // Factor}$$

Out[∘]=
$$-\frac{(-1+T)^2\,T\,(1+T^2)\,\hbar}{(1-T+T^2)^3}$$

In[∘]:=
$$-\frac{(-1+T)^2\,T\,(1+T^2)\,\hbar}{(1-T+T^2)^3}*-T\,(T-1)^{-2}\left(1-T+T^2\right)^3T^{-3}\text{ // Expand}$$

Out[∘]=
$$\frac{1}{T}+T$$

In[∘]:= $\mathbb{E}_{\{\}\to\{1,2\}}[0,0,y_2\,x_1]$ // $tm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,0,(1-T_1+x_1\,y_1)+(2\,a_1\,T_1+x_1\,y_1)\,\epsilon+O[\epsilon]^2\right]$

In[∘]:= $R_{1,2}\,R_{3,4}$ // $dm_{1,3\to5}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{2,4,5\}}\left[a_2\,b_5+a_4\,b_5,\;x_2\,y_5+x_4\,y_5,\;1+\left(-a_2\,x_4\,y_5-\frac{1}{4}x_2^2\,y_5^2-\frac{1}{4}x_4^2\,y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\overline{kR}_{1,2}\,\overline{kR}_{3,4}$ // $tm_{1,4\to5}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{2,3,5\}}\left[-t\,a_2-t\,a_5,\;-\frac{x_5\,y_3}{T}-\frac{x_2\,y_5}{T},\;1+\left(-a_2\,a_5-a_3\,a_5-\frac{a_3\,x_5\,y_3}{T}-\frac{a_5\,x_5\,y_3}{T}-\frac{3\,x_5^2\,y_3^2}{4\,T^2}-\frac{a_2\,x_2\,y_5}{T}-\frac{a_5\,x_2\,y_5}{T}-\frac{3\,x_2^2\,y_5^2}{4\,T^2}\right)\epsilon+O[\epsilon]^2\right]$$

$\overline{kR}_{1,2}\,\overline{kR}_{3,4}$ // $tm_{1,4\to5}$

In[∘]:=
```
(*Working Casimir, not unique!*)
Define[ωi = 𝔼{}→{i}[0, 0, Series[y e^(ϵ a) x + (e^(ϵ (a+1)) + e^(-ϵ a) T)/(e^ϵ - 1) - (T + 1) ϵ^-1, {ϵ, 0, 3}] /.
    {a → ai, T → Ti, x → xi, y → yi}]
]
ωsq = ω1 ω2 // tm1,2→1;
ωcub = ωsq ω2 // tm1,2→1;
ω4 = ωcub ω2 // tm1,2→1;
(*Cleaned versions*)
ωc = ω1[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωsqc = ωsq[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωcubc = ωcub[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ω4c = ω4[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
```