Introducing the q-Casimir w to all orders and expressing and computing the invariant in terms of it.

Pensieve header: The "Speedy" engine.

```
In[∘]:= Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

```
In[∘]:= PP_ = Identity; $k = 2; γ = 1; ℏ = 1;
```

The "Speedy" Engine

Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
          Expand[ℰ] //. e^x_ e^y_ ⧴ e^(x+y) /. e^x_ ⧴ e^CCF[x]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := Module[
           {vs = Cases[ℰ, (y | b | t | a | w | x | η | β | τ | α | ω | ξ)_, ∞] ⋃
              {y, b, t, a, w, x, η, β, τ, α, ω, ξ}},
           Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) ⧴ CCF[c] (Times @@ vs^ps)]
          ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

In[∘]:=
```
{t*, b*, y*, a*, w*, x*, z*} = {τ, β, η, α, ω, ξ, ζ};
{τ*, β*, η*, α*, ω*, ξ*, ζ*} = {t, b, y, a, w, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

In[∘]:=
```
U2l = {B_i_^(p_.) :> e^(-p ħ γ b_i), B^(p_.) :> e^(-p ħ γ b), T_i_^(p_.) :> e^(-p ħ t_i),
   T^(p_.) :> e^(-p ħ t), 𝒜_i_^(p_.) :> e^(p γ α_i), 𝒜^(p_.) :> e^(p γ α), Ω_i_^(p_.) :> e^(p ω_i), Ω^(p_.) :> e^(p ω)};
l2U = {e^(c_. b_i_+d_.) :> B_i^(-c/(ħ γ)) e^d, e^(c_. b+d_.) :> B^(-c/(ħ γ)) e^d,
   e^(c_. t_i_+d_.) :> T_i^(-c/ħ) e^d, e^(c_. t+d_.) :> T^(-c/ħ) e^d,
   e^(c_. α_i_+d_.) :> 𝒜_i^(c/γ) e^d, e^(c_. α+d_.) :> 𝒜^(c/γ) e^d,
   e^(c_. ω_i_+d_.) :> Ω_i^c e^d, e^(c_. ω+d_.) :> Ω^c e^d,
   e^ℰ_ :> e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

In[∘]:=
```
D_b[f_] := ∂_b f - ħ γ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ħ γ B_i ∂_B_i f;
D_t[f_] := ∂_t f - ħ T ∂_T f; D_t_i_[f_] := ∂_t_i f - ħ T_i ∂_T_i f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + γ 𝒜_i ∂_𝒜_i f;
D_ω[f_] := ∂_ω f + Ω ∂_Ω f; D_ω_i_[f_] := ∂_ω_i f + Ω_i ∂_Ω_i f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^(d_.) :> (D_{ζ*,d}[f])) /. ζ* → 0 /.
  ((ζ* /. {b → B, t → T, α → 𝒜, ω → Ω}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[ ]:=
```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_{ζs}[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the "*L*-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single"*P*". Here the *z*'s are *b* and *α* and the *ζ*'s are *β* and *a*.

In[ ]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜, ω → Ω};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule, zrule /.
      r_Rule :> ((U = r⟦1⟧ /. {b → B, t → T, α → 𝒜, ω → Ω}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^(-Q1) D_{Thread[{zs,{ps}}]}[e^Q1]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
     Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ] ];
```

In[ ]:=
```
TZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, Zs, c, ys, ηs,
     lt, zrule, Zrule, ζrule, Q1, EEQ, EQ, Lnew = L, Qnew = Q, Pnew = P},
   zrule = Table[ζ* → Coefficient[L, ζ], {ζ, ζs}];
   ζrule = Table[ζ → 0, {ζ, ζs}];
   Lnew = L /. U2l /. zrule /. ζrule;
   Qnew = Q /. U2l /. zrule /. ζrule; (**)
   Pnew = P /. U2l /. zrule /. ζrule;
   (𝔼[Lnew, Qnew, Pnew] //. l2U)
  ];
```

```
In[∘]:= B{} [L_, R_] := L R;
B{is__} [L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | w | x | y)i → vn@i, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ω | Ω | ξ | η)i → vn@i, {i, {is}}]
    ] // TZipJoin@@Table[{τn@i},{i,{is}}] // LZipJoin@@Table[{wn@i,βn@i,an@i},{i,{is}}] //
   QZipJoin@@Table[{ξn@i,yn@i},{i,{is}}] ];(**)
Bis___ [L_, R_] := B{is} [L, R];
```

𝔼 morphisms with domain and range.

```
In[∘]:= Bis_List [𝔼d1_→r1_ [L1_, Q1_, P1_], 𝔼d2_→r2_ [L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis [𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_ [L1_, Q1_, P1_] // 𝔼d2_→r2_ [L2_, Q2_, P2_] :=
  Br1⋂d2 [𝔼d1→r1 [L1, Q1, P1], 𝔼d2→r2 [L2, Q2, P2]];
𝔼d1_→r1_ [L1_, Q1_, P1_] ≡ 𝔼d2_→r2_ [L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_ [L1_, Q1_, P1_] 𝔼d2_→r2_ [L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼dr_ [L_, Q_, P_]$k_ := 𝔼dr @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

𝔼[Λ]

```
In[∘]:= 𝔼dr_ [Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, e^(Λ-Λ0)]$k /. l2U]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
In[∘]:= SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
     SD[opnisp,$k_Integer, Block[{i, j, k}, opisp,$k = ℰ; opnis,$k]];
     SD[opisp, op{is},$k]; SD[opsis__, op{sis}];
    ] /. {SD → SetDelayed,
     isp → {is} /. {i → i_, j → j_, k → k_},
     nis → {is} /. {i → ii, j → jj, k → kk},
     nisp → {is} /. {i → ii_, j → jj_, k → kk_}
    }] ]]
```

Symmetric Algebra Objects

In[∘]:=

$$\mathtt{sm}_{i\_,j\_\to k\_} := \mathbb{E}_{\{i,j\}\to\{k\}}[b_k\,(\beta_i+\beta_j)+t_k\,(\tau_i+\tau_j)+a_k\,(\alpha_i+\alpha_j)+y_k\,(\eta_i+\eta_j)+x_k\,(\xi_i+\xi_j)];$$
$$\mathtt{s\Delta}_{i\_\to j\_,k\_} := \mathbb{E}_{\{i\}\to\{j,k\}}\left[\beta_i\,\left(b_j+b_k\right)+\tau_i\,\left(t_j+t_k\right)+\alpha_i\,(a_j+a_k)+\eta_i\,(y_j+y_k)+\xi_i\,(x_j+x_k)\right];$$
$$\mathtt{sS}_{i\_} := \mathbb{E}_{\{i\}\to\{i\}}[-\beta_i\,b_i-\tau_i\,t_i-\alpha_i\,a_i-\eta_i\,y_i-\xi_i\,x_i];$$
$$\mathtt{s\epsilon}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}[0];$$
$$\mathtt{s\eta}_{i\_} := \mathbb{E}_{\{i\}\to\{\}}[0];$$

In[∘]:=

$$\mathtt{s\sigma}_{i\_\to j\_} := \mathbb{E}_{\{i\}\to\{j\}}[\beta_i\,b_j+\tau_i\,t_j+\alpha_i\,a_j+\eta_i\,y_j+\xi_i\,x_j];$$
$$\mathtt{s\Upsilon}_{i\_\to j\_,k\_,l\_,m\_} := \mathbb{E}_{\{i\}\to\{j,k,l,m\}}[\beta_i\,b_k+\tau_i\,t_k+\alpha_i\,a_l+\eta_i\,y_j+\xi_i\,x_m];$$

Booting Up QU

In[∘]:=

$$\mathtt{Define}[\mathtt{a\sigma}_{i\to j}=\mathbb{E}_{\{i\}\to\{j\}}[a_j\,\alpha_i+x_j\,\xi_i],\ \mathtt{b\sigma}_{i\to j}=\mathbb{E}_{\{i\}\to\{j\}}[b_j\,\beta_i+y_j\,\eta_i]]$$

In[∘]:=

$$\mathtt{Define}\left[\mathtt{am}_{i,j\to k}=\mathbb{E}_{\{i,j\}\to\{k\}}\left[(\alpha_i+\alpha_j)\,a_k+\left(\mathcal{A}_j^{-1}\,\xi_i+\xi_j\right)\,x_k\right],\right.$$
$$\left.\mathtt{bm}_{i,j\to k}=\mathbb{E}_{\{i,j\}\to\{k\}}\left[(\beta_i+\beta_j)\,b_k+\left(\eta_i+e^{-\epsilon\,\beta_i}\,\eta_j\right)\,y_k\right]\right]$$

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$.

$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=

$$\mathtt{Define}\left[R_{i,j}=\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\,a_j\,b_i+\sum_{k=1}^{\$k+1}\frac{\left(1-e^{\gamma\,\epsilon\,\hbar}\right)^k\,(\hbar\,y_i\,x_j)^k}{k\,\left(1-e^{k\,\gamma\,\epsilon\,\hbar}\right)}\right],\right.$$
$$\overline{R}_{i,j}=\mathtt{CF}@\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\,a_j\,b_i,\ -\hbar\,x_j\,y_i\,/\,B_i,\ 1+\mathtt{If}\left[\$k==0,\ 0,\ (\overline{R}_{\{i,j\},\$k-1})_{\$k}[3]-\right.\right.$$
$$\left.\left.\left(\left(\left(\overline{R}_{\{i,j\},0}\right)_{\$k}\,R_{1,2}\,\left(\overline{R}_{\{3,4\},\$k-1}\right)_{\$k}\right)\ //\ \left(\mathtt{bm}_{i,1\to i}\,\mathtt{am}_{j,2\to j}\right)\ //\ \left(\mathtt{bm}_{i,3\to i}\,\mathtt{am}_{j,4\to j}\right)\right)[3]\right]\right],$$
$$P_{i,j}=\mathbb{E}_{\{i,j\}\to\{\}}[\beta_i\,\alpha_j\,/\,\hbar,\ \eta_i\,\xi_j\,/\,\hbar,\ 1+\mathtt{If}[\$k==0,\ 0,\ (P_{\{i,j\},\$k-1})_{\$k}[3]-$$
$$\left.(R_{1,2}\ //\ ((P_{\{1,j\},0})_{\$k}\,(P_{\{i,2\},\$k-1})_{\$k}))[3]]]\right]$$

In[∘]:=

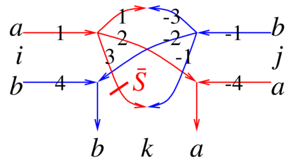
$$\mathtt{Define}\left[\mathtt{aS}_i=\left(\mathtt{a\sigma}_{i\to 2}\,\overline{R}_{1,i}\right)\ //\ P_{1,2},\right.$$
$$\overline{\mathtt{aS}}_i=\mathbb{E}_{\{i\}\to\{i\}}\left[-a_i\,\alpha_i,\ -x_i\,\mathcal{A}_i\,\xi_i,\ 1+\mathtt{If}\left[\$k==0,\ 0,\ (\overline{\mathtt{aS}}_{\{i\},\$k-1})_{\$k}[3]-\right.\right.$$
$$\left.\left.\left.\left((\overline{\mathtt{aS}}_{\{i\},0})_{\$k}\ //\ \mathtt{aS}_i\ //\ (\overline{\mathtt{aS}}_{\{i\},\$k-1})_{\$k}\right)[3]\right]\right]\right]$$

(was $\mathrm{aS}_j=\overline{R}_{i,j}\sim B_i\sim P_{i,j}$).

In[∘]:=

$$\mathtt{Define}\left[\mathtt{bS}_i=\mathtt{b\sigma}_{i\to 1}\,R_{i,2}\ //\ \mathtt{aS}_2\ //\ P_{1,2},\right.$$
$$\overline{\mathtt{bS}}_i=\mathtt{b\sigma}_{i\to 1}\,R_{i,2}\ //\ \overline{\mathtt{aS}}_2\ //\ P_{1,2},$$
$$\mathtt{a\Delta}_{i\to j,k}=(R_{1,j}\,R_{2,k})\ //\ \mathtt{bm}_{1,2\to 3}\ //\ P_{3,i},$$
$$\left.\mathtt{b\Delta}_{i\to j,k}=(R_{j,1}\,R_{k,2})\ //\ \mathtt{am}_{1,2\to 3}\ //\ P_{i,3}\right]$$

(was $bS_i = R_{i,1} \sim B_1 \sim aS_1 \sim B_1 \sim P_{i,1}$, $\overline{bS}_i = R_{i,1} \sim B_1 \sim \overline{aS}_1 \sim B_1 \sim P_{i,1}$).

The Drinfel'd double: 

In[∘]:=
```
Define[
  dm_{i,j→k} = ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // \overline{aS}_3) (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{-1,3} P_{-3,1} am_{2,-4→k} bm_{4,-2→k})]
```

In[∘]:=
```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
  dϵ_i = sϵ_i, dη_i = sη_i,
  dS_i = sΥ_{i→1,1,2,2} // (\overline{bS}_1 aS_2) // dm_{2,1→i},
  \overline{dS}_i = sΥ_{i→1,1,2,2} // (bS_1 \overline{aS}_2) // dm_{2,1→i},
  dΔ_{i→j,k} = (bΔ_{i→3,1} aΔ_{i→2,4}) // (dm_{3,4→k} dm_{1,2→j})]
```

In[∘]:=
```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} ℯ^{-ℏ ϵ a_i/2}]_{$k},
  \overline{C}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} ℯ^{ℏ ϵ a_i/2}]_{$k},
  c_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/4} ℯ^{-ℏ ϵ a_i/4}]_{$k},
  \overline{c}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/4} ℯ^{ℏ ϵ a_i/4}]_{$k},
  Kink_i = (R_{1,3} \overline{C}_2) // dm_{1,2→1} // dm_{1,3→i},
  \overline{Kink}_i = (\overline{R}_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i},
  ρ_i = (c_1 \overline{c}_3 dS_i) // dm_{1,i→i} // dm_{i,3→i}] (*ρ reverses a strand*)
```

Note. $t = -\epsilon a + \gamma b$ and $b = t/\gamma + \epsilon a/\gamma$

In[∘]:=
```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i + t_i) / γ + ξ_i x_i + η_i y_i],
  t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (-ϵ a_i + γ b_i) + ξ_i x_i + η_i y_i]]
```

The t-Tensors

In[∘]:=
```
Define[tR_{i,j} = R_{i,j} // (b2t_i b2t_j),
  \overline{tR}_{i,j} = \overline{R}_{i,j} // (b2t_i b2t_j) ,
  tm_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k ),
  tC_i = (C_i // b2t_i ),
  \overline{tC}_i = (\overline{C}_i // b2t_i ),
  tKink_i = Kink_i // b2t_i ,
  \overline{tKink}_i = \overline{Kink}_i // b2t_i ,
  tΔ_{i→j,k} = t2b_i // dΔ_{i->j,k} // (b2t_j b2t_k),
  tS_i = t2b_i // dS_i // b2t_i]
```

In[∘]:=
```
tΔ_{i→j,k} // tS_k // tm_{j,k→i}
```

Out[∘]=
$$\mathbb{E}_{\{i\}\to\{i\}}\left[0,\ 0,\ 1+O[\epsilon]^1\right]$$

In[∘]:=
$\left(\overline{\text{tKink}}_8\ \overline{\text{tKink}}_9\ \overline{\text{tKink}}_{10}\ \text{tR}_{5,1}\ \text{tR}_{2,6}\ \text{tR}_{7,3}\ \text{tC}_4\ //\ \text{tm}_{1,2\to1}\ //\ \text{tm}_{1,3\to1}\ //\ \text{tm}_{1,4\to1}\ //\ \text{tm}_{1,5\to1}\ //\ \text{tm}_{1,6\to1}\ //\ \text{tm}_{1,7\to1}\ //\ \text{tm}_{1,8\to1}\ //\ \text{tm}_{1,9\to1}\ //\ \text{tm}_{1,10\to1}\right)\ //.\ \text{l2U}\ //\ \text{CF}$

Out[∘]=
$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T_1}{1-T_1+T_1^2}+O[\epsilon]^1\right]$

The Knot Tensors

In[∘]:=
$\text{Define}\Big[\text{kR}_{i,j}=\left(R_{i,j}\ //\ \left(\text{b2t}_i\ \text{b2t}_j\right)\ /.\ t_{i|j}\to t\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\overline{\text{kR}}_{i,j}=\left(\overline{R}_{i,j}\ //\ \left(\text{b2t}_i\ \text{b2t}_j\right)\ /.\ \{t_{i|j}\to t,\ T_{i|j}\to T\}\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\text{km}_{i,j\to k}=\left(\left(\left(\text{t2b}_i\ \text{t2b}_j\right)\ //\ \text{dm}_{i,j\to k}\ //\ \text{b2t}_k\right)\ /.\ \{t_k\to t,\ T_k\to T,\ \tau_{i|j}\to 0\}\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\text{kC}_i=\left(\left(C_i\ //\ \text{b2t}_i\right)\ /.\ T_i\to T\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\overline{\text{kC}}_i=\left(\left(\overline{C}_i\ //\ \text{b2t}_i\right)\ /.\ T_i\to T\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\text{kKink}_i=\left(\text{Kink}_i\ //\ \text{b2t}_i\ /.\ \{t_i\to t,\ T_i\to T\}\right)\ //.\ \text{l2U}\ //\ \text{CF},$
$\overline{\text{kKink}}_i=\left(\overline{\text{Kink}}_i\ //\ \text{b2t}_i\ /.\ \{t_i\to t,\ T_i\to T\}\right)\ //.\ \text{l2U}\ //\ \text{CF}\Big]$

In[∘]:= $R_{i,j}$

Out[∘]=
$\mathbb{E}_{\{\}\to\{i,j\}}\left[a_j\,b_i,\ x_j\,y_i,\ 1-\frac{1}{4}\left(x_j^2\,y_i^2\right)\epsilon+\left(\frac{1}{9}x_j^3\,y_i^3+\frac{1}{32}x_j^4\,y_i^4\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= $\text{kR}_{i,j}$

Out[∘]=
$\mathbb{E}_{\{\}\to\{i,j\}}\left[t\,a_j,\ x_j\,y_i,\ 1+\left(a_i\,a_j-\frac{1}{4}x_j^2\,y_i^2\right)\epsilon+\left(\frac{1}{2}a_i^2\,a_j^2-\frac{1}{4}a_i\,a_j\,x_j^2\,y_i^2+\frac{1}{9}x_j^3\,y_i^3+\frac{1}{32}x_j^4\,y_i^4\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= kKink_1

Out[∘]=
$\mathbb{E}_{\{\}\to\{1\}}\Big[t\,a_1,\ x_1\,y_1,\ \frac{1}{\sqrt{T}}+\left(\frac{a_1}{\sqrt{T}}+\frac{a_1^2}{\sqrt{T}}-\frac{x_1^2\,y_1^2}{4\sqrt{T}}\right)\epsilon+$
$\left(\frac{a_1^2}{2\sqrt{T}}+\frac{a_1^3}{\sqrt{T}}+\frac{a_1^4}{2\sqrt{T}}-\frac{a_1\,x_1^2\,y_1^2}{4\sqrt{T}}-\frac{a_1^2\,x_1^2\,y_1^2}{4\sqrt{T}}+\frac{x_1^3\,y_1^3}{9\sqrt{T}}+\frac{x_1^4\,y_1^4}{32\sqrt{T}}\right)\epsilon^2+O[\epsilon]^3\Big]$

In[∘]:=
```
kKink̄_8 kKink̄_9 kKink̄_10 kR_{5,1} kR_{2,6} kR_{7,3} kC_4 // km_{1,2→1} // km_{1,3→1} // km_{1,4→1} // km_{1,5→1} // km_{1,6→1} //
  km_{1,7→1} // km_{1,8→1} // km_{1,9→1} // km_{1,10→1} // Timing
```

Out[∘]=

$$\left\{12.2,\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{-T^2+2T^3-3T^4+2T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T+2T^3\right)a_1}{1-2T+3T^2-2T^3+T^4}+\frac{\left(-2T-2T^2\right)x_1 y_1}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+\left(\frac{T^2-2T^3+4T^4-2T^5+6T^7-11T^8+4T^9}{2-10T+30T^2-60T^3+90T^4-102T^5+90T^6-60T^7+30T^8-10T^9+2T^{10}}+\frac{\left(4T^2-10T^3+16T^4-14T^5-4T^6+4T^7\right)a_1}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\frac{\left(2T+2T^2-12T^3+2T^4+2T^5\right)a_1^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T+2T^2+4T^3+4T^4+2T^5-2T^6\right)x_1 y_1}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\frac{\left(4T+12T^2-12T^3-8T^4\right)a_1 x_1 y_1}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(3T+9T^2+3T^3\right)x_1^2 y_1^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\epsilon^2+O[\epsilon]^3\right]\right\}$$

a2w and w2a and q-Casimir

In[∘]:=
```
CenteqSol1 = {}; CenteqSol2 = {}; KinkSol1 = {}; KinkSol2 = {};
```

In[]:= **(*Working Casimir, not unique! But has the right specialization when ϵ=0.*)**

$$\mathtt{Cas}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\Big[0,\,0,\,\mathtt{Series}\Big[y\,e^{\epsilon\hbar a}\,x + \frac{e^{\hbar\epsilon\,(a+1)} + e^{-\hbar\epsilon a}\,T}{\hbar\,(e^{\hbar\epsilon}-1)} - \frac{1+T}{\hbar\,(e^{\hbar\epsilon}-1)} - \frac{1+T}{2} + \epsilon\,\mathtt{Sum}[\theta1_{u,v}\,y^v\,a^{u-v}\,x^v,\ \{u,0,2\},\{v,0,u\}] + \epsilon^2\,\mathtt{Sum}[\theta2_{u,v}\,y^v\,a^{u-v}\,x^v,\ \{u,0,4\},\{v,0,u\}],\ \{\epsilon,0,\$k\}\Big]\ /.\ \mathtt{CenteqSol1}\ /.\ \mathtt{CenteqSol2}\ /.\ \mathtt{KinkSol1}\ /.\ \{a\to a_i,\ T\to T,\ x\to x_i,\ y\to y_i\}\Big]$$

$$\mathtt{Cas}_i$$

Out[]=

$$\mathbb{E}_{\{\}\to\{i\}}\Big[0,\,0,\,\left(\frac{1}{2}-\frac{T}{2}+a_i-T\,a_i+x_i\,y_i\right)+\frac{1}{2}\left(a_i+T\,a_i+a_i^2+T\,a_i^2+2\,a_i\,x_i\,y_i+2\,\theta1_{0,0}+2\,a_i\,\theta1_{1,0}+2\,x_i\,y_i\,\theta1_{1,1}+2\,a_i^2\,\theta1_{2,0}+2\,a_i\,x_i\,y_i\,\theta1_{2,1}+2\,x_i^2\,y_i^2\,\theta1_{2,2}\right)\epsilon+$$
$$\Bigg(\frac{a_i}{12}-\frac{T\,a_i}{12}+\frac{a_i^2}{4}-\frac{T\,a_i^2}{4}+\frac{a_i^3}{6}-\frac{T\,a_i^3}{6}+\frac{1}{2}\,a_i^2\,x_i\,y_i+\theta2_{0,0}+x_i\,y_i\,\theta2_{1,1}+a_i\,x_i\,y_i\,\theta2_{2,1}+a_i^2\,x_i\,y_i\,\theta2_{3,1}+$$
$$a_i\left(-\frac{2\,T-t\,T-2\,T^2-t\,T^2}{3\,t\,(-1+T)}-(-1+T)\,\theta2_{1,1}-\frac{1}{2}\,(-1+T)\,\theta2_{2,1}-\frac{1}{6}\,(-1+T)\,\theta2_{3,1}\right)+$$
$$a_i\,x_i^2\,y_i^2\left(-\frac{-2+t+2\,t\,T+2\,T^2+t\,T^2}{2\,t\,(-1+T)^3}-\frac{\theta2_{3,1}}{-1+T}-\theta2_{4,2}\right)+$$
$$x_i^2\,y_i^2\left(-\frac{2-t-2\,t\,T-2\,T^2-t\,T^2}{4\,t\,(-1+T)^3}-\frac{\theta2_{2,1}}{2\,(-1+T)}+\frac{\theta2_{3,1}}{2\,(-1+T)}+\frac{\theta2_{4,2}}{3}\right)-$$
$$\frac{1}{6}\,(-1+2\,T-T^2)\,a_i^4\,\theta2_{4,2}-\frac{2}{3}\,(-1+T)\,a_i^3\,x_i\,y_i\,\theta2_{4,2}+a_i^2\,x_i^2\,y_i^2\,\theta2_{4,2}-\frac{2\,a_i\,x_i^3\,y_i^3\,\theta2_{4,2}}{3\,(-1+T)}+$$
$$\frac{x_i^4\,y_i^4\,\theta2_{4,2}}{6\,(-1+T)^2}+x_i^3\,y_i^3\left(-\frac{6-3\,t-8\,T-2\,t\,T+2\,T^2+t\,T^2}{6\,t\,(-1+T)^4}+\frac{\theta2_{3,1}}{3\,(-1+T)^2}+\frac{2\,\theta2_{4,2}}{3\,(-1+T)}\right)+$$
$$a_i^3\left(\frac{2\,(-2\,T+t\,T+2\,T^2+t\,T^2)}{3\,t\,(-1+T)}-\frac{1}{3}\,(-1+T)\,\theta2_{3,1}-\frac{1}{3}\,(-1+2\,T-T^2)\,\theta2_{4,2}\right)+$$
$$a_i^2\left(-\frac{2\,T-t\,T-2\,T^2-t\,T^2}{t\,(-1+T)}-\frac{1}{2}\,(-1+T)\,\theta2_{2,1}-\frac{1}{2}\,(-1+T)\,\theta2_{3,1}-\frac{1}{6}\,(-1+2\,T-T^2)\,\theta2_{4,2}\right)\Bigg)\epsilon^2+O[\epsilon]^3\Big]$$

In[]:=

$$\theta1s = \mathtt{Flatten@Table}[\theta1_{u,v},\ \{u,0,2\},\ \{v,0,u\}]$$
$$\theta2s = \mathtt{Flatten@Table}[\theta2_{u,v},\ \{u,0,4\},\ \{v,0,u\}]$$

Out[]= $\{\theta1_{0,0},\ \theta1_{1,0},\ \theta1_{1,1},\ \theta1_{2,0},\ \theta1_{2,1},\ \theta1_{2,2}\}$

Out[]= $\{\theta2_{0,0},\ \theta2_{1,0},\ \theta2_{1,1},\ \theta2_{2,0},\ \theta2_{2,1},\ \theta2_{2,2},\ \theta2_{3,0},\ \theta2_{3,1},\ \theta2_{3,2},\ \theta2_{3,3},\ \theta2_{4,0},\ \theta2_{4,1},\ \theta2_{4,2},\ \theta2_{4,3},\ \theta2_{4,4}\}$

In[]:=

$$(\mathtt{Cas}_i\ //\ \mathtt{km}_{i,j\to k}) \equiv (\mathtt{Cas}_i\ //\ \mathtt{km}_{j,i\to k})$$
$$\mathtt{Centeq1} = \%[[1]];$$

In[∘]:=
$$\text{CenteqSol1} = \text{Solve}\Big[\text{Thread}\Big[\Big\{\text{Coefficient}\big[\text{Centeq1}, \epsilon\, y_k^2\, \eta_j^2\big], \text{Coefficient}[\text{Centeq1}, \epsilon\, a_k\, y_k\, \eta_j], \text{Coefficient}[\text{Centeq1}, \epsilon\, y_k\, \eta_j] \;/.\; a_k \to 0, \text{Coefficient}\big[\text{Centeq1}, \epsilon\, \mathcal{A}_j^{-1}\, y_k^2\, x_k\, \eta_j\big]\Big\} == 0\Big], \theta 1s\Big][[1]]$$

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=
$$\left\{\theta 1_{1,0} \to -(-1+T)\,\theta 1_{1,1} + \theta 1_{2,0},\ \theta 1_{2,1} \to -\frac{2\,\theta 1_{2,0}}{-1+T},\ \theta 1_{2,2} \to \frac{\theta 1_{2,0}}{(-1+T)^2}\right\}$$

In[∘]:=
$$(\text{Cas}_i \,//\, \text{km}_{i,j\to k}) \equiv (\text{Cas}_i \,//\, \text{km}_{j,i\to k});$$
$$\text{Centeq2} = \%[[1]] \,//\, \text{Simplify} \,//\, \text{Numerator} \,//\, \text{Expand}$$

In[∘]:=
$$\text{Thread}\Big[\Big\{\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 x_k^2 y_k^3 \eta_j\big] \;/.\; \mathcal{A}_j \to 0,$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^4 \mathcal{A}_j^3 \eta_j^4\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^4 x_k \mathcal{A}_j^2 \eta_j^3\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^3 \mathcal{A}_j^2 \eta_j^2 x_k a_k\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^3 x_k^2 a_k \mathcal{A}_j \eta_j\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^3 x_k^2 \mathcal{A}_j \eta_j\big] \;/.\; a_k \to 0,$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 a_k y_k^2 x_k \mathcal{A}_j^2 \eta_j\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 \mathcal{A}_j^3 a_k y_k \eta_j\big],$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k \eta_j \mathcal{A}_j^3\big] \;/.\; a_k \to 0,$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 y_k^2 \mathcal{A}_j^2 x_k \eta_j\big] \;/.\; a_k \to 0,$$
$$\text{Coefficient}\big[\text{Centeq2}, \epsilon^2 a_k^2 y_k \mathcal{A}_j^3 \eta_j\big]\Big\} == 0\Big]$$

Out[∘]=
$$\{\text{True},\ 4 t\,\theta 2_{4,0} - 8 t T\,\theta 2_{4,0} + 4 t T^2\,\theta 2_{4,0} - 4 t\,\theta 2_{4,1} + 12 t T\,\theta 2_{4,1} - 12 t T^2\,\theta 2_{4,1} + 4 t T^3\,\theta 2_{4,1} + 4 t\,\theta 2_{4,2} - 16 t T\,\theta 2_{4,2} + 24 t T^2\,\theta 2_{4,2} - 16 t T^3\,\theta 2_{4,2} + 4 t T^4\,\theta 2_{4,2} - 4 t\,\theta 2_{4,3} + 20 t T\,\theta 2_{4,3} - 40 t T^2\,\theta 2_{4,3} + 40 t T^3\,\theta 2_{4,3} - 20 t T^4\,\theta 2_{4,3} + 4 t T^5\,\theta 2_{4,3} + 4 t\,\theta 2_{4,4} - 24 t T\,\theta 2_{4,4} + 60 t T^2\,\theta 2_{4,4} - 80 t T^3\,\theta 2_{4,4} + 60 t T^4\,\theta 2_{4,4} - 24 t T^5\,\theta 2_{4,4} + 4 t T^6\,\theta 2_{4,4} == 0,$$
$$-4 t\,\theta 2_{4,1} + 8 t T\,\theta 2_{4,1} - 4 t T^2\,\theta 2_{4,1} + 8 t\,\theta 2_{4,2} - 24 t T\,\theta 2_{4,2} + 24 t T^2\,\theta 2_{4,2} - 8 t T^3\,\theta 2_{4,2} - 12 t\,\theta 2_{4,3} + 48 t T\,\theta 2_{4,3} - 72 t T^2\,\theta 2_{4,3} + 48 t T^3\,\theta 2_{4,3} - 12 t T^4\,\theta 2_{4,3} + 16 t\,\theta 2_{4,4} - 80 t T\,\theta 2_{4,4} + 160 t T^2\,\theta 2_{4,4} - 160 t T^3\,\theta 2_{4,4} + 80 t T^4\,\theta 2_{4,4} - 16 t T^5\,\theta 2_{4,4} == 0,$$
$$12 t\,\theta 2_{4,1} - 24 t T\,\theta 2_{4,1} + 12 t T^2\,\theta 2_{4,1} - 16 t\,\theta 2_{4,2} + 48 t T\,\theta 2_{4,2} - 48 t T^2\,\theta 2_{4,2} + 16 t T^3\,\theta 2_{4,2} + 12 t\,\theta 2_{4,3} - 48 t T\,\theta 2_{4,3} + 72 t T^2\,\theta 2_{4,3} - 48 t T^3\,\theta 2_{4,3} + 12 t T^4\,\theta 2_{4,3} == 0,$$
$$-8 t\,\theta 2_{4,2} + 16 t T\,\theta 2_{4,2} - 8 t T^2\,\theta 2_{4,2} + 12 t\,\theta 2_{4,3} - 36 t T\,\theta 2_{4,3} + 36 t T^2\,\theta 2_{4,3} - 12 t T^3\,\theta 2_{4,3} == 0,$$
$$8 - 4 t - 8 T - 4 t T - 4 t\,\theta 2_{3,2} + 8 t T\,\theta 2_{3,2} - 4 t T^2\,\theta 2_{3,2} + 12 t\,\theta 2_{3,3} - 36 t T\,\theta 2_{3,3} + 36 t T^2\,\theta 2_{3,3} - 12 t T^3\,\theta 2_{3,3} + 4 t\,\theta 2_{4,2} - 8 t T\,\theta 2_{4,2} + 4 t T^2\,\theta 2_{4,2} == 0,$$
$$8 - 4 t - 8 t T - 8 T^2 - 4 t T^2 - 8 t\,\theta 2_{3,1} + 16 t T\,\theta 2_{3,1} - 8 t T^2\,\theta 2_{3,1} + 8 t\,\theta 2_{3,2} - 24 t T\,\theta 2_{3,2} + 24 t T^2\,\theta 2_{3,2} - 8 t T^3\,\theta 2_{3,2} + 12 t\,\theta 2_{4,1} - 24 t T\,\theta 2_{4,1} + 12 t T^2\,\theta 2_{4,1} == 0,$$
$$-8 t\,\theta 2_{2,0} + 16 t T\,\theta 2_{2,0} - 8 t T^2\,\theta 2_{2,0} + 4 t\,\theta 2_{2,1} - 12 t T\,\theta 2_{2,1} + 12 t T^2\,\theta 2_{2,1} - 4 t T^3\,\theta 2_{2,1} + 12 t\,\theta 2_{3,0} - 24 t T\,\theta 2_{3,0} + 12 t T^2\,\theta 2_{3,0} - 16 t\,\theta 2_{4,0} + 32 t T\,\theta 2_{4,0} - 16 t T^2\,\theta 2_{4,0} == 0,$$
$$-4 t\,\theta 2_{1,0} + 8 t T\,\theta 2_{1,0} - 4 t T^2\,\theta 2_{1,0} + 4 t\,\theta 2_{1,1} - 12 t T\,\theta 2_{1,1} + 12 t T^2\,\theta 2_{1,1} - 4 t T^3\,\theta 2_{1,1} + 4 t\,\theta 2_{2,0} - 8 t T\,\theta 2_{2,0} + 4 t T^2\,\theta 2_{2,0} - 4 t\,\theta 2_{3,0} + 8 t T\,\theta 2_{3,0} - 4 t T^2\,\theta 2_{3,0} + 4 t\,\theta 2_{4,0} - 8 t T\,\theta 2_{4,0} + 4 t T^2\,\theta 2_{4,0} == 0,$$
$$-4 + 2 t + 4 t T + 4 T^2 + 2 t T^2 - 4 t\,\theta 2_{2,1} + 8 t T\,\theta 2_{2,1} - 4 t T^2\,\theta 2_{2,1} + 8 t\,\theta 2_{2,2} - 24 t T\,\theta 2_{2,2} + 24 t T^2\,\theta 2_{2,2} - 8 t T^3\,\theta 2_{2,2} + 4 t\,\theta 2_{3,1} - 8 t T\,\theta 2_{3,1} + 4 t T^2\,\theta 2_{3,1} - 4 t\,\theta 2_{4,1} + 8 t T\,\theta 2_{4,1} - 4 t T^2\,\theta 2_{4,1} == 0,$$
$$16 T - 8 t T - 32 T^2 + 16 T^3 + 8 t T^3 - 12 t\,\theta 2_{3,0} + 24 t T\,\theta 2_{3,0} - 12 t T^2\,\theta 2_{3,0} + 4 t\,\theta 2_{3,1} - 12 t T\,\theta 2_{3,1} + 12 t T^2\,\theta 2_{3,1} - 4 t T^3\,\theta 2_{3,1} + 24 t\,\theta 2_{4,0} - 48 t T\,\theta 2_{4,0} + 24 t T^2\,\theta 2_{4,0} == 0\}$$

In[∘]:=
```
CenteqSol2 = Solve[Thread[{Coefficient[Centeq2, ϵ^2 x_k^2 y_k^3 η_j],
    Coefficient[Centeq2, ϵ^2 y_k^4 𝒜_j^3 η_j^4],
    Coefficient[Centeq2, ϵ^2 y_k^4 x_k 𝒜_j^2 η_j^3],
    Coefficient[Centeq2, ϵ^2 y_k^3 𝒜_j^2 η_j^2 x_k a_k],
    Coefficient[Centeq2, ϵ^2 y_k^3 x_k^2 a_k 𝒜_j η_j],
    Coefficient[Centeq2, ϵ^2 y_k^3 x_k^2 𝒜_j η_j] /. a_k → 0,
    Coefficient[Centeq2, ϵ^2 a_k y_k^2 x_k 𝒜_j^2 η_j],
    Coefficient[Centeq2, ϵ^2 𝒜_j^3 a_k y_k η_j],
    Coefficient[Centeq2, ϵ^2 y_k η_j 𝒜_j^3] /. a_k → 0,
    Coefficient[Centeq2, ϵ^2 y_k^2 𝒜_j^2 x_k η_j] /. a_k → 0,
    Coefficient[Centeq2, ϵ^2 a_k^2 y_k 𝒜_j^3 η_j]} == 0], θ2s][[1]]
```

Solve: Equations may not give solutions for all "solve" variables.

Out[∘]=

$$\Big\{\theta 2_{1,0} \to -\frac{2T - tT - 2T^2 - tT^2}{3t(-1+T)} - (-1+T)\,\theta 2_{1,1} - \frac{1}{2}(-1+T)\,\theta 2_{2,1} - \frac{1}{6}(-1+T)\,\theta 2_{3,1},$$

$$\theta 2_{2,0} \to -\frac{2T - tT - 2T^2 - tT^2}{t(-1+T)} - \frac{1}{2}(-1+T)\,\theta 2_{2,1} - \frac{1}{2}(-1+T)\,\theta 2_{3,1} - \frac{1}{6}(-1+2T-T^2)\,\theta 2_{4,2},$$

$$\theta 2_{2,2} \to -\frac{2 - t - 2tT - 2T^2 - tT^2}{4t(-1+T)^3} - \frac{\theta 2_{2,1}}{2(-1+T)} + \frac{\theta 2_{3,1}}{2(-1+T)} + \frac{\theta 2_{4,2}}{3},$$

$$\theta 2_{3,0} \to \frac{2(-2T + tT + 2T^2 + tT^2)}{3t(-1+T)} - \frac{1}{3}(-1+T)\,\theta 2_{3,1} - \frac{1}{3}(-1+2T-T^2)\,\theta 2_{4,2},$$

$$\theta 2_{3,2} \to -\frac{-2 + t + 2tT + 2T^2 + tT^2}{2t(-1+T)^3} - \frac{\theta 2_{3,1}}{-1+T} - \theta 2_{4,2},$$

$$\theta 2_{3,3} \to -\frac{6 - 3t - 8T - 2tT + 2T^2 + tT^2}{6t(-1+T)^4} + \frac{\theta 2_{3,1}}{3(-1+T)^2} + \frac{2\,\theta 2_{4,2}}{3(-1+T)},\ \theta 2_{4,0} \to -\frac{1}{6}(-1+2T-T^2)\,\theta 2_{4,2},$$

$$\theta 2_{4,1} \to -\frac{2}{3}(-1+T)\,\theta 2_{4,2},\ \theta 2_{4,3} \to -\frac{2\,\theta 2_{4,2}}{3(-1+T)},\ \theta 2_{4,4} \to \frac{\theta 2_{4,2}}{6(-1+T)^2}\Big\}$$

In[∘]:=
```
(Cas_i // km_{i,j→k}) ≡ (Cas_i // km_{j,i→k})
```

In[∘]:=
```
Φ1[λ] = Sum[ϕ_{u,v}[λ] y_i^v a_i^(u-v) x_i^v, {u, 0, 2}, {v, 0, u}]
Φ[λ] = 1 + ϵ Φ1[λ]
ϕs = Flatten@ Table[ϕ_{u,v}[λ], {u, 0, 2}, {v, 0, u}]
```

Out[∘]= $\phi_{0,0}[\lambda] + a_i\,\phi_{1,0}[\lambda] + x_i\,y_i\,\phi_{1,1}[\lambda] + a_i^2\,\phi_{2,0}[\lambda] + a_i\,x_i\,y_i\,\phi_{2,1}[\lambda] + x_i^2\,y_i^2\,\phi_{2,2}[\lambda]$

Out[∘]= $1 + \epsilon\,(\phi_{0,0}[\lambda] + a_i\,\phi_{1,0}[\lambda] + x_i\,y_i\,\phi_{1,1}[\lambda] + a_i^2\,\phi_{2,0}[\lambda] + a_i\,x_i\,y_i\,\phi_{2,1}[\lambda] + x_i^2\,y_i^2\,\phi_{2,2}[\lambda])$

Out[∘]= $\{\phi_{0,0}[\lambda],\ \phi_{1,0}[\lambda],\ \phi_{1,1}[\lambda],\ \phi_{2,0}[\lambda],\ \phi_{2,1}[\lambda],\ \phi_{2,2}[\lambda]\}$

In[ ]:=
```
eqns1 = Select[CoefficientRules[
    Coefficient[∂_λ Φ[λ] + (∂_λ ((1 - T) (1/2 + a_i) λ + (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i)) Φ[λ] -
      ((𝔼_{}→{i}[(1 - T) (1/2 + a_i) λ, (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i, Φ[λ]] Cas_j) // km_{i,j→i})[[3]], ϵ],
    {y_i, a_i, x_i}], (#[[1, 1]] + 2 #[[1, 2]] + #[[1, 1]]) ≤ 4 &] /.
  (u_ → v_) :> (v == 0) //. U2l // Simplify
```

Out[ ]=
$$\left\{e^{\left(-1+e^{-t}\right)\lambda}\phi_{2,1}[\lambda]+\phi_{2,2}'[\lambda]==\frac{e^{2\left(t+\left(-1+e^{-t}\right)\lambda\right)}\theta 1_{2,0}}{\left(-1+e^{t}\right)^2},\right.$$
$$2\,e^{\left(-1+e^{-t}\right)\lambda}\phi_{2,0}[\lambda]+\phi_{2,1}'[\lambda]==\frac{e^{\left(-1+e^{-t}\right)\lambda}\left(-1+e^{t}+2\,e^{t}\theta 1_{2,0}\right)}{-1+e^{t}},$$
$$e^{-\lambda}\left(-e^{e^{-t}\lambda}\theta 1_{1,1}+e^{e^{-t}\lambda}\phi_{1,0}[\lambda]-e^{e^{-t}\lambda}\phi_{2,0}[\lambda]+e^{\lambda}\phi_{1,1}'[\lambda]\right)==0,$$
$$\left.1+e^{-t}+2\,\theta 1_{2,0}==2\,\phi_{2,0}'[\lambda],\ 1+e^{-t}+\left(2-2\,e^{-t}\right)\theta 1_{1,1}+2\,\theta 1_{2,0}==2\,\phi_{1,0}'[\lambda],\ \theta 1_{0,0}==\phi_{0,0}'[\lambda]\right\}$$

In[ ]:=
```
Φ[λ] = Φ[λ] /. DSolve[Join[Table[(f /. λ → 0) == 0, {f, ϕs}], eqns1], ϕs, λ][[1]]
```

Out[ ]=
$$1+\epsilon\left(\lambda\,\theta 1_{0,0}+e^{\left(-1+e^{-t}\right)\lambda}\lambda\,x_i\,y_i\,\theta 1_{1,1}+\frac{1}{2}e^{-t}\lambda\,a_i^2\left(1+e^{t}+2\,e^{t}\theta 1_{2,0}\right)+\right.$$
$$\frac{1}{2}e^{-t}\lambda\,a_i\left(1+e^{t}-2\,\theta 1_{1,1}+2\,e^{t}\theta 1_{1,1}+2\,e^{t}\theta 1_{2,0}\right)+\frac{1}{\left(-1+e^{t}\right)^2}a_i\,x_i\,y_i\left(-2\,e^{t}+2\,e^{t+\left(-1+e^{-t}\right)\lambda}-\right.$$
$$\left.e^{\left(-1+e^{-t}\right)\lambda}\lambda+e^{2t+\left(-1+e^{-t}\right)\lambda}\lambda-2\,e^{t+\left(-1+e^{-t}\right)\lambda}\lambda\,\theta 1_{2,0}+2\,e^{2t+\left(-1+e^{-t}\right)\lambda}\lambda\,\theta 1_{2,0}\right)+$$
$$\frac{1}{4\left(-1+e^{t}\right)^3}e^{t}x_i^2\,y_i^2\left(3\,e^{t}-e^{2t}-8\,e^{t+\left(-1+e^{-t}\right)\lambda}+5\,e^{t+2\left(-1+e^{-t}\right)\lambda}+e^{2t+2\left(-1+e^{-t}\right)\lambda}-\right.$$
$$\left.\left.2\,e^{2\left(-1+e^{-t}\right)\lambda}\lambda+2\,e^{2t+2\left(-1+e^{-t}\right)\lambda}\lambda-4\,e^{t+2\left(-1+e^{-t}\right)\lambda}\lambda\,\theta 1_{2,0}+4\,e^{2t+2\left(-1+e^{-t}\right)\lambda}\lambda\,\theta 1_{2,0}\right)\right)$$

In[ ]:=
```
Expw1 =
  (𝔼_{i}→{i}[(1 - T) (a_i + 1/2) ω_i, (e^(ω_i (1-T)) - 1)/(1 - T) e^(-(1-T) ω_i) y_i x_i, Φ[λ]] /. ω_i → t/(1 - T) /. λ → t/(1 - T) //
    Simplify) //. l2U // Simplify
```

Out[ ]=
$$\mathbb{E}_{\{i\}\to\{i\}}\left[t\left(\frac{1}{2}+a_i\right),\ x_i\,y_i,\ 1+\frac{1}{4\left(-1+T\right)^4}\right.$$
$$\epsilon\left(-4\,t\left(-1+T\right)^3\theta 1_{0,0}-4\,t\left(-1+T\right)^3 T\,x_i\,y_i\,\theta 1_{1,1}-2\,t\left(-1+T\right)^3 a_i^2\left(1+T+2\,\theta 1_{2,0}\right)+2\,t\left(1-T\right)^3 a_i\right.$$
$$\left(1+T-2\left(-1+T\right)\theta 1_{1,1}+2\,\theta 1_{2,0}\right)+4\left(-1+T\right)^2 T\,a_i\,x_i\,y_i\left(-2+t+2\,T+t\,T+2\,t\,\theta 1_{2,0}\right)-$$
$$\left.\left.T^{-\frac{4}{-1+T}}\left(-T^{\frac{1}{-1+T}}+T^{\frac{T}{-1+T}}\right)x_i^2\,y_i^2\left(-T^{\frac{3}{-1+T}}+\left(5+2\,t\right)T^{\frac{3T}{-1+T}}+3\,T^{\frac{2+T}{-1+T}}+\left(-7+2\,t\right)T^{\frac{1+2T}{-1+T}}+4\,t\,T^{\frac{1+2T}{-1+T}}\theta 1_{2,0}\right)\right)\right]$$

In[∘]:= **kKink$_i$**

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[t\,a_i,\ x_i\,y_i,\ \frac{1}{\sqrt{T}}+\left(\frac{a_i}{\sqrt{T}}+\frac{a_i^2}{\sqrt{T}}-\frac{x_i^2\,y_i^2}{4\sqrt{T}}\right)\epsilon+\left(\frac{a_i^2}{2\sqrt{T}}+\frac{a_i^3}{\sqrt{T}}+\frac{a_i^4}{2\sqrt{T}}-\frac{a_i\,x_i^2\,y_i^2}{4\sqrt{T}}-\frac{a_i^2\,x_i^2\,y_i^2}{4\sqrt{T}}+\frac{x_i^3\,y_i^3}{9\sqrt{T}}+\frac{x_i^4\,y_i^4}{32\sqrt{T}}\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:= **Coefficient[Expw1[[3]], $x_i^2\,y_i^2\,\epsilon$] == $T^{1/2}$ Coefficient[kKink$_i$[[3]], $x_i^2\,y_i^2\,\epsilon$]**
Coefficient[Expw1[[3]], $a_i^2\,\epsilon$] == $T^{1/2}$ Coefficient[kKink$_i$[[3]], $a_i^2\,\epsilon$]
(Coefficient[Expw1[[3]], $a_i\,\epsilon$] /. $x_i \to 0$) == $T^{1/2}$ Coefficient[kKink$_i$[[3]], $a_i\,\epsilon$]
(Coefficient[Expw1[[3]], ${a_i}^2\,\epsilon$] /. $x_i \to 0$) == $T^{1/2}$ Coefficient[kKink$_i$[[3]], ${a_i}^2\,\epsilon$]
(Coefficient[Expw1[[3]], ϵ] /. {$a_i \to 0$, $x_i \to 0$}) == $T^{1/2}$ Coefficient[kKink$_i$[[3]], ${a_i}^2\,\epsilon$]

Out[∘]= $-\frac{T^{-\frac{4}{-1+T}}\left(-T^{\frac{1}{-1+T}}+T^{\frac{T}{-1+T}}\right)\left(-T^{\frac{3}{-1+T}}+(5+2t)\,T^{\frac{3T}{-1+T}}+3\,T^{\frac{2+T}{-1+T}}+(-7+2t)\,T^{\frac{1+2T}{-1+T}}+4\,t\,T^{\frac{1+2T}{-1+T}}\,\theta 1_{2,0}\right)}{4\,(-1+T)^4} == -\frac{1}{4}$

Out[∘]= $-\frac{t\,(1+T+2\,\theta 1_{2,0})}{2\,(-1+T)} == 1$

Out[∘]= $\frac{t\,(1-T)^3\,(1+T-2\,(-1+T)\,\theta 1_{1,1}+2\,\theta 1_{2,0})}{2\,(-1+T)^4} == 1$

Out[∘]= $-\frac{t\,(1+T+2\,\theta 1_{2,0})}{2\,(-1+T)} == 1$

Out[∘]= $-\frac{t\,\theta 1_{0,0}}{-1+T} == 1$

In[∘]:= **KinkSol1 = Solve[{(Coefficient[Expw1[[3]], $a_i\,\epsilon$] /. $x_i \to 0$) == $T^{1/2}$ Coefficient[kKink$_i$[[3]], $a_i\,\epsilon$],**
(Coefficient[Expw1[[3]], ${a_i}^2\,\epsilon$] /. $x_i \to 0$) == $T^{1/2}$ Coefficient[kKink$_i$[[3]], ${a_i}^2\,\epsilon$],
(Coefficient[Expw1[[3]], ϵ] /. {$a_i \to 0$, $x_i \to 0$}) == 0}, {$\theta 1_{0,0}$, $\theta 1_{1,1}$, $\theta 1_{2,0}$}][[1]]

Out[∘]= $\left\{\theta 1_{0,0}\to 0,\ \theta 1_{1,1}\to 0,\ \theta 1_{2,0}\to -\frac{-2+t+2T+tT}{2t}\right\}$

In[∘]:= **Expw1 /. KinkSol1 //. U2l // Simplify**

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\left[t\left(\frac{1}{2}+a_i\right),\ x_i\,y_i,\ 1+\epsilon\,a_i+\epsilon\,a_i^2-\frac{1}{4}\,\epsilon\,x_i^2\,y_i^2\right]$

In[∘]:= **Cas$_i$ /. KinkSol1**

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\Big[0, 0, \left(\frac{1}{2}-\frac{T}{2}+a_i-T\,a_i+x_i\,y_i\right)+\frac{1}{2}\left(a_i+T\,a_i-\frac{(-2+t+2T+tT)\,a_i}{t}+a_i^2+T\,a_i^2-\frac{(-2+t+2T+tT)\,a_i^2}{t}+2\,a_i\,x_i\,y_i+\frac{2\,(-2+t+2T+tT)\,a_i\,x_i\,y_i}{t\,(-1+T)}-\frac{(-2+t+2T+tT)\,x_i^2\,y_i^2}{t\,(-1+T)^2}\right)\epsilon+\left(\frac{a_i}{12}-\frac{T\,a_i}{12}+\frac{a_i^2}{4}-\frac{T\,a_i^2}{4}+\frac{a_i^3}{6}-\frac{T\,a_i^3}{6}+\frac{1}{2}a_i^2\,x_i\,y_i+\theta2_{0,0}+a_i\,\theta2_{1,0}+x_i\,y_i\,\theta2_{1,1}+a_i^2\,\theta2_{2,0}+a_i\,x_i\,y_i\,\theta2_{2,1}+x_i^2\,y_i^2\,\theta2_{2,2}+a_i^3\,\theta2_{3,0}+a_i^2\,x_i\,y_i\,\theta2_{3,1}+a_i\,x_i^2\,y_i^2\,\theta2_{3,2}+x_i^3\,y_i^3\,\theta2_{3,3}+a_i^4\,\theta2_{4,0}+a_i^3\,x_i\,y_i\,\theta2_{4,1}+a_i^2\,x_i^2\,y_i^2\,\theta2_{4,2}+a_i\,x_i^3\,y_i^3\,\theta2_{4,3}+x_i^4\,y_i^4\,\theta2_{4,4}\right)\epsilon^2+O[\epsilon]^3\Big]$

Out[∘]= { }

In[∘]:= **Φ[λ] = Φ[λ] /. CenteqSol1 /. KinkSol1 //. l2U**

Out[∘]= $1+\epsilon\left(\frac{1}{2}T\left(1+\frac{1}{T}-\frac{-2+t+2T+tT}{t\,T}\right)\lambda\,a_i+\frac{1}{2}T\left(1+\frac{1}{T}-\frac{-2+t+2T+tT}{t\,T}\right)\lambda\,a_i^2+\frac{\left(-\frac{2}{T}+\frac{2\,e^{-\lambda+e^{-t}\lambda}}{T}-e^{-\lambda+e^{-t}\lambda}\lambda+\frac{e^{-\lambda+e^{-t}\lambda}\lambda}{T^2}-\frac{e^{-\lambda+e^{-t}\lambda}(-2+t+2T+tT)\lambda}{t\,T^2}+\frac{e^{-\lambda+e^{-t}\lambda}(-2+t+2T+tT)\lambda}{t\,T}\right)a_i\,x_i\,y_i}{\left(-1+\frac{1}{T}\right)^2}+\frac{1}{4\left(-1+\frac{1}{T}\right)^3T}\left(-\frac{1}{T^2}+\frac{e^{-2\lambda+2e^{-t}\lambda}}{T^2}+\frac{3}{T}-\frac{8\,e^{-\lambda+e^{-t}\lambda}}{T}+\frac{5\,e^{-2\lambda+2e^{-t}\lambda}}{T}-2\,e^{-2\lambda+2e^{-t}\lambda}\lambda+\frac{2\,e^{-2\lambda+2e^{-t}\lambda}\lambda}{T^2}-\frac{2\,e^{-2\lambda+2e^{-t}\lambda}(-2+t+2T+tT)\lambda}{t\,T^2}+\frac{2\,e^{-2\lambda+2e^{-t}\lambda}(-2+t+2T+tT)\lambda}{t\,T}\right)x_i^2\,y_i^2\right)$

In[∘]:= **Collect[Φ[λ] //. l2U // Simplify, ϵ]**

Out[∘]= $1+\epsilon\left(-\frac{(-1+T)\,\lambda\,a_i^2}{t}-\frac{e^{-2\lambda}\left(-8\,e^{\lambda+e^{-t}\lambda}\,t\,T+e^{2\lambda}\,t\,(-1+3T)+e^{2e^{-t}\lambda}\left(t+5\,t\,T+4\,(-1+T)^2\lambda\right)\right)x_i^2\,y_i^2}{4\,t\,(-1+T)^3}-\frac{e^{-\lambda}\,a_i\left(e^{\lambda}(-1+T)^3\lambda-2\left(-e^{\lambda}\,t\,T+e^{e^{-t}\lambda}\left(t\,T+(-1+T)^2\lambda\right)\right)x_i\,y_i\right)}{t\,(-1+T)^2}\right)$

In[∘]:= **Cas$_1$**

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0, 0, \left(\frac{1}{2}-\frac{T}{2}+a_1-T\,a_1+x_1\,y_1\right)+\frac{1}{2}\left(a_1+T\,a_1+a_1^2+T\,a_1^2+2\,a_1\,x_1\,y_1+2\,\theta_{0,0}+2\,x_1\,y_1\,\theta_{1,1}-(-1+T)\,a_1^2\,\theta_{2,1}+2\,a_1\,x_1\,y_1\,\theta_{2,1}-\frac{x_1^2\,y_1^2\,\theta_{2,1}}{-1+T}+2\,a_1\left(-(-1+T)\,\theta_{1,1}-\frac{1}{2}(-1+T)\,\theta_{2,1}\right)\right)\epsilon+O[\epsilon]^2\Big]$

In[∘]:=
```
Φ₂[λ] = Sum[φ_{u,v}[λ] y_i^v a_i^(u-v) x_i^v, {u, 0, 4}, {v, 0, u}]
Φ[λ] += ϵ^2 Φ₂[λ]
φs = Flatten@ Table[φ_{u,v}[λ], {u, 0, 4}, {v, 0, u}]
```

Out[∘]= $\phi_{0,0}[\lambda] + a_i\,\phi_{1,0}[\lambda] + x_i\,y_i\,\phi_{1,1}[\lambda] + a_i^2\,\phi_{2,0}[\lambda] + a_i\,x_i\,y_i\,\phi_{2,1}[\lambda] + x_i^2\,y_i^2\,\phi_{2,2}[\lambda] + a_i^3\,\phi_{3,0}[\lambda] + a_i^2\,x_i\,y_i\,\phi_{3,1}[\lambda] + a_i\,x_i^2\,y_i^2\,\phi_{3,2}[\lambda] + x_i^3\,y_i^3\,\phi_{3,3}[\lambda] + a_i^4\,\phi_{4,0}[\lambda] + a_i^3\,x_i\,y_i\,\phi_{4,1}[\lambda] + a_i^2\,x_i^2\,y_i^2\,\phi_{4,2}[\lambda] + a_i\,x_i^3\,y_i^3\,\phi_{4,3}[\lambda] + x_i^4\,y_i^4\,\phi_{4,4}[\lambda]$

Out[∘]= $$1 + \epsilon\left(\frac{1}{2}\,T\left(1+\frac{1}{T}-\frac{-2+t+2\,T+t\,T}{t\,T}\right)\lambda\,a_i + \frac{1}{2}\,T\left(1+\frac{1}{T}-\frac{-2+t+2\,T+t\,T}{t\,T}\right)\lambda\,a_i^2 + \frac{\left(-\frac{2}{T}+\frac{2\,e^{-\lambda+e^{-t}\lambda}}{T}-e^{-\lambda+e^{-t}\lambda}\lambda+\frac{e^{-\lambda+e^{-t}\lambda}\lambda}{T^2}-\frac{e^{-\lambda+e^{-t}\lambda}(-2+t+2\,T+t\,T)\lambda}{t\,T^2}+\frac{e^{-\lambda+e^{-t}\lambda}(-2+t+2\,T+t\,T)\lambda}{t\,T}\right)a_i\,x_i\,y_i}{\left(-1+\frac{1}{T}\right)^2} + \frac{1}{4\left(-1+\frac{1}{T}\right)^3 T}\left(-\frac{1}{T^2}+\frac{e^{-2\lambda+2e^{-t}\lambda}}{T^2}+\frac{3}{T}-\frac{8\,e^{-\lambda+e^{-t}\lambda}}{T}+\frac{5\,e^{-2\lambda+2e^{-t}\lambda}}{T}-2\,e^{-2\lambda+2e^{-t}\lambda}\lambda+\frac{2\,e^{-2\lambda+2e^{-t}\lambda}\lambda}{T^2}-\frac{2\,e^{-2\lambda+2e^{-t}\lambda}(-2+t+2\,T+t\,T)\lambda}{t\,T^2}+\frac{2\,e^{-2\lambda+2e^{-t}\lambda}(-2+t+2\,T+t\,T)\lambda}{t\,T}\right)x_i^2\,y_i^2\right) + \epsilon^2\left(\phi_{0,0}[\lambda] + a_i\,\phi_{1,0}[\lambda] + x_i\,y_i\,\phi_{1,1}[\lambda] + a_i^2\,\phi_{2,0}[\lambda] + a_i\,x_i\,y_i\,\phi_{2,1}[\lambda] + x_i^2\,y_i^2\,\phi_{2,2}[\lambda] + a_i^3\,\phi_{3,0}[\lambda] + a_i^2\,x_i\,y_i\,\phi_{3,1}[\lambda] + a_i\,x_i^2\,y_i^2\,\phi_{3,2}[\lambda] + x_i^3\,y_i^3\,\phi_{3,3}[\lambda] + a_i^4\,\phi_{4,0}[\lambda] + a_i^3\,x_i\,y_i\,\phi_{4,1}[\lambda] + a_i^2\,x_i^2\,y_i^2\,\phi_{4,2}[\lambda] + a_i\,x_i^3\,y_i^3\,\phi_{4,3}[\lambda] + x_i^4\,y_i^4\,\phi_{4,4}[\lambda]\right)$$

Out[∘]= $\{\phi_{0,0}[\lambda], \phi_{1,0}[\lambda], \phi_{1,1}[\lambda], \phi_{2,0}[\lambda], \phi_{2,1}[\lambda], \phi_{2,2}[\lambda], \phi_{3,0}[\lambda], \phi_{3,1}[\lambda], \phi_{3,2}[\lambda], \phi_{3,3}[\lambda], \phi_{4,0}[\lambda], \phi_{4,1}[\lambda], \phi_{4,2}[\lambda], \phi_{4,3}[\lambda], \phi_{4,4}[\lambda]\}$

In[∘]:=
```
Φ[λ] = Φ[λ] //. U2l
```

In[∘]:=
```
eq2 = ∂_λ Φ[λ] + (∂_λ ((1 - T) (1/2 + a_i) λ + (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i)) Φ[λ] -
   ((𝔼_{}→{i}[(1 - T) (1/2 + a_i) λ, (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i, Φ[λ]] Cas_j) // km_{i,j→i})[[3]];
```

Out[∘]= $Aborted

In[∘]:=
```
eqns2 = Select[CoefficientRules[
      Coefficient[∂_λ Φ[λ] + (∂_λ ((1 - T) (1/2 + a_i) λ + (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i)) Φ[λ] -
        ((𝔼_{}→{i}[(1 - T) (1/2 + a_i) λ, (e^(λ (1-T)) - 1)/(1 - T) e^(-(1-T) λ) y_i x_i, Φ[λ]] Cas_j) // km_{i,j→i})[[3]],
       ϵ^2], {y_i, a_i, x_i}], (#[[1, 1]] + 2 #[[1, 2]] + #[[1, 1]]) ≤ 4 * 2 &] /.
   (u_ → v_) :> (v == 0) //. U2l // Simplify
```

Out[∘]= $$\left\{\frac{1}{(-1+e^t)\,t}\, e^{-\lambda}\left(24\,e^{4\,(t+\lambda)}\,t - 18\,e^{3\,t+4\,\lambda}\,t - 6\,e^{5\,t+4\,\lambda}\,t + 24\,e^{4\,(t+e^{-t}\lambda)}\,t + 144\,e^{2\,(2\,t+\lambda+e^{-t}\lambda)}\,t + 108\,e^{3\,t+\lambda+3\,e^{-t}\lambda}\,t - 96\,e^{4\,t+\lambda+3\,e^{-t}\lambda}\,t - 12\,e^{5\,t+\lambda+3\,e^{-t}\lambda}\,t - 30\,e^{3\,t+4\,e^{-t}\lambda}\,t + 6\,e^{5\,t+4\,e^{-t}\lambda}\,t - 144\,e^{3\,t+2\,(1+e^{-t})\lambda}\,t + \right.\right.$$

$$84\, e^{3t+(3+e^{-t})\lambda}\, t - 96\, e^{4t+(3+e^{-t})\lambda}\, t + 12\, e^{5t+(3+e^{-t})\lambda}\, t - 6\, e^{4(t+\lambda)}\, t^2 - 9\, e^{3t+4\lambda}\, t^2 +$$
$$3\, e^{5t+4\lambda}\, t^2 - 18\, e^{4(t+e^{-t}\lambda)}\, t^2 - 72\, e^{2(2t+\lambda+e^{-t}\lambda)}\, t^2 + 54\, e^{3t+\lambda+3e^{-t}\lambda}\, t^2 + 60\, e^{4t+\lambda+3e^{-t}\lambda}\, t^2 +$$
$$6\, e^{5t+\lambda+3e^{-t}\lambda}\, t^2 - 15\, e^{3t+4e^{-t}\lambda}\, t^2 - 3\, e^{5t+4e^{-t}\lambda}\, t^2 - 72\, e^{3t+2(1+e^{-t})\lambda}\, t^2 + 42\, e^{3t+(3+e^{-t})\lambda}\, t^2 +$$
$$36\, e^{4t+(3+e^{-t})\lambda}\, t^2 - 6\, e^{5t+(3+e^{-t})\lambda}\, t^2 - 72\, e^{4(t+e^{-t}\lambda)}\, \lambda - 24\, e^{2(t+\lambda+e^{-t}\lambda)}\, \lambda - 72\, e^{2(2t+\lambda+e^{-t}\lambda)}\, \lambda +$$
$$48\, e^{2t+\lambda+3e^{-t}\lambda}\, \lambda - 144\, e^{3t+\lambda+3e^{-t}\lambda}\, \lambda + 144\, e^{4t+\lambda+3e^{-t}\lambda}\, \lambda - 48\, e^{5t+\lambda+3e^{-t}\lambda}\, \lambda -$$
$$24\, e^{2t+4e^{-t}\lambda}\, \lambda + 72\, e^{3t+4e^{-t}\lambda}\, \lambda + 24\, e^{5t+4e^{-t}\lambda}\, \lambda + 72\, e^{3t+2(1+e^{-t})\lambda}\, \lambda + 24\, e^{5t+2(1+e^{-t})\lambda}\, \lambda +$$
$$12\, e^{4(t+e^{-t}\lambda)}\, t\, \lambda - 12\, e^{2(t+\lambda+e^{-t}\lambda)}\, t\, \lambda + 12\, e^{2(2t+\lambda+e^{-t}\lambda)}\, t\, \lambda + 24\, e^{2t+\lambda+3e^{-t}\lambda}\, t\, \lambda -$$
$$24\, e^{3t+\lambda+3e^{-t}\lambda}\, t\, \lambda - 24\, e^{4t+\lambda+3e^{-t}\lambda}\, t\, \lambda + 24\, e^{5t+\lambda+3e^{-t}\lambda}\, t\, \lambda - 12\, e^{2t+4e^{-t}\lambda}\, t\, \lambda +$$
$$12\, e^{3t+4e^{-t}\lambda}\, t\, \lambda - 12\, e^{5t+4e^{-t}\lambda}\, t\, \lambda + 12\, e^{3t+2(1+e^{-t})\lambda}\, t\, \lambda - 12\, e^{5t+2(1+e^{-t})\lambda}\, t\, \lambda +$$
$$6\, e^{2t}\, t\, \Big(3\, e^{t+4\lambda}\, t - 4\, e^{2t+4\lambda}\, t + e^{3t+4\lambda}\, t - 11\, e^{t+(3+e^{-t})\lambda}\, t + 12\, e^{2t+(3+e^{-t})\lambda}\, t - e^{3t+(3+e^{-t})\lambda}\, t +$$
$$e^{t+2(1+e^{-t})\lambda}\, (13\, t - 12\, \lambda) + 4\, e^{2t+\lambda+3e^{-t}\lambda}\, (t - 3\, \lambda) - 12\, e^{2(t+\lambda+e^{-t}\lambda)}\, (t - \lambda) + 4\, e^{2(1+e^{-t})\lambda}\, \lambda -$$
$$4\, e^{\lambda+3e^{-t}\lambda}\, \lambda + e^{3t+\lambda+3e^{-t}\lambda}\, (t + 4\, \lambda) - e^{3t+2(1+e^{-t})\lambda}\, (t + 4\, \lambda) + e^{t+\lambda+3e^{-t}\lambda}\, (-5\, t + 12\, \lambda)\Big)\, \theta 1_{2,1} -$$
$$6\, e^{t+2\lambda}\, (-1 + e^t)^2\, t\, \Big(e^{2(t+\lambda)}\, t - 3\, e^{t+2\lambda}\, t + 8\, e^{t+\lambda+e^{-t}\lambda}\, t - 4\, e^{2e^{-t}\lambda}\, \lambda - e^{2(t+e^{-t}\lambda)}\, (t + 4\, \lambda) +$$
$$e^{t+2e^{-t}\lambda}\, (-5\, t + 8\, \lambda)\Big)\, \theta 1_{2,2} - 12\, e^{4(t+e^{-t}\lambda)}\, t^2\, \theta 2_{4,2} - 4\, e^{2t+4e^{-t}\lambda}\, t^2\, \theta 2_{4,2} +$$
$$12\, e^{3t+4e^{-t}\lambda}\, t^2\, \theta 2_{4,2} + 4\, e^{5t+4e^{-t}\lambda}\, t^2\, \theta 2_{4,2} + 24\, e^{(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] - 120\, e^{t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] +$$
$$240\, e^{2t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] - 240\, e^{3t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] + 120\, e^{4t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] -$$
$$24\, e^{5t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,3}[\lambda] + 24\, e^{4\lambda}\, t^2\, \phi_{4,4}'[\lambda] + 120\, e^{4(t+\lambda)}\, t^2\, \phi_{4,4}'[\lambda] - 120\, e^{t+4\lambda}\, t^2\, \phi_{4,4}'[\lambda] +$$
$$240\, e^{2t+4\lambda}\, t^2\, \phi_{4,4}'[\lambda] - 240\, e^{3t+4\lambda}\, t^2\, \phi_{4,4}'[\lambda] - 24\, e^{5t+4\lambda}\, t^2\, \phi_{4,4}'[\lambda]\Big) == 0,$$

$$\frac{1}{(-1 + e^t)\, t}\, e^{-\lambda}\, \Big(-6\, e^{4(t+\lambda)}\, t - 42\, e^{2t+4\lambda}\, t + 48\, e^{3t+4\lambda}\, t - 150\, e^{2(t+\lambda+e^{-t}\lambda)}\, t + 6\, e^{2(2t+\lambda+e^{-t}\lambda)}\, t +$$
$$54\, e^{2t+\lambda+3e^{-t}\lambda}\, t - 48\, e^{3t+\lambda+3e^{-t}\lambda}\, t - 6\, e^{4t+\lambda+3e^{-t}\lambda}\, t + 144\, e^{3t+2(1+e^{-t})\lambda}\, t + 138\, e^{2t+(3+e^{-t})\lambda}\, t -$$
$$144\, e^{3t+(3+e^{-t})\lambda}\, t + 6\, e^{4t+(3+e^{-t})\lambda}\, t + 3\, e^{4(t+\lambda)}\, t^2 - 21\, e^{2t+4\lambda}\, t^2 - 18\, e^{3t+4\lambda}\, t^2 -$$
$$99\, e^{2(t+\lambda+e^{-t}\lambda)}\, t^2 - 3\, e^{2(2t+\lambda+e^{-t}\lambda)}\, t^2 + 42\, e^{2t+\lambda+3e^{-t}\lambda}\, t^2 + 18\, e^{3t+\lambda+3e^{-t}\lambda}\, t^2 - 54\, e^{3t+2(1+e^{-t})\lambda}\, t^2 +$$
$$78\, e^{2t+(3+e^{-t})\lambda}\, t^2 + 54\, e^{3t+(3+e^{-t})\lambda}\, t^2 + 216\, e^{2(t+\lambda+e^{-t}\lambda)}\, \lambda + 72\, e^{2(2t+\lambda+e^{-t}\lambda)}\, \lambda +$$
$$48\, e^{t+\lambda+3e^{-t}\lambda}\, \lambda - 144\, e^{2t+\lambda+3e^{-t}\lambda}\, \lambda + 144\, e^{3t+\lambda+3e^{-t}\lambda}\, \lambda - 48\, e^{4t+\lambda+3e^{-t}\lambda}\, \lambda - 72\, e^{t+2(1+e^{-t})\lambda}\, \lambda -$$
$$216\, e^{3t+2(1+e^{-t})\lambda}\, \lambda + 24\, e^{t+(3+e^{-t})\lambda}\, \lambda - 72\, e^{2t+(3+e^{-t})\lambda}\, \lambda + 72\, e^{3t+(3+e^{-t})\lambda}\, \lambda - 24\, e^{4t+(3+e^{-t})\lambda}\, \lambda +$$
$$36\, e^{2(t+\lambda+e^{-t}\lambda)}\, t\, \lambda - 36\, e^{2(2t+\lambda+e^{-t}\lambda)}\, t\, \lambda + 36\, e^{t+\lambda+3e^{-t}\lambda}\, t\, \lambda - 60\, e^{2t+\lambda+3e^{-t}\lambda}\, t\, \lambda +$$
$$12\, e^{3t+\lambda+3e^{-t}\lambda}\, t\, \lambda + 12\, e^{4t+\lambda+3e^{-t}\lambda}\, t\, \lambda - 36\, e^{t+2(1+e^{-t})\lambda}\, t\, \lambda + 36\, e^{3t+2(1+e^{-t})\lambda}\, t\, \lambda +$$
$$12\, e^{t+(3+e^{-t})\lambda}\, t\, \lambda - 12\, e^{2t+(3+e^{-t})\lambda}\, t\, \lambda - 12\, e^{3t+(3+e^{-t})\lambda}\, t\, \lambda + 12\, e^{4t+(3+e^{-t})\lambda}\, t\, \lambda +$$
$$3\, e^{t+2\lambda}\, (-1 + e^t)\, t\, \Big(e^{2(t+\lambda)}\, t - 11\, e^{t+2\lambda}\, t + 8\, e^{t+\lambda+e^{-t}\lambda}\, (3\, t - 2\, \lambda) - 12\, e^{2e^{-t}\lambda}\, \lambda +$$
$$8\, e^{\lambda+e^{-t}\lambda}\, \lambda + 8\, e^{2t+\lambda+e^{-t}\lambda}\, \lambda - e^{2(t+e^{-t}\lambda)}\, (t + 12\, \lambda) + e^{t+2e^{-t}\lambda}\, (-13\, t + 24\, \lambda)\Big)\, \theta 1_{2,1} -$$
$$24\, e^{3\lambda}\, (-1 + e^t)^2\, t\, \Big(-e^{t+\lambda}\, t + e^{t+e^{-t}\lambda}\, (t - 2\, \lambda) + e^{e^{-t}\lambda}\, \lambda + e^{2t+e^{-t}\lambda}\, \lambda\Big)\, \theta 1_{2,2} +$$
$$8\, e^{t+\lambda+3e^{-t}\lambda}\, t^2\, \theta 2_{4,2} - 24\, e^{2t+\lambda+3e^{-t}\lambda}\, t^2\, \theta 2_{4,2} + 24\, e^{3t+\lambda+3e^{-t}\lambda}\, t^2\, \theta 2_{4,2} - 8\, e^{4t+\lambda+3e^{-t}\lambda}\, t^2\, \theta 2_{4,2} +$$
$$24\, e^{(3+e^{-t})\lambda}\, t^2\, \phi_{4,2}[\lambda] - 96\, e^{t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,2}[\lambda] + 144\, e^{2t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,2}[\lambda] -$$
$$96\, e^{3t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,2}[\lambda] + 24\, e^{4t+(3+e^{-t})\lambda}\, t^2\, \phi_{4,2}[\lambda] + 12\, e^{4\lambda}\, t^2\, \phi_{4,3}'[\lambda] +$$
$$12\, e^{4(t+\lambda)}\, t^2\, \phi_{4,3}'[\lambda] - 48\, e^{t+4\lambda}\, t^2\, \phi_{4,3}'[\lambda] + 72\, e^{2t+4\lambda}\, t^2\, \phi_{4,3}'[\lambda] - 48\, e^{3t+4\lambda}\, t^2\, \phi_{4,3}'[\lambda]\Big) == 0,$$

$$\frac{1}{(-1 + e^t)\, t}\, e^{-\lambda}\, \Big(120\, e^{3(t+\lambda)}\, t - 90\, e^{2t+3\lambda}\, t - 30\, e^{4t+3\lambda}\, t - 88\, e^{3(t+e^{-t}\lambda)}\, t - 294\, e^{2t+\lambda+2e^{-t}\lambda}\, t +$$
$$264\, e^{3t+\lambda+2e^{-t}\lambda}\, t + 30\, e^{4t+\lambda+2e^{-t}\lambda}\, t + 118\, e^{2t+3e^{-t}\lambda}\, t - 30\, e^{4t+3e^{-t}\lambda}\, t + 258\, e^{2t+(2+e^{-t})\lambda}\, t -$$
$$264\, e^{3t+(2+e^{-t})\lambda}\, t + 6\, e^{4t+(2+e^{-t})\lambda}\, t - 30\, e^{3(t+\lambda)}\, t^2 - 45\, e^{2t+3\lambda}\, t^2 + 15\, e^{4t+3\lambda}\, t^2 + 74\, e^{3(t+e^{-t}\lambda)}\, t^2 -$$
$$147\, e^{2t+\lambda+2e^{-t}\lambda}\, t^2 - 162\, e^{3t+\lambda+2e^{-t}\lambda}\, t^2 - 15\, e^{4t+\lambda+2e^{-t}\lambda}\, t^2 + 59\, e^{2t+3e^{-t}\lambda}\, t^2 + 15\, e^{4t+3e^{-t}\lambda}\, t^2 +$$
$$129\, e^{2t+(2+e^{-t})\lambda}\, t^2 + 126\, e^{3t+(2+e^{-t})\lambda}\, t^2 - 3\, e^{4t+(2+e^{-t})\lambda}\, t^2 + 360\, e^{3(t+e^{-t}\lambda)}\, \lambda - 120\, e^{t+\lambda+2e^{-t}\lambda}\, \lambda +$$
$$360\, e^{2t+\lambda+2e^{-t}\lambda}\, \lambda - 360\, e^{3t+\lambda+2e^{-t}\lambda}\, \lambda + 120\, e^{4t+\lambda+2e^{-t}\lambda}\, \lambda + 120\, e^{t+3e^{-t}\lambda}\, \lambda - 360\, e^{2t+3e^{-t}\lambda}\, \lambda -$$

$$120\, e^{4t+3e^{-t}\lambda}\lambda - 60\, e^{3(t+e^{-t}\lambda)}\, t\,\lambda - 60\, e^{t+\lambda+2e^{-t}\lambda}\, t\,\lambda + 60\, e^{2t+\lambda+2e^{-t}\lambda}\, t\,\lambda + 60\, e^{3t+\lambda+2e^{-t}\lambda}\, t\,\lambda -$$
$$60\, e^{4t+\lambda+2e^{-t}\lambda}\, t\,\lambda + 60\, e^{t+3e^{-t}\lambda}\, t\,\lambda - 60\, e^{2t+3e^{-t}\lambda}\, t\,\lambda + 60\, e^{4t+3e^{-t}\lambda}\, t\,\lambda + 6\, e^{2t}\left(e^{\lambda} - e^{e^{-t}\lambda}\right) t$$
$$\left(e^{2(t+\lambda)}\, t - 3\, e^{t+2\lambda}\, t + 8\, e^{t+\lambda+e^{-t}\lambda}\, t - 4\, e^{2e^{-t}\lambda}\lambda - e^{2(t+e^{-t}\lambda)}\left(t + 4\lambda\right) + e^{t+2e^{-t}\lambda}\left(-5t + 8\lambda\right)\right)$$
$$\theta1_{1,0} + 12\, e^{t+\lambda}\left(-1 + e^{t}\right) t \left(e^{2(t+\lambda)}\, t - 3\, e^{t+2\lambda}\, t + 8\, e^{t+\lambda+e^{-t}\lambda}\, t - 4\, e^{2e^{-t}\lambda}\lambda -\right.$$
$$\left. e^{2(t+e^{-t}\lambda)}\left(t + 4\lambda\right) + e^{t+2e^{-t}\lambda}\left(-5t + 8\lambda\right)\right)\theta1_{2,1} - 16\, e^{3(t+e^{-t}\lambda)}\, t^2\,\theta2_{3,1} +$$
$$8\, e^{2t+3e^{-t}\lambda}\, t^2\,\theta2_{3,1} + 8\, e^{4t+3e^{-t}\lambda}\, t^2\,\theta2_{3,1} + 48\, e^{3(t+e^{-t}\lambda)}\, t^2\,\theta2_{4,2} + 16\, e^{t+3e^{-t}\lambda}\, t^2\,\theta2_{4,2} -$$
$$48\, e^{2t+3e^{-t}\lambda}\, t^2\,\theta2_{4,2} - 16\, e^{4t+3e^{-t}\lambda}\, t^2\,\theta2_{4,2} - 24\, e^{(2+e^{-t})\lambda}\, t^2\,\phi_{3,2}[\lambda] + 96\, e^{t+(2+e^{-t})\lambda}\, t^2\,\phi_{3,2}[\lambda] -$$
$$144\, e^{2t+(2+e^{-t})\lambda}\, t^2\,\phi_{3,2}[\lambda] + 96\, e^{3t+(2+e^{-t})\lambda}\, t^2\,\phi_{3,2}[\lambda] - 24\, e^{4t+(2+e^{-t})\lambda}\, t^2\,\phi_{3,2}[\lambda] +$$
$$24\, e^{(2+e^{-t})\lambda}\, t^2\,\phi_{4,2}[\lambda] - 96\, e^{t+(2+e^{-t})\lambda}\, t^2\,\phi_{4,2}[\lambda] + 144\, e^{2t+(2+e^{-t})\lambda}\, t^2\,\phi_{4,2}[\lambda] -$$
$$96\, e^{3t+(2+e^{-t})\lambda}\, t^2\,\phi_{4,2}[\lambda] + 24\, e^{4t+(2+e^{-t})\lambda}\, t^2\,\phi_{4,2}[\lambda] - 24\, e^{3\lambda}\, t^2\,\phi_{3,3}'[\lambda] +$$
$$\left. 96\, e^{3(t+\lambda)}\, t^2\,\phi_{3,3}'[\lambda] + 96\, e^{t+3\lambda}\, t^2\,\phi_{3,3}'[\lambda] - 144\, e^{2t+3\lambda}\, t^2\,\phi_{3,3}'[\lambda] - 24\, e^{4t+3\lambda}\, t^2\,\phi_{3,3}'[\lambda]\right) == 0,$$

$$4\, e^{2\lambda}\left(-2\, e^{t+e^{-t}\lambda}\left(t - 3\lambda\right) + 2\, e^{t+\lambda}\left(t - \lambda\right) + e^{\lambda}\lambda - 3\, e^{e^{-t}\lambda}\lambda + e^{2t+\lambda}\lambda - 3\, e^{2t+e^{-t}\lambda}\lambda\right)\theta1_{2,1} +$$
$$\frac{1}{\left(-1 + e^{t}\right) t}$$
$$e^{-\lambda}\left(19\, e^{t+4\lambda}\, t - 20\, e^{2t+4\lambda}\, t + e^{3t+4\lambda}\, t - 20\, e^{2(t+\lambda+e^{-t}\lambda)}\, t + 21\, e^{t+2(1+e^{-t})\lambda}\, t - e^{3t+2(1+e^{-t})\lambda}\, t -\right.$$
$$40\, e^{t+(3+e^{-t})\lambda}\, t + 40\, e^{2t+(3+e^{-t})\lambda}\, t + 8\, e^{t+4\lambda}\, t^2 + 8\, e^{2t+4\lambda}\, t^2 + 16\, e^{t+2(1+e^{-t})\lambda}\, t^2 - 24\, e^{t+(3+e^{-t})\lambda}\, t^2 -$$
$$8\, e^{2t+(3+e^{-t})\lambda}\, t^2 + 4\, e^{4\lambda}\lambda + 24\, e^{2(1+e^{-t})\lambda}\lambda - 24\, e^{(3+e^{-t})\lambda}\lambda - 12\, e^{t+4\lambda}\lambda + 12\, e^{2t+4\lambda}\lambda -$$
$$4\, e^{3t+4\lambda}\lambda + 72\, e^{2(t+\lambda+e^{-t}\lambda)}\lambda - 72\, e^{t+2(1+e^{-t})\lambda}\lambda - 24\, e^{3t+2(1+e^{-t})\lambda}\lambda + 72\, e^{t+(3+e^{-t})\lambda}\lambda -$$
$$72\, e^{2t+(3+e^{-t})\lambda}\lambda + 24\, e^{3t+(3+e^{-t})\lambda}\lambda + 2\, e^{4\lambda}\, t\,\lambda + 18\, e^{2(1+e^{-t})\lambda}\, t\,\lambda - 12\, e^{(3+e^{-t})\lambda}\, t\,\lambda -$$
$$2\, e^{t+4\lambda}\, t\,\lambda - 2\, e^{2t+4\lambda}\, t\,\lambda + 2\, e^{3t+4\lambda}\, t\,\lambda + 14\, e^{2(t+\lambda+e^{-t}\lambda)}\, t\,\lambda - 34\, e^{t+2(1+e^{-t})\lambda}\, t\,\lambda +$$
$$2\, e^{3t+2(1+e^{-t})\lambda}\, t\,\lambda + 12\, e^{t+(3+e^{-t})\lambda}\, t\,\lambda + 12\, e^{2t+(3+e^{-t})\lambda}\, t\,\lambda - 12\, e^{3t+(3+e^{-t})\lambda}\, t\,\lambda -$$
$$4\, e^{2(1+e^{-t})\lambda}\left(-1 + e^{t}\right)^3 t^2\,\theta2_{4,2} + 12\, e^{(3+e^{-t})\lambda}\left(-1 + e^{t}\right)^3 t^2\,\phi_{4,1}[\lambda] - 4\, e^{4\lambda}\, t^2\,\phi_{4,2}'[\lambda] +$$
$$\left. 12\, e^{t+4\lambda}\, t^2\,\phi_{4,2}'[\lambda] - 12\, e^{2t+4\lambda}\, t^2\,\phi_{4,2}'[\lambda] + 4\, e^{3t+4\lambda}\, t^2\,\phi_{4,2}'[\lambda]\right) == 4\, e^{-t+3\lambda}\left(-1 + e^{t}\right)^3\lambda\,\theta1_{2,2},$$

$$\frac{1}{\left(-1 + e^{t}\right) t}\, e^{-t-\lambda}\left(-48\, e^{2(t+\lambda)}\, t + 48\, e^{3t+2\lambda}\, t - 56\, e^{2(t+e^{-t}\lambda)}\, t + 96\, e^{2t+\lambda+e^{-t}\lambda}\, t - 96\, e^{3t+\lambda+e^{-t}\lambda}\, t +\right.$$
$$48\, e^{3t+2e^{-t}\lambda}\, t + 8\, e^{4t+2e^{-t}\lambda}\, t - 27\, e^{2(t+\lambda)}\, t^2 - 26\, e^{3t+2\lambda}\, t^2 + e^{4t+2\lambda}\, t^2 - 49\, e^{2(t+e^{-t}\lambda)}\, t^2 +$$
$$72\, e^{2t+\lambda+e^{-t}\lambda}\, t^2 + 40\, e^{3t+\lambda+e^{-t}\lambda}\, t^2 - 22\, e^{3t+2e^{-t}\lambda}\, t^2 - 5\, e^{4t+2e^{-t}\lambda}\, t^2 - 24\, e^{2(t+\lambda)}\lambda +$$
$$8\, e^{t+2\lambda}\lambda + 24\, e^{3t+2\lambda}\lambda - 8\, e^{4t+2\lambda}\lambda + 216\, e^{2(t+e^{-t}\lambda)}\lambda + 32\, e^{t+\lambda+e^{-t}\lambda}\lambda - 96\, e^{2t+\lambda+e^{-t}\lambda}\lambda +$$
$$96\, e^{3t+\lambda+e^{-t}\lambda}\lambda - 32\, e^{4t+\lambda+e^{-t}\lambda}\lambda - 72\, e^{t+2e^{-t}\lambda}\lambda - 216\, e^{3t+2e^{-t}\lambda}\lambda + 72\, e^{4t+2e^{-t}\lambda}\lambda -$$
$$4\, e^{2(t+\lambda)}\, t\,\lambda + 4\, e^{t+2\lambda}\, t\,\lambda - 4\, e^{3t+2\lambda}\, t\,\lambda + 4\, e^{4t+2\lambda}\, t\,\lambda + 88\, e^{2(t+e^{-t}\lambda)}\, t\,\lambda + 16\, e^{t+\lambda+e^{-t}\lambda}\, t\,\lambda -$$
$$16\, e^{2t+\lambda+e^{-t}\lambda}\, t\,\lambda - 16\, e^{3t+\lambda+e^{-t}\lambda}\, t\,\lambda + 16\, e^{4t+\lambda+e^{-t}\lambda}\, t\,\lambda - 56\, e^{t+2e^{-t}\lambda}\, t\,\lambda - 8\, e^{3t+2e^{-t}\lambda}\, t\,\lambda -$$
$$24\, e^{4t+2e^{-t}\lambda}\, t\,\lambda + 2\, e^{2t}\, t\left(e^{2(t+\lambda)}\, t - 11\, e^{t+2\lambda}\, t + 8\, e^{t+\lambda+e^{-t}\lambda}\left(3t - 2\lambda\right) - 12\, e^{2e^{-t}\lambda}\lambda +\right.$$
$$\left. 8\, e^{\lambda+e^{-t}\lambda}\lambda + 8\, e^{2t+\lambda+e^{-t}\lambda}\lambda - e^{2(t+e^{-t}\lambda)}\left(t + 12\lambda\right) + e^{t+2e^{-t}\lambda}\left(-13t + 24\lambda\right)\right)\theta1_{1,0} +$$
$$8\, e^{t+\lambda}\left(-1 + e^{t}\right) t\left(2\, e^{t+e^{-t}\lambda}\left(t - \lambda\right) + e^{\lambda}\lambda + e^{e^{-t}\lambda}\lambda + e^{2t+\lambda}\lambda + e^{2t+e^{-t}\lambda}\lambda - 2\, e^{t+\lambda}\left(t + \lambda\right)\right)\theta1_{2,1} -$$
$$8\, e^{2\lambda}\, t\,\lambda\,\theta1_{2,2} - 48\, e^{2(t+\lambda)}\, t\,\lambda\,\theta1_{2,2} + 32\, e^{t+2\lambda}\, t\,\lambda\,\theta1_{2,2} + 32\, e^{3t+2\lambda}\, t\,\lambda\,\theta1_{2,2} -$$
$$8\, e^{4t+2\lambda}\, t\,\lambda\,\theta1_{2,2} - 8\, e^{2(t+e^{-t}\lambda)}\, t^2\,\theta2_{3,1} + 16\, e^{3t+2e^{-t}\lambda}\, t^2\,\theta2_{3,1} - 8\, e^{4t+2e^{-t}\lambda}\, t^2\,\theta2_{3,1} +$$
$$24\, e^{2(t+e^{-t}\lambda)}\, t^2\,\theta2_{4,2} - 8\, e^{t+2e^{-t}\lambda}\, t^2\,\theta2_{4,2} - 24\, e^{3t+2e^{-t}\lambda}\, t^2\,\theta2_{4,2} + 8\, e^{4t+2e^{-t}\lambda}\, t^2\,\theta2_{4,2} -$$
$$16\, e^{t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{3,1}[\lambda] + 48\, e^{2t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{3,1}[\lambda] - 48\, e^{3t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{3,1}[\lambda] +$$
$$16\, e^{4t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{3,1}[\lambda] + 24\, e^{t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{4,1}[\lambda] - 72\, e^{2t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{4,1}[\lambda] +$$
$$72\, e^{3t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{4,1}[\lambda] - 24\, e^{4t+\lambda+e^{-t}\lambda}\, t^2\,\phi_{4,1}[\lambda] + 24\, e^{2(t+\lambda)}\, t^2\,\phi_{3,2}'[\lambda] -$$
$$\left. 8\, e^{t+2\lambda}\, t^2\,\phi_{3,2}'[\lambda] - 24\, e^{3t+2\lambda}\, t^2\,\phi_{3,2}'[\lambda] + 8\, e^{4t+2\lambda}\, t^2\,\phi_{3,2}'[\lambda]\right) == 0,$$

$$\begin{aligned}
&\frac{1}{(-1+e^{t})\,t}\,e^{-\lambda}\,(-18\,e^{t+4\lambda}\,t+24\,e^{2t+4\lambda}\,t-6\,e^{3t+4\lambda}\,t+24\,e^{2(t+\lambda+e^{-t}\lambda)}\,t-36\,e^{t+2(1+e^{-t})\lambda}\,t+\\
&\quad 12\,e^{3t+2(1+e^{-t})\lambda}\,t+48\,e^{t+(3+e^{-t})\lambda}\,t-48\,e^{2t+(3+e^{-t})\lambda}\,t-9\,e^{t+4\lambda}\,t^{2}-6\,e^{2t+4\lambda}\,t^{2}+\\
&\quad 3\,e^{3t+4\lambda}\,t^{2}-24\,e^{2(t+\lambda+e^{-t}\lambda)}\,t^{2}-18\,e^{t+2(1+e^{-t})\lambda}\,t^{2}-6\,e^{3t+2(1+e^{-t})\lambda}\,t^{2}+24\,e^{t+(3+e^{-t})\lambda}\,t^{2}+\\
&\quad 24\,e^{2t+(3+e^{-t})\lambda}\,t^{2}-48\,e^{2(1+e^{-t})\lambda}\,\lambda-144\,e^{2(t+\lambda+e^{-t}\lambda)}\,\lambda+144\,e^{t+2(1+e^{-t})\lambda}\,\lambda+48\,e^{3t+2(1+e^{-t})\lambda}\,\lambda-\\
&\quad 24\,e^{2(1+e^{-t})\lambda}\,t\,\lambda+24\,e^{2(t+\lambda+e^{-t}\lambda)}\,t\,\lambda+24\,e^{t+2(1+e^{-t})\lambda}\,t\,\lambda-24\,e^{3t+2(1+e^{-t})\lambda}\,t\,\lambda+6\,e^{t+2\lambda}\,t\\
&\quad (e^{2(t+\lambda)}\,t-3\,e^{t+2\lambda}\,t+8\,e^{t+\lambda+e^{-t}\lambda}\,t-4\,e^{2e^{-t}\lambda}\,\lambda-e^{2(t+e^{-t}\lambda)}\,(t+4\lambda)+e^{t+2e^{-t}\lambda}\,(-5t+8\lambda))\\
&\quad \theta 1_{1,0}+6\,e^{t+2(1+e^{-t})\lambda}\,(-1+e^{t})^{2}\,t^{2}\,\theta 2_{2,1}+12\,e^{2(t+\lambda+e^{-t}\lambda)}\,t^{2}\,\theta 2_{3,1}-6\,e^{t+2(1+e^{-t})\lambda}\,t^{2}\,\theta 2_{3,1}-\\
&\quad 6\,e^{3t+2(1+e^{-t})\lambda}\,t^{2}\,\theta 2_{3,1}-4\,e^{2(1+e^{-t})\lambda}\,t^{2}\,\theta 2_{4,2}-12\,e^{2(t+\lambda+e^{-t}\lambda)}\,t^{2}\,\theta 2_{4,2}+\\
&\quad 12\,e^{t+2(1+e^{-t})\lambda}\,t^{2}\,\theta 2_{4,2}+4\,e^{3t+2(1+e^{-t})\lambda}\,t^{2}\,\theta 2_{4,2}+12\,e^{(3+e^{-t})\lambda}\,t^{2}\,\phi_{2,1}[\lambda]-\\
&\quad 36\,e^{t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{2,1}[\lambda]+36\,e^{2t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{2,1}[\lambda]-12\,e^{3t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{2,1}[\lambda]-\\
&\quad 12\,e^{(3+e^{-t})\lambda}\,t^{2}\,\phi_{3,1}[\lambda]+36\,e^{t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{3,1}[\lambda]-36\,e^{2t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{3,1}[\lambda]+\\
&\quad 12\,e^{3t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{3,1}[\lambda]+12\,e^{(3+e^{-t})\lambda}\,t^{2}\,\phi_{4,1}[\lambda]-36\,e^{t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{4,1}[\lambda]+\\
&\quad 36\,e^{2t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{4,1}[\lambda]-12\,e^{3t+(3+e^{-t})\lambda}\,t^{2}\,\phi_{4,1}[\lambda]+12\,e^{4\lambda}\,t^{2}\,\phi_{2,2}'[\lambda]-\\
&\quad 36\,e^{t+4\lambda}\,t^{2}\,\phi_{2,2}'[\lambda]+36\,e^{2t+4\lambda}\,t^{2}\,\phi_{2,2}'[\lambda]-12\,e^{3t+4\lambda}\,t^{2}\,\phi_{2,2}'[\lambda])==0,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(-1+e^{t})\,t}\,e^{-t-\lambda}\,(-3\,e^{\lambda}\,(-1+e^{t})^{2}\,t\,\lambda\,\theta 1_{2,1}-2\,e^{e^{-t}\lambda}\,(-1+e^{t})^{2}\,t^{2}\,\theta 2_{4,2}+3\,(2\,e^{t+\lambda}\,t-2\,e^{t+e^{-t}\lambda}\,t+\\
&\quad 2\,e^{\lambda}\,\lambda-4\,e^{e^{-t}\lambda}\,\lambda-4\,e^{t+\lambda}\,\lambda+2\,e^{2t+\lambda}\,\lambda+8\,e^{t+e^{-t}\lambda}\,\lambda-4\,e^{2t+e^{-t}\lambda}\,\lambda+e^{\lambda}\,t\,\lambda-2\,e^{e^{-t}\lambda}\,t\,\lambda-\\
&\quad e^{2t+\lambda}\,t\,\lambda+2\,e^{t+e^{-t}\lambda}\,t\,\lambda+4\,e^{t+e^{-t}\lambda}\,(-1+e^{t})\,t^{2}\,\phi_{4,0}[\lambda]+e^{t+\lambda}\,(-1+e^{t})\,t^{2}\,\phi_{4,1}'[\lambda]))==0,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(-1+e^{t})\,t}\,e^{-t-\lambda}\,(2\,e^{t+\lambda}\,t^{2}+2\,e^{2t+\lambda}\,t^{2}-3\,e^{t+e^{-t}\lambda}\,t^{2}-e^{3t+e^{-t}\lambda}\,t^{2}-2\,e^{\lambda}\,\lambda-6\,e^{e^{-t}\lambda}\,\lambda+\\
&\quad 6\,e^{t+\lambda}\,\lambda-6\,e^{2t+\lambda}\,\lambda+2\,e^{3t+\lambda}\,\lambda+18\,e^{t+e^{-t}\lambda}\,\lambda-18\,e^{2t+e^{-t}\lambda}\,\lambda+6\,e^{3t+e^{-t}\lambda}\,\lambda-e^{\lambda}\,t\,\lambda-\\
&\quad 7\,e^{e^{-t}\lambda}\,t\,\lambda+e^{t+\lambda}\,t\,\lambda+e^{2t+\lambda}\,t\,\lambda-e^{3t+\lambda}\,t\,\lambda+11\,e^{t+e^{-t}\lambda}\,t\,\lambda-e^{2t+e^{-t}\lambda}\,t\,\lambda-3\,e^{3t+e^{-t}\lambda}\,t\,\lambda+\\
&\quad 2\,e^{t}\,t\,(-2\,e^{t+e^{-t}\lambda}\,(t-3\lambda)+2\,e^{t+\lambda}\,(t-\lambda)+e^{\lambda}\,\lambda-3\,e^{e^{-t}\lambda}\,\lambda+e^{2t+\lambda}\,\lambda-3\,e^{2t+e^{-t}\lambda}\,\lambda)\,\theta 1_{1,0}-\\
&\quad 2\,e^{\lambda}\,(-1+e^{t})^{3}\,t\,\lambda\,\theta 1_{2,1}-2\,e^{t+e^{-t}\lambda}\,t^{2}\,\theta 2_{3,1}+4\,e^{2t+e^{-t}\lambda}\,t^{2}\,\theta 2_{3,1}-2\,e^{3t+e^{-t}\lambda}\,t^{2}\,\theta 2_{3,1}+\\
&\quad 6\,e^{t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]-12\,e^{2t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]+6\,e^{3t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]-\\
&\quad 12\,e^{t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]+24\,e^{2t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]-12\,e^{3t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]+\\
&\quad 2\,e^{t+\lambda}\,t^{2}\,\phi_{3,1}'[\lambda]-4\,e^{2t+\lambda}\,t^{2}\,\phi_{3,1}'[\lambda]+2\,e^{3t+\lambda}\,t^{2}\,\phi_{3,1}'[\lambda])==0,
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(-1+e^{t})\,t}\,e^{-t-\lambda}\,(-4\,e^{t+\lambda}\,t+4\,e^{2t+\lambda}\,t+4\,e^{t+e^{-t}\lambda}\,t-4\,e^{2t+e^{-t}\lambda}\,t-2\,e^{t+\lambda}\,t^{2}-2\,e^{2t+\lambda}\,t^{2}+2\,e^{t+e^{-t}\lambda}\,t^{2}+\\
&\quad 2\,e^{2t+e^{-t}\lambda}\,t^{2}+2\,e^{\lambda}\,\lambda+2\,e^{e^{-t}\lambda}\,\lambda-6\,e^{t+\lambda}\,\lambda+6\,e^{2t+\lambda}\,\lambda-2\,e^{3t+\lambda}\,\lambda-6\,e^{t+e^{-t}\lambda}\,\lambda+6\,e^{2t+e^{-t}\lambda}\,\lambda-\\
&\quad 2\,e^{3t+e^{-t}\lambda}\,\lambda+e^{\lambda}\,t\,\lambda+e^{e^{-t}\lambda}\,t\,\lambda-e^{t+\lambda}\,t\,\lambda-e^{2t+\lambda}\,t\,\lambda+e^{3t+\lambda}\,t\,\lambda-e^{t+e^{-t}\lambda}\,t\,\lambda-e^{2t+e^{-t}\lambda}\,t\,\lambda+\\
&\quad e^{3t+e^{-t}\lambda}\,t\,\lambda+2\,e^{t}\,t\,(2\,e^{t+e^{-t}\lambda}\,(t-\lambda)+e^{\lambda}\,\lambda+e^{e^{-t}\lambda}\,\lambda+e^{2t+\lambda}\,\lambda+e^{2t+e^{-t}\lambda}\,\lambda-2\,e^{t+\lambda}\,(t+\lambda))\,\theta 1_{1,0}-\\
&\quad 2\,e^{t+e^{-t}\lambda}\,(-1+e^{t})^{2}\,t^{2}\,\theta 2_{2,1}+4\,e^{t+e^{-t}\lambda}\,t^{2}\,\phi_{2,0}[\lambda]-8\,e^{2t+e^{-t}\lambda}\,t^{2}\,\phi_{2,0}[\lambda]+\\
&\quad 4\,e^{3t+e^{-t}\lambda}\,t^{2}\,\phi_{2,0}[\lambda]-6\,e^{t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]+12\,e^{2t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]-\\
&\quad 6\,e^{3t+e^{-t}\lambda}\,t^{2}\,\phi_{3,0}[\lambda]+8\,e^{t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]-16\,e^{2t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]+\\
&\quad 8\,e^{3t+e^{-t}\lambda}\,t^{2}\,\phi_{4,0}[\lambda]+2\,e^{t+\lambda}\,t^{2}\,\phi_{2,1}'[\lambda]-4\,e^{2t+\lambda}\,t^{2}\,\phi_{2,1}'[\lambda]+2\,e^{3t+\lambda}\,t^{2}\,\phi_{2,1}'[\lambda])==0,
\end{aligned}$$

$$e^{-\lambda}\,(-e^{e^{-t}\lambda}\,\theta 2_{1,1}+e^{e^{-t}\lambda}\,\phi_{1,0}[\lambda]-e^{e^{-t}\lambda}\,\phi_{2,0}[\lambda]+e^{e^{-t}\lambda}\,\phi_{3,0}[\lambda]-e^{e^{-t}\lambda}\,\phi_{4,0}[\lambda]+e^{\lambda}\,\phi_{1,1}'[\lambda])==0,$$

$$\frac{e^{-t}\,((-1+e^{t})^{2}\,t^{2}\,\theta 2_{4,2}+6\,((-1+e^{t})^{2}\,\lambda-e^{2t}\,t^{2}\,\phi_{4,0}'[\lambda]))}{t}==0,$$

$$\frac{1}{(-1+e^t)\,t} e^{-t}\Big(-8 e^t t + 8 e^{2t} t - 3 e^t t^2 - 6 e^{2t} t^2 + e^{3t} t^2 - 6\lambda + 18 e^t \lambda - 18 e^{2t}\lambda + 6 e^{3t}\lambda + 3 t\lambda - 3 e^t t\lambda - 3 e^{2t} t\lambda + 3 e^{3t} t\lambda + 6 e^t (-1+e^t)^2 t\lambda\,\theta1_{1,0} + 2 e^t (-1+e^t)^2 t^2\,\theta2_{3,1} - 2 t^2\,\theta2_{4,2} + 6 e^t t^2\,\theta2_{4,2} - 6 e^{2t} t^2\,\theta2_{4,2} + 2 e^{3t} t^2\,\theta2_{4,2} + 6 e^{2t} t^2\,\phi_{3,0}'[\lambda] - 6 e^{3t} t^2\,\phi_{3,0}'[\lambda]\Big) == 0,$$

$$\frac{1}{(-1+e^t)\,t} e^{-t}\Big(-24 e^t + 24 e^{2t} - 9 e^t t - 18 e^{2t} t + 3 e^{3t} t + 6\lambda - 6 e^t\lambda - 6 e^{2t}\lambda + 6 e^{3t}\lambda + 12 e^t (-1+e^t)^2 \lambda\,\theta1_{1,0} + 6 e^t (-1+e^t)^2 t\,\theta2_{2,1} + 6 e^t t\,\theta2_{3,1} - 12 e^{2t} t\,\theta2_{3,1} + 6 e^{3t} t\,\theta2_{3,1} - 2 t\,\theta2_{4,2} + 6 e^t t\,\theta2_{4,2} - 6 e^{2t} t\,\theta2_{4,2} + 2 e^{3t} t\,\theta2_{4,2} + 12 e^{2t} t\,\phi_{2,0}'[\lambda] - 12 e^{3t} t\,\phi_{2,0}'[\lambda]\Big) == 0,$$

$$\frac{1}{(-1+e^t)\,t} e^{-t}\Big(-8 + 8 e^t - 3 t - 6 e^t t + e^{2t} t + 12 (-1+e^t)^2 t\,\theta2_{1,1} + 6 (-1+e^t)^2 t\,\theta2_{2,1} + 2 t\,\theta2_{3,1} - 4 e^t t\,\theta2_{3,1} + 2 e^{2t} t\,\theta2_{3,1} + 12 e^t t\,\phi_{1,0}'[\lambda] - 12 e^{2t} t\,\phi_{1,0}'[\lambda]\Big) == 0,\ \theta2_{0,0} == \phi_{0,0}'[\lambda]\Big\}$$

In[∘]:=
```
Φ[λ] = Φ[λ] /. DSolve[Join[Table[(f /. λ → 0) == 0, {f, ϕs}], eqns2], ϕs, λ][[1]]
```

Out[∘]=

$$1 + \epsilon\,(\cdots 1 \cdots) + \epsilon^2\Big(\lambda\,\theta2_{0,0} + e^{(-1+e^{-t})\lambda}\lambda\,x_i\,y_i\,\theta2_{1,1} + \frac{e^{-t}\,\cdots 3\cdots\,(-8 e^{2t} t + \cdots 67\cdots + 4\,\cdots 3\cdots\,\theta2_{\cdots 1\cdots})}{4\,(-1+e^t)^3\,t^2} + \cdots 9\cdots + \frac{x_i^3 y_i^3\,(\cdots 1\cdots)}{144\,\cdots 1\cdots^5\,t^2} +$$
$$\frac{a_i\,x_i^3\,\cdots 1\cdots\,(-66 e^{3t} t + 72 e^{4t} t - 6 e^{5t} t + \cdots 152\cdots + 96\,\cdots 4\cdots - 64\,e^{\cdots 1\cdots}\,t^2\lambda\,\theta2_{4,2} + 16\,e^{5t+3(-1+\cdots 1\cdots)\lambda}\,t^2\lambda\,\theta2_{4,2})}{24\,(-1+e^t)^5\,t^2} +$$
$$\frac{x_i^4 y_i^4\,(36 e^{4t} t - 48 e^{5t} t + \cdots 209\cdots + 16\,e^{6t+4(-1+e^{-t})\lambda}\,t^2\lambda\,\theta2_{4,2})}{96\,(-1+e^t)^6\,t^2}\Big)$$

large output | show less | show more | show all | set size limit...

$OutputSizeLimit: This output can only be updated in the same kernel session that generated it.

```
Φ[λ] =
```

In[∘]:=
```
Expw2 =
 (𝔼{i}→{i}[(1 - T) (a_i + 1/2) ω_i, ((e^(ω_i (1-T)) - 1)/(1 - T)) e^(-(1-T) ω_i) y_i x_i, Φ[λ]] /. ω_i → t/(1 - T) /. λ → t/(1 - T) //
    Simplify) //. l2U // Simplify
```

$R_{1,2}$

In[∘]:=
```
w2a_i_ := 𝔼{i}→{i}[(1 - T) (a_i + 1/2) ω_i,
   ((e^(ω_i (1-T)) - 1)/(1 - T)) e^(-(1-T) ω_i) y_i x_i + ξ_i x_i + η_i y_i, (Φ[λ] /. λ :> ω_i) + O[ϵ]^3] //. U2l
```

In[∘]:=
```
w2a_i_ := 𝔼{i}→{i}[(1 - T) (a_i + 1/2) ω_i, ((e^(ω_i (1-T)) - 1)/(1 - T)) e^(-(1-T) ω_i) y_i x_i + ξ_i x_i + η_i y_i,
   (Φ[λ] /. KinkSol /. λ :> ω_i) + O[ϵ]^2] //. U2l // Simplify
```

In[∘]:= w2a$_i$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\left[\left(1-e^{-t}\right)\left(\frac{1}{2}+a_i\right)\omega_i,\ y_i\,\eta_i+x_i\left(-\frac{e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)y_i}{-1+e^t}+\xi_i\right),\right.$
$1+\frac{1}{4\left(-1+e^t\right)^3 t}e^{-t}\left(4\left(-1+e^t\right)^4 a_i^2\,\omega_i+e^{2t}x_i^2 y_i^2\left(e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)\right.\right.$
$\left.\left(-3+e^t+5\,e^{\left(-1+e^{-t}\right)\omega_i}+e^{t+\left(-1+e^{-t}\right)\omega_i}\right)t+4\,e^{2\left(-1+e^{-t}\right)\omega_i}\left(-1+e^t\right)^2\omega_i\right)+4\left(-1+e^t\right)a_i$
$\left.\left.\left(\left(-1+e^t\right)^3\omega_i+2\,e^{t-\omega_i}x_i\,y_i\left(e^t\left(-e^{\omega_i}+e^{e^{-t}\omega_i}\right)t+e^{e^{-t}\omega_i}\left(-1+e^t\right)^2\omega_i\right)\right)\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:=
```
FF_z_ := If[z == 0, 1, Sum[f_{u,v}[α_i] y_i^v w_i^(u-v) x_i^v, {u, 0, 2 z}, {v, 0, u}]]
F = FF_0 + ϵ FF_1 + O[ϵ]^2
fs_z_ := Flatten@ Table[f_{u,v}[α_i], {u, 0, 2 z}, {v, 0, u}]
```

Out[∘]= $1+\left(f_{0,0}[\alpha_i]+w_i\,f_{1,0}[\alpha_i]+x_i\,y_i\,f_{1,1}[\alpha_i]+w_i^2\,f_{2,0}[\alpha_i]+w_i\,x_i\,y_i\,f_{2,1}[\alpha_i]+x_i^2\,y_i^2\,f_{2,2}[\alpha_i]\right)\epsilon+O[\epsilon]^2$

In[∘]:= a2w$_{i_}$:= $\mathbb{E}_{\{i\}\to\{i\}}\left[\alpha_i\left(\frac{-1}{2}+\frac{w_i}{1-T}\right),\ \left(e^{-\alpha_i}-1\right)\frac{y_i\,x_i}{1-T}+\xi_i\,x_i+\eta_i\,y_i,\ F+O[\epsilon]^2\right]$ //. U21

(*1+ $\epsilon\left(-\frac{e^{-t-\alpha_i}\left(-1+e^{\alpha_i}\right)x_i\,y_i}{\left(-1+e^{-t}\right)^2}-\frac{2\,e^{-t-\alpha_i}\left(-1+e^{\alpha_i}\right)w_i\,x_i\,y_i}{\left(-1+e^{-t}\right)^3}+\frac{e^{-2\alpha_i}\left(-1+3\,e^{-t}\right)\left(-1+e^{2\alpha_i}\right)x_i^2\,y_i^2}{4\left(-1+e^{-t}\right)^3}-\frac{\left(1+e^{-t}\right)\alpha_i}{24\left(-1+e^{-t}\right)}+\frac{\left(1+e^{-t}\right)w_i^2\,\alpha_i}{2\left(-1+e^{-t}\right)^3}\right)$*)

In[∘]:=
```
((a2w_i // w2a_i) //. U21 // CF // Simplify // Normal) /. ϵ → 0
((w2a_i // a2w_i) //. U21 // CF // Simplify // Normal) /. ϵ → 0
```

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\left[a_i\,\alpha_i,\ y_i\,\eta_i+x_i\left(\frac{e^{t-\frac{e^t\alpha_i}{-1+e^t}}\left(e^{\frac{\alpha_i}{-1+e^t}}-\left(e^{\frac{e^t\alpha_i}{-1+e^t}}\right)^{e^{-t}}\right)y_i}{-1+e^t}+\xi_i\right),\ 1\right]$

Out[∘]= \$Aborted

In[∘]:=
```
Sol1 = FullSimplify /@
  (Solve[Thread[Flatten@CoefficientList[Coefficient[((a2w_i // w2a_i) //. U21 // Simplify)⟦
      3⟧, ϵ], {x_i, a_i, y_i}] == 0], fs_1]⟦1⟧ /. (e^ww_)^w_ -> e^(ww w))
```

Out[∘]= $\left\{f_{0,0}[\alpha_i]\to\frac{\alpha_i}{4\,t},\ f_{1,0}[\alpha_i]\to 0,\ f_{1,1}[\alpha_i]\to\frac{e^t\left(-1+e^{-\alpha_i}\right)}{\left(-1+e^t\right)^2},\ f_{2,0}[\alpha_i]\to-\frac{e^{2t}\alpha_i}{\left(-1+e^t\right)^2 t},\right.$
$\left.f_{2,1}[\alpha_i]\to\frac{2\,e^{2t}\left(1-e^{-\alpha_i}\right)}{\left(-1+e^t\right)^3},\ f_{2,2}[\alpha_i]\to-\frac{e^{2t}\left(-3+e^t\right)\left(-1+e^{-2\alpha_i}\right)}{4\left(-1+e^t\right)^3}\right\}$

In[∘]:=
```
Clear[F]
F = FF_0 + ϵ (FF_1 /. Sol1) + O[ϵ]^2
```

Out[∘]= $1+\left(\frac{e^t\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{\left(-1+e^t\right)^2}+\frac{2\,e^{2t}\left(1-e^{-\alpha_i}\right)w_i\,x_i\,y_i}{\left(-1+e^t\right)^3}-\right.$
$\left.\frac{e^{2t}\left(-3+e^t\right)\left(-1+e^{-2\alpha_i}\right)x_i^2\,y_i^2}{4\left(-1+e^t\right)^3}+\frac{\alpha_i}{4\,t}-\frac{e^{2t}\,w_i^2\,\alpha_i}{\left(-1+e^t\right)^2 t}\right)\epsilon+O[\epsilon]^2$

```
In[]:= F = FF_0 + ϵ (FF_1 /. Sol1) + ϵ^2 (FF_2) + O[ϵ]^3
Sol2 = Solve[Thread[Flatten@CoefficientList[Coefficient[
      ((a2w_i // w2a_i) //. U2l // CF // Simplify)[[3]], ϵ^2], {x_i, a_i, y_i}] == 0], fs_2][[1]]
```

Out[]=

$$1+\left(-\frac{e^{t-\alpha_i}\left(-1+e^{\alpha_i}\right)x_i y_i}{\left(-1+e^t\right)^2}+\frac{2e^{2t-\alpha_i}\left(-1+e^{\alpha_i}\right)w_i x_i y_i}{\left(-1+e^t\right)^3}+\frac{e^{2t-2\alpha_i}\left(-3+e^t\right)\left(-1+e^{2\alpha_i}\right)x_i^2 y_i^2}{4\left(-1+e^t\right)^3}+\frac{\left(1+e^t\right)\alpha_i}{8\left(-1+e^t\right)}-\frac{e^{2t}\left(1+e^t\right)w_i^2\alpha_i}{2\left(-1+e^t\right)^3}\right)\epsilon+$$
$$\left(f_{0,0}[\alpha_i]+w_i f_{1,0}[\alpha_i]+x_i y_i f_{1,1}[\alpha_i]+w_i^2 f_{2,0}[\alpha_i]+w_i x_i y_i f_{2,1}[\alpha_i]+x_i^2 y_i^2 f_{2,2}[\alpha_i]+w_i^3 f_{3,0}[\alpha_i]+w_i^2 x_i y_i f_{3,1}[\alpha_i]+w_i x_i^2 y_i^2 f_{3,2}[\alpha_i]+x_i^3 y_i^3 f_{3,3}[\alpha_i]+w_i^4 f_{4,0}[\alpha_i]+w_i^3 x_i y_i f_{4,1}[\alpha_i]+w_i^2 x_i^2 y_i^2 f_{4,2}[\alpha_i]+w_i x_i^3 y_i^3 f_{4,3}[\alpha_i]+x_i^4 y_i^4 f_{4,4}[\alpha_i]\right)\epsilon^2+O[\epsilon]^3$$

Out[]=

$$\Big\{f_{0,0}[\alpha_i]\to\frac{\left(1+e^t\right)^2\alpha_i^2}{128\left(-1+e^t\right)^2},\ f_{1,0}[\alpha_i]\to-\frac{e^t\left(1+4e^t+e^{2t}\right)\alpha_i}{12\left(-1+e^t\right)^3},$$
$$f_{1,1}[\alpha_i]\to-\frac{e^{t-\alpha_i}\left(1+e^t\right)\left(-1+e^{\alpha_i}\right)\left(2+\alpha_i\right)}{8\left(-1+e^t\right)^3},$$
$$f_{2,0}[\alpha_i]\to-\frac{e^{2t}\left(1+e^t\right)^2\alpha_i^2}{16\left(-1+e^t\right)^4},\ f_{2,1}[\alpha_i]\to\frac{e^{2t-\alpha_i}\left(1+e^t\right)\left(-1+e^{\alpha_i}\right)\left(8+\alpha_i\right)}{4\left(-1+e^t\right)^4},$$
$$f_{2,2}[\alpha_i]\to\frac{e^{2t-2\alpha_i}\left(-1+e^{\alpha_i}\right)\left(-80-48e^{\alpha_i}-3\alpha_i-2e^t\alpha_i+e^{2t}\alpha_i-3e^{\alpha_i}\alpha_i-2e^{t+\alpha_i}\alpha_i+e^{2t+\alpha_i}\alpha_i\right)}{32\left(-1+e^t\right)^4},$$
$$f_{3,0}[\alpha_i]\to\frac{e^{3t}\left(1+4e^t+e^{2t}\right)\alpha_i}{3\left(-1+e^t\right)^5},\ f_{3,1}[\alpha_i]\to\frac{e^{3t-\alpha_i}\left(1+e^t\right)\left(-1+e^{\alpha_i}\right)\left(-6+\alpha_i\right)}{2\left(-1+e^t\right)^5},$$
$$f_{3,2}[\alpha_i]\to\frac{e^{3t-2\alpha_i}\left(-5+4e^{\alpha_i}+e^{2\alpha_i}\right)}{\left(-1+e^t\right)^5},\ f_{3,3}[\alpha_i]\to$$
$$-\frac{e^{3t-3\alpha_i}\left(-1+e^{\alpha_i}\right)\left(67-29e^t+4e^{2t}+40e^{\alpha_i}+13e^{2\alpha_i}-20e^{t+\alpha_i}+4e^{2t+\alpha_i}-11e^{t+2\alpha_i}+4e^{2t+2\alpha_i}\right)}{36\left(-1+e^t\right)^5},$$
$$f_{4,0}[\alpha_i]\to\frac{e^{4t}\left(1+e^t\right)^2\alpha_i^2}{8\left(-1+e^t\right)^6},\ f_{4,1}[\alpha_i]\to-\frac{e^{4t-\alpha_i}\left(1+e^t\right)\left(-1+e^{\alpha_i}\right)\alpha_i}{\left(-1+e^t\right)^6},$$
$$f_{4,2}[\alpha_i]\to-\frac{e^{4t-2\alpha_i}\left(-1+e^{\alpha_i}\right)\left(16-16e^{\alpha_i}-3\alpha_i-2e^t\alpha_i+e^{2t}\alpha_i-3e^{\alpha_i}\alpha_i-2e^{t+\alpha_i}\alpha_i+e^{2t+\alpha_i}\alpha_i\right)}{8\left(-1+e^t\right)^6},$$
$$f_{4,3}[\alpha_i]\to\frac{e^{4t-3\alpha_i}\left(-3+e^t\right)\left(-1+e^{\alpha_i}\right)^2\left(1+e^{\alpha_i}\right)}{2\left(-1+e^t\right)^6},\ f_{4,4}[\alpha_i]\to\frac{e^{4t-4\alpha_i}\left(-3+e^t\right)^2\left(-1+e^{2\alpha_i}\right)^2}{32\left(-1+e^t\right)^6}\Big\}$$

```
In[]:= Clear[F]
F = FF_0 + ϵ (FF_1 /. Sol1) + ϵ^2 (FF_2 /. Sol2) + O[ϵ]^3
```

In[∘]:= `F = FF0 + ∈ (FF1 /. Sol1)`

Out[∘]= $1+\epsilon\left(\frac{e^{-\frac{(1+3e^t)t}{-1+e^t}}\left(2e^{\frac{(1+3\cdots)t}{-1+e^t}}+2e^{2t+\frac{2(\cdots)t}{\cdots}}-e^{2t+\cdots}-4e^{t+\frac{\cdots}{\cdots}}+e^{2t+\frac{(1+\cdots)t}{-1+e^t}}\right)\alpha_i}{4(-1+e^t)^2 t}+\cdots 8\cdots\right)$

large output | show less | show more | show all | set size limit...

In[∘]:= `(a2wi // w2ai) //. U2l // CF // FullSimplify`
`(w2ai // a2wi) //. U2l // CF // Simplify`

Out[∘]= $Aborted

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\left[w_i\,\omega_i,\ y_i\,\eta_i+x_i\,\xi_i,\ 1+O[\epsilon]^2\right]$

In[∘]:= `Simplify[(`$e^{\frac{\alpha_i}{-1+e^t}}-\left(e^{\frac{e^t\alpha_i}{-1+e^t}}e^{-t}\right)$`), Assumptions → t > 0]`

Out[∘]= 0

In[∘]:= `F`

Out[∘]= $$1+\left(\frac{e^t\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{\left(-1+e^t\right)^2}+\frac{2\,e^{2t}\left(1-e^{-\alpha_i}\right)w_i\,x_i\,y_i}{\left(-1+e^t\right)^3}-\frac{e^{2t}\left(-3+e^t\right)\left(-1+e^{-2\alpha_i}\right)x_i^2\,y_i^2}{4\left(-1+e^t\right)^3}+\frac{\alpha_i}{4t}-\frac{e^{2t}\,w_i^2\,\alpha_i}{\left(-1+e^t\right)^2 t}\right)\epsilon+O[\epsilon]^2$$

In[∘]:= `(kKinki // a2wi // Simplify) /. {((1/T)^z_)^zz_ -> (1/T)^(z zz)} // Simplify`

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\left[-\frac{t}{2}+\frac{t\,w_i}{1-T},\ 0,\ \frac{1}{\sqrt{T}}+O[\epsilon]^2\right]$$

In[∘]:= `TrigToExp /@ (a2wi // FullSimplify) // FullSimplify`

Out[∘]= $$\mathbb{E}_{\{i\}\to\{i\}}\left[\frac{1}{2}\left(-1+\frac{2\,e^t\,w_i}{-1+e^t}\right)\alpha_i,\ y_i\,\eta_i+x_i\left(\frac{e^t\left(-1+e^{-\alpha_i}\right)y_i}{-1+e^t}+\xi_i\right),\right.$$
$$1+\frac{1}{4\left(-1+e^t\right)^3 t}e^{-\frac{2e^t\alpha_i}{-1+e^t}}\left(4\,e^t\left(e^{\frac{2e^t\alpha_i}{-1+e^t}}-e^{\mathrm{Coth}\left[\frac{t}{2}\right]\alpha_i}\right)t\left(1+e^t\left(-1+2\,w_i\right)\right)x_i\,y_i-\right.$$
$$\left.\left.e^{2t}\left(-3+e^t\right)\left(e^{\frac{2\alpha_i}{-1+e^t}}-e^{\frac{2e^t\alpha_i}{-1+e^t}}\right)t\,x_i^2\,y_i^2-e^{\frac{2e^t\alpha_i}{-1+e^t}}\left(-1+e^t\right)\left(-\left(-1+e^t\right)^2+4\,e^{2t}\,w_i^2\right)\alpha_i\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= **w2a$_i$**

Out[∘]= $$\mathbb{E}_{\{i\}\to\{i\}}\Big[\left(1-e^{-t}\right)\left(\frac{1}{2}+a_i\right)\omega_i,\ y_i\,\eta_i+x_i\left(-\frac{e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)y_i}{-1+e^t}+\xi_i\right),$$
$$1+\frac{1}{4\left(-1+e^t\right)^3 t}e^{-t}\Big(4\left(-1+e^t\right)^4 a_i^2\,\omega_i+e^{2t}x_i^2 y_i^2\Big(e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)\left(-3+e^t+5\,e^{\left(-1+e^{-t}\right)\omega_i}+e^{t+\left(-1+e^{-t}\right)\omega_i}\right)t+4\,e^{2\left(-1+e^{-t}\right)\omega_i}\left(-1+e^t\right)^2\omega_i\Big)+4\left(-1+e^t\right)a_i\Big(\left(-1+e^t\right)^3\omega_i+2\,e^{t-\omega_i}x_i\,y_i\Big(e^t\left(-e^{\omega_i}+e^{e^{-t}\omega_i}\right)t+e^{e^{-t}\omega_i}\left(-1+e^t\right)^2\omega_i\Big)\Big)\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= **a2w$_i$**

Out[∘]= $$\mathbb{E}_{\{i\}\to\{i\}}\Big[\left(-\frac{1}{2}+\frac{w_i}{1-e^{-t}}\right)\alpha_i,\ \frac{\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{1-e^{-t}}+y_i\,\eta_i+x_i\,\xi_i,$$
$$1+\left(\frac{e^t\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{\left(-1+e^t\right)^2}+\frac{2\,e^{2t}\left(1-e^{-\alpha_i}\right)w_i\,x_i\,y_i}{\left(-1+e^t\right)^3}-\frac{e^{2t}\left(-3+e^t\right)\left(-1+e^{-2\alpha_i}\right)x_i^2 y_i^2}{4\left(-1+e^t\right)^3}+\frac{\alpha_i}{4t}-\frac{e^{2t}w_i^2\,\alpha_i}{\left(-1+e^t\right)^2 t}\right)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:=
$$\texttt{w2a}_{i\_}:=\mathbb{E}_{\{i\}\to\{i\}}\Big[\left(1-e^{-t}\right)\left(\frac{1}{2}+a_i\right)\omega_i,\ y_i\,\eta_i+x_i\left(-\frac{e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)y_i}{-1+e^t}+\xi_i\right),\ 1+\frac{1}{4\left(-1+e^t\right)^3 t}$$
$$e^{-t}\Big(4\left(-1+e^t\right)^4 a_i^2\,\omega_i+e^{2t}x_i^2 y_i^2\Big(e^t\left(-1+e^{\left(-1+e^{-t}\right)\omega_i}\right)\left(-3+e^t+5\,e^{\left(-1+e^{-t}\right)\omega_i}+e^{t+\left(-1+e^{-t}\right)\omega_i}\right)t+4\,e^{2\left(-1+e^{-t}\right)\omega_i}\left(-1+e^t\right)^2\omega_i\Big)+4\left(-1+e^t\right)a_i\Big(\left(-1+e^t\right)^3\omega_i+2\,e^{t-\omega_i}x_i\,y_i\Big(e^t\left(-e^{\omega_i}+e^{e^{-t}\omega_i}\right)t+e^{e^{-t}\omega_i}\left(-1+e^t\right)^2\omega_i\Big)\Big)\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:=
$$\texttt{a2w}_{i\_}:=\mathbb{E}_{\{i\}\to\{i\}}\Big[\left(-\frac{1}{2}+\frac{w_i}{1-e^{-t}}\right)\alpha_i,\ \frac{\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{1-e^{-t}}+y_i\,\eta_i+x_i\,\xi_i,\ 1+\left(\frac{e^t\left(-1+e^{-\alpha_i}\right)x_i\,y_i}{\left(-1+e^t\right)^2}+\frac{2\,e^{2t}\left(1-e^{-\alpha_i}\right)w_i\,x_i\,y_i}{\left(-1+e^t\right)^3}-\frac{e^{2t}\left(-3+e^t\right)\left(-1+e^{-2\alpha_i}\right)x_i^2 y_i^2}{4\left(-1+e^t\right)^3}+\frac{\alpha_i}{4t}-\frac{e^{2t}w_i^2\,\alpha_i}{\left(-1+e^t\right)^2 t}\right)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= **Z@Knot[3, 1] // a2w$_0$**

Out[∘]= $$\mathbb{E}_{\{\}\to\{0\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{-T+2T^2-2T^3+2T^4-T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T-2T^2\right)w_0}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

wTensors

In[∘]:=
```
Define[wR_{i,j} = (tR_{i,j} // (a2w_i a2w_j)) //. l2U // CF,
  wR̄_{i,j} = (tR̄_{i,j} // (a2w_i a2w_j)) //. l2U // CF ,
  wm_{i,j→k} = ((w2a_i w2a_j) // tm_{i,j→k} // a2w_k ) //. l2U // CF,
  wC_i = (tC_i // a2w_i) //. l2U // CF ,
  wC̄_i = (tC̄_i // a2w_i) //. l2U // CF,
  wKink_i = (tKink_i // a2w_i) //. l2U // CF ,
  wKink̄_i = (tKink̄_i // a2w_i) //. l2U // CF,
  wΔ_{i→j,k} = w2a_i // tΔ_{i->j,k} // (a2w_j a2w_k) //. l2U // CF,
  wS_i = w2a_i // tS_i // a2w_i //. l2U // CF]
```

In[∘]:=
```
Define[wkR_{i,j} = kR_{i,j} // (a2w_i a2w_j) //. l2U // CF,
  wkR̄_{i,j} = (kR̄_{i,j} // (a2w_i a2w_j)) //. l2U // CF ,
  wkm_{i,j→k} = ((w2a_i w2a_j) // (km_{i,j→k} //. l2U) // a2w_k ) //. l2U // CF,
  wkC_i = (kC_i // a2w_i) //. l2U // CF ,
  wkC̄_i = (kC̄_i // a2w_i) //. l2U // CF,
  wkKink_i = (kKink_i // a2w_i) //. l2U // CF ,
  wkKink̄_i = (kKink̄_i // a2w_i) //. l2U // CF,
  uwkR_{i,j} = (𝔼_{{}→{i,j1}}[0, 0, 1/Sqrt[T]] wkR_{i,j1}) wkKink̄_{j2} // wkm_{j1,j2→j},
  uwkR̄_{i,j} = 𝔼_{{}→{i,j1}}[0, 0, Sqrt[T]] wkR̄_{i,j1} wkKink_{j2} // wkm_{j1,j2→j}
]
```

In[∘]:= `Simplify /@ (𝔼_{{}→{i}}[0, 0, w_i] // w2a_i // CF)`

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i\}}\Big[0,\,0,\,\Big(\frac{1}{2}-\frac{T}{2}-(-1+T)\,a_i+x_i\,y_i\Big)-\frac{\Big(2\,(-1+T)^3\,a_i^2+(-2+t+2\,T+t\,T)\,x_i^2\,y_i^2+2\,(-1+T)\,a_i\,\big((-1+T)^2-2\,(-1+T+t\,T)\,x_i\,y_i\big)\Big)\,\epsilon}{2\,\big(t\,(-1+T)^2\big)}+O[\epsilon]^2\Big]$$

In[∘]:= `Simplify /@ (𝔼_{{}→{i}}[0, 0, a_i] // a2w_i // CF)`

Out[∘]=

$$\mathbb{E}_{\{\}\to\{i\}}\Big[0,\,0,\,-\frac{-1+T+2\,w_i-2\,x_i\,y_i}{2\,(-1+T)}+\frac{\Big((-1+T)^3-4\,(-1+T)\,w_i^2-4\,t\,(-1+T)\,T\,x_i\,y_i-8\,t\,T\,w_i\,x_i\,y_i+2\,t\,(-1+3\,T)\,x_i^2\,y_i^2\Big)\,\epsilon}{4\,t\,(-1+T)^3}+O[\epsilon]^2\Big]$$

In[∘]:= (wkKink₈ //. U2l // FullSimplify) //. l2U
($\overline{\text{wkKink}_8}$ //. U2l // FullSimplify) //. l2U

Out[∘]= $\mathbb{E}_{\{\}\to\{8\}}\left[-\frac{t}{2}+\frac{t\,w_8}{\left(-1+\frac{1}{T}\right)T},\ 0,\ \frac{1}{\sqrt{T}}+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{8\}}\left[\frac{t}{2}+\frac{t\,w_8}{-1+T},\ 0,\ \sqrt{T}+O[\epsilon]^2\right]$

In[∘]:= $\text{wkm}_{1,2\to3}$

Out[∘]= $$\mathbb{E}_{\{1,2\}\to\{3\}}\Big[w_3\,\omega_1+w_3\,\omega_2,\ y_3\,\eta_1+y_3\,\eta_2+x_3\,\xi_1+(1-T)\,\eta_2\,\xi_1+x_3\,\xi_2,\ 1+\Big(-T\,\eta_2\,\xi_1-\frac{2\,T\,w_3\,\eta_2\,\xi_1}{-1+T}+\frac{(-1+3\,T)\,x_3\,y_3\,\eta_2\,\xi_1}{-1+T}+\frac{1}{2}(1-3\,T)\,y_3\,\eta_2^2\,\xi_1+\frac{1}{2}(1-3\,T)\,x_3\,\eta_2\,\xi_1^2+\frac{1}{4}\left(1-4\,T+3\,T^2\right)\eta_2^2\,\xi_1^2\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= $\text{wkm}_{1,2\to3}$ /. $\epsilon\to0$

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{3\}}\left[w_3\,\omega_1+w_3\,\omega_2,\ y_3\,\eta_1+y_3\,\eta_2+x_3\,\xi_1+(1-T)\,\eta_2\,\xi_1+x_3\,\xi_2,\ 1\right]$

In[∘]:= $\text{wkm}_{1,2\to3}$ /. {ω_ → 0, Ω_ → 1}

Out[∘]= $$\mathbb{E}_{\{1,2\}\to\{3\}}\Big[0,\ y_3\,\eta_1+y_3\,\eta_2+x_3\,\xi_1+(1-T)\,\eta_2\,\xi_1+x_3\,\xi_2,\ 1+\Big(-T\,\eta_2\,\xi_1-\frac{2\,T\,w_3\,\eta_2\,\xi_1}{-1+T}+\frac{(-1+3\,T)\,x_3\,y_3\,\eta_2\,\xi_1}{-1+T}+\frac{1}{2}(1-3\,T)\,y_3\,\eta_2^2\,\xi_1+\frac{1}{2}(1-3\,T)\,x_3\,\eta_2\,\xi_1^2+\frac{1}{4}\left(1-4\,T+3\,T^2\right)\eta_2^2\,\xi_1^2\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= $\text{km}_{1,2\to3}$

Out[∘]= $$\mathbb{E}_{\{1,2\}\to\{3\}}\Big[a_3\,\alpha_1+a_3\,\alpha_2,\ y_3\,\eta_1+\frac{y_3\,\eta_2}{\mathcal{A}_1}+\frac{x_3\,\xi_1}{\mathcal{A}_2}+(1-T)\,\eta_2\,\xi_1+x_3\,\xi_2,\ 1+\Big(2\,T\,a_3\,\eta_2\,\xi_1+\frac{x_3\,y_3\,\eta_2\,\xi_1}{\mathcal{A}_1\,\mathcal{A}_2}+\frac{(1-3\,T)\,y_3\,\eta_2^2\,\xi_1}{2\,\mathcal{A}_1}+\frac{(1-3\,T)\,x_3\,\eta_2\,\xi_1^2}{2\,\mathcal{A}_2}+\frac{1}{4}\left(1-4\,T+3\,T^2\right)\eta_2^2\,\xi_1^2\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= $\mathbf{wkR_{5,1}}$
$\overline{\mathbf{wkR}}_{5,1}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{5,1\}}\left[-\frac{t}{2}+\frac{t\,w_1}{1-T},\ -x_1 y_1+x_1 y_5,\ 1+\left(\frac{1}{2}+\frac{w_1}{-2+2T}-\frac{w_1^2}{1-2T+T^2}+\frac{w_5}{-2+2T}+\frac{w_1 w_5}{1-2T+T^2}+\frac{T x_1 y_1}{-2+2T}+\frac{2T w_1 x_1 y_1}{1-2T+T^2}-\frac{T w_5 x_1 y_1}{1-2T+T^2}+\frac{\left(1-2T-3T^2\right)x_1^2 y_1^2}{4-8T+4T^2}-\frac{x_5 y_5}{-2+2T}-\frac{w_1 x_5 y_5}{1-2T+T^2}+\frac{T x_1 x_5 y_1 y_5}{1-2T+T^2}-\frac{1}{4}x_1^2 y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{5,1\}}\left[\frac{t}{2}+\frac{t\,w_1}{-1+T},\ \frac{x_1 y_1}{T}-\frac{x_1 y_5}{T},\ 1+\left(-\frac{1}{2}-\frac{w_1}{-2+2T}+\frac{w_1^2}{1-2T+T^2}-\frac{w_5}{-2+2T}-\frac{w_1 w_5}{1-2T+T^2}+\frac{(1-2T)\,x_1 y_1}{-2T+2T^2}-\frac{2 w_1 x_1 y_1}{1-2T+T^2}+\frac{w_5 x_1 y_1}{T-2T^2+T^3}+\frac{\left(-1+2T+3T^2\right)x_1^2 y_1^2}{4T^2-8T^3+4T^4}+\frac{x_1 y_5}{T}+\frac{w_1 x_1 y_5}{-T+T^2}+\frac{w_5 x_1 y_5}{-T+T^2}+\frac{x_5 y_5}{-2+2T}+\frac{w_1 x_5 y_5}{1-2T+T^2}+\frac{x_1^2 y_1 y_5}{T^2-T^3}-\frac{x_1 x_5 y_1 y_5}{T-2T^2+T^3}-\frac{3x_1^2 y_5^2}{4T^2}+\frac{x_1 x_5 y_5^2}{T-T^2}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\mathbf{wkR_{5,1}\,wkR_{2,6}\,wkR_{7,3}\,wkC_4\ //\ wkm_{1,2\to1}\ //\ wkm_{1,3\to1}\ //\ wkm_{1,4\to1}\ //\ wkm_{1,5\to1}\ //\ wkm_{1,6\to1}\ //\ wkm_{1,7\to1}}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[-\frac{3t}{2}-\frac{3t\,w_1}{-1+T},\ 0,\ \frac{1}{\sqrt{T}-T^{3/2}+T^{5/2}}+\left(\frac{1-2T+2T^2-2T^3+T^4}{\sqrt{T}-3T^{3/2}+6T^{5/2}-7T^{7/2}+6T^{9/2}-3T^{11/2}+T^{13/2}}+\frac{(-2-2T)\,w_1}{\sqrt{T}-2T^{3/2}+3T^{5/2}-2T^{7/2}+T^{9/2}}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\mathbf{wkR_{5,1}}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{5,1\}}\left[-\frac{t}{2}+\frac{t\,w_1}{1-T},\ -x_1 y_1+x_1 y_5,\ 1+\left(\frac{1}{2}+\frac{w_1}{-2+2T}-\frac{w_1^2}{1-2T+T^2}+\frac{w_5}{-2+2T}+\frac{w_1 w_5}{1-2T+T^2}+\frac{T x_1 y_1}{-2+2T}+\frac{2T w_1 x_1 y_1}{1-2T+T^2}-\frac{T w_5 x_1 y_1}{1-2T+T^2}+\frac{\left(1-2T-3T^2\right)x_1^2 y_1^2}{4-8T+4T^2}-\frac{x_5 y_5}{-2+2T}-\frac{w_1 x_5 y_5}{1-2T+T^2}+\frac{T x_1 x_5 y_1 y_5}{1-2T+T^2}-\frac{1}{4}x_1^2 y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\mathbf{\overline{wkKink}_8\,\overline{wkKink}_9\,\overline{wkKink}_{10}\,wkR_{5,1}\,wkR_{2,6}\,wkR_{7,3}\,wkC_4\ //\ wkm_{1,2\to1}\ //\ wkm_{1,3\to1}\ //\ wkm_{1,4\to1}\ //\ wkm_{1,5\to1}\ //\ wkm_{1,6\to1}\ //\ wkm_{1,7\to1}\ //\ wkm_{1,8\to1}\ //\ wkm_{1,9\to1}\ //\ wkm_{1,10\to1}\ //\ Timing}$

Out[∘]= $$\left\{0.604,\ \mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{T-2T^2+2T^3-2T^4+T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{\left(-2T-2T^2\right)w_1}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]\right\}$$

In[∘]:=
```
kKink̄8 kKink̄9 kKink̄10 kR5,1 kR2,6 kR7,3 kC4 // km1,2→1 // km1,3→1 // km1,4→1 // km1,5→1 // km1,6→1 //
    km1,7→1 // km1,8→1 // km1,9→1 // km1,10→1
% //
 a2w1
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{T}{1-T+T^2}+\left(\frac{-T^2+2T^3-3T^4+2T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(-2T+2T^3)\,a_1}{1-2T+3T^2-2T^3+T^4}+\frac{(-2T-2T^2)\,x_1 y_1}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{T}{1-T+T^2}+\left(\frac{T-2T^2+2T^3-2T^4+T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(-2T-2T^2)\,w_1}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]  PositiveQ@x
                           Xm[x[[2]], x[[1]]]  True ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]]; {1 - l, k + 1, l})
             })],
     Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := 𝔼{}→{i}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kCj, rot[i, n + 1] kC̄j] // kmi,j→i];
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{{}→{0}}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#[[1]], #[[2]], #[[1]] - 1, #[[2]] - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
     Xp, (kR_{i,j} OverBar[kKink_k]) // km_{j,k→j},
     Xm, (OverBar[kR_{i,j}] kKink_k) // km_{j,k→j}
    ];
   ξ1 = (rot[k, rots[[i]]] ξ1) // km_{k,i→i}; rots[[i]] = 0;
   ξ1 = (ξ1 rot[k, rots[[i + 1]]]) // km_{i,k→i}; rots[[i + 1]] = 0;
   ξ1 = (rot[k, rots[[j]]] ξ1) // km_{k,j→j}; rots[[j]] = 0;
   ξ1 = (ξ1 rot[k, rots[[j + 1]]]) // km_{j,k→j}; rots[[j + 1]] = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st[[i + 2]] → st[[i + 1]]];
   If[MemberQ[done, i - 1], ξ = ξ // km_{st[[i]],i→st[[i]]}; st = st /. st[[i + 1]] → st[[i]]];
   If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st[[j + 2]] → st[[j + 1]]];
   If[MemberQ[done, j - 1], ξ = ξ // km_{st[[j]],j→st[[j]]}; st = st /. st[[j + 1]] → st[[j]]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ (*/. {x0→x,y0→y,a0→a}*))
 ]
```

In[∘]:=
```
Z31 = Z@Knot[3, 1] // Timing
```

Out[∘]=

$$\Big\{20.54,\ \mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ \frac{T}{1-T+T^2}+\left(\frac{-2T+3T^2-2T^3+T^4}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(-2T+2T^3)\,a_0}{1-2T+3T^2-2T^3+T^4}+\frac{(-2T-2T^2)\,x_0\,y_0}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+$$

$$\left(\frac{4T-11T^2+6T^3-2T^5+4T^6-2T^7+T^8}{2-10T+30T^2-60T^3+90T^4-102T^5+90T^6-60T^7+30T^8-10T^9+2T^{10}}+\frac{(4T-4T^2-14T^3+16T^4-10T^5+4T^6)\,a_0}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\right.$$

$$\frac{(2T+2T^2-12T^3+2T^4+2T^5)\,a_0^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(2T-2T^2-4T^3-4T^4-2T^5+2T^6)\,x_0\,y_0}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+$$

$$\left.\frac{(4T+12T^2-12T^3-8T^4)\,a_0\,x_0\,y_0}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(3T+9T^2+3T^3)\,x_0^2\,y_0^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\epsilon^2+O[\epsilon]^3\Big]\Big\}$$

In[∘]:=
```
Z31⟦2⟧ // a2w0
Collect[%⟦3⟧, {ϵ, w0}, Factor]
```

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0, 0, \frac{T}{1-T+T^2} + \left(\frac{-T+2T^2-2T^3+2T^4-T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6} + \frac{(-2T-2T^2)\,w_0}{1-2T+3T^2-2T^3+T^4}\right)\epsilon + \left(\frac{T-7T^2+13T^3-26T^4+38T^5-26T^6+13T^7-7T^8+T^9}{4-20T+60T^2-120T^3+180T^4-204T^5+180T^6-120T^7+60T^8-20T^9+4T^{10}} + \frac{(2T-2T^2-4T^3-4T^4-2T^5+2T^6)\,w_0}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8} + \frac{(3T+9T^2+3T^3)\,w_0^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\frac{T}{1-T+T^2} + \epsilon\left(-\frac{(-1+T)^2\,T\,(1+T^2)}{(1-T+T^2)^3} - \frac{2T(1+T)\,w_0}{(1-T+T^2)^2}\right) + \epsilon^2\left(\frac{(-1+T)^2\,T\,(1-5T+2T^2-17T^3+2T^4-5T^5+T^6)}{4(1-T+T^2)^5} + \frac{2T(1+T)(1-2T-2T^3+T^4)\,w_0}{(1-T+T^2)^4} + \frac{3T(1+3T+T^2)\,w_0^2}{(1-T+T^2)^3}\right)$

In[∘]:=
```
Z41 = Z@Knot[4, 1]
Z41 // a2w0
Collect[%⟦3⟧, {ϵ, w0}, Factor]
```

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0, 0, -\frac{T}{1-3T+T^2} + \left(\frac{T-T^3}{1-6T+11T^2-6T^3+T^4} + \frac{(2T-2T^3)\,a_0}{1-6T+11T^2-6T^3+T^4} + \frac{(2T+2T^2)\,x_0\,y_0}{1-6T+11T^2-6T^3+T^4}\right)\epsilon + \left(\frac{-T+12T^3-22T^4+12T^5-T^7}{2-24T+116T^2-288T^3+390T^4-288T^5+116T^6-24T^7+2T^8} + \frac{(-2T-6T^2+12T^3-6T^4-2T^5)\,a_0}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6} + \frac{(-2T-6T^2+12T^3-6T^4-2T^5)\,a_0^2}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6} + \frac{(-4T-20T^2+12T^3+8T^4)\,a_0\,x_0\,y_0}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6} + \frac{(-3T-11T^2-3T^3)\,x_0^2\,y_0^2}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6}\right)\epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0, 0, -\frac{T}{1-3T+T^2} + \frac{(2T+2T^2)\,w_0\,\epsilon}{1-6T+11T^2-6T^3+T^4} + \left(\frac{T-4T^2-4T^3+14T^4-4T^5-4T^6+T^7}{4-48T+232T^2-576T^3+780T^4-576T^5+232T^6-48T^7+4T^8} + \frac{(-3T-11T^2-3T^3)\,w_0^2}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6}\right)\epsilon^2 + O[\epsilon]^3\right]$

Out[∘]= $-\frac{T}{1-3T+T^2} + \frac{2T(1+T)\,\epsilon\,w_0}{(1-3T+T^2)^2} + \epsilon^2\left(\frac{(-1+T)^2\,T\,(1-2T-9T^2-2T^3+T^4)}{4(1-3T+T^2)^4} - \frac{T(3+11T+3T^2)\,w_0^2}{(1-3T+T^2)^3}\right)$

In[∘]:=
```
Z51 = Z@Knot[5, 1]
Z51 // a2w0
Collect[%[[3]], {ϵ, w0}, Factor]
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ \frac{T^2}{1-T+T^2-T^3+T^4}+\left(\frac{-4T^2+7T^3-8T^4+8T^5-6T^6+4T^7-2T^8+T^9}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\frac{(-4T^2+2T^3-2T^5+4T^6)\,a_0}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}+\frac{(-4T^2-2T^3-2T^4-4T^5)\,x_0\,y_0}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}\right)\epsilon+\Big((16T^2-55T^3+102T^4-135T^5+108T^6-59T^7+18T^8+4T^9-10T^{10}+16T^{11}-14T^{12}+13T^{13}-8T^{14}+5T^{15}-2T^{16}+T^{17})\Big/(2-10T+30T^2-70T^3+140T^4-242T^5+370T^6-510T^7+640T^8-730T^9+762T^{10}-730T^{11}+640T^{12}-510T^{13}+370T^{14}-242T^{15}+140T^{16}-70T^{17}+30T^{18}-10T^{19}+2T^{20})+\big((16T^2-34T^3+32T^4+2T^5-92T^6+130T^7-128T^8+102T^9-68T^{10}+38T^{11}-16T^{12}+6T^{13})\,a_0\big)\Big/(1-4T+10T^2-20T^3+35T^4-52T^5+68T^6-80T^7+85T^8-80T^9+68T^{10}-52T^{11}+35T^{12}-20T^{13}+10T^{14}-4T^{15}+T^{16})+\frac{(8T^2-6T^3-6T^4+28T^5-60T^6+28T^7-6T^8-6T^9+8T^{10})\,a_0^2}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\big((8T^2-12T^3+12T^4-6T^5-18T^6-4T^7-4T^8-18T^9-6T^{10}+12T^{11}-12T^{12}+8T^{13})\,x_0\,y_0\big)\Big/(1-4T+10T^2-20T^3+35T^4-52T^5+68T^6-80T^7+85T^8-80T^9+68T^{10}-52T^{11}+35T^{12}-20T^{13}+10T^{14}-4T^{15}+T^{16})+\frac{(16T^2+12T^3-4T^4+60T^5-60T^6-4T^7-8T^8-24T^9)\,a_0\,x_0\,y_0}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\frac{(10T^2+15T^3+15T^4+45T^5+15T^6+15T^7+10T^8)\,x_0^2\,y_0^2}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ \frac{T^2}{1-T+T^2-T^3+T^4}+\left(\frac{-2T^2+4T^3-5T^4+6T^5-6T^6+6T^7-5T^8+4T^9-2T^{10}}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\frac{(-4T^2-2T^3-2T^4-4T^5)\,w_0}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}\right)\epsilon+\Big((6T^2-29T^3+65T^4-115T^5+165T^6-241T^7+344T^8-450T^9+510T^{10}-450T^{11}+344T^{12}-241T^{13}+165T^{14}-115T^{15}+65T^{16}-29T^{17}+6T^{18})\Big/(4-20T+60T^2-140T^3+280T^4-484T^5+740T^6-1020T^7+1280T^8-1460T^9+1524T^{10}-1460T^{11}+1280T^{12}-1020T^{13}+740T^{14}-484T^{15}+280T^{16}-140T^{17}+60T^{18}-20T^{19}+4T^{20})+\big((8T^2-12T^3+12T^4-6T^5-18T^6-4T^7-4T^8-18T^9-6T^{10}+12T^{11}-12T^{12}+8T^{13})\,w_0\big)\Big/(1-4T+10T^2-20T^3+35T^4-52T^5+68T^6-80T^7+85T^8-80T^9+68T^{10}-52T^{11}+35T^{12}-20T^{13}+10T^{14}-4T^{15}+T^{16})+\frac{(10T^2+15T^3+15T^4+45T^5+15T^6+15T^7+10T^8)\,w_0^2}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]=
$$\frac{T^2}{1-T+T^2-T^3+T^4}+\epsilon\left(-\frac{(-1+T)^2 T^2 (1+T^2)(2+T^2+2T^4)}{(1-T+T^2-T^3+T^4)^3}-\frac{2T^2(1+T)(2-T+2T^2)\,w_0}{(1-T+T^2-T^3+T^4)^2}\right)+\epsilon^2\left(\frac{1}{4(1-T+T^2-T^3+T^4)^5}(-1+T)^2T^2(6-17T+25T^2-48T^3+44T^4-105T^5+90T^6-165T^7+90T^8-105T^9+44T^{10}-48T^{11}+25T^{12}-17T^{13}+6T^{14})+\frac{2T^2(1+T)(2-2T+5T^2-2T^3+2T^4)(2-3T-3T^5+2T^6)\,w_0}{(1-T+T^2-T^3+T^4)^4}+\frac{5T^2(2+3T+3T^2+9T^3+3T^4+3T^5+2T^6)\,w_0^2}{(1-T+T^2-T^3+T^4)^3}\right)$$

In[∘]:=
```
Z52 = Z@Knot[5, 2];
Z52 // a2w0
Collect[%[[3]], {ϵ, w0}, Factor]
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,0,\frac{T}{2-3T+2T^2}+\left(\frac{-5T+14T^2-18T^3+14T^4-5T^5}{8-36T+78T^2-99T^3+78T^4-36T^5+8T^6}+\frac{(-4T-4T^2)\,w_0}{4-12T+17T^2-12T^3+4T^4}\right)\epsilon+\left(\frac{18T-124T^2+373T^3-709T^4+884T^5-709T^6+373T^7-124T^8+18T^9}{64-480T+1760T^2-4080T^3+6580T^4-7686T^5+6580T^6-4080T^7+1760T^8-480T^9+64T^{10}}+\frac{(20T-32T^2-32T^5+20T^6)\,w_0}{16-96T+280T^2-504T^3+609T^4-504T^5+280T^6-96T^7+16T^8}+\frac{(12T+38T^2+12T^3)\,w_0^2}{8-36T+78T^2-99T^3+78T^4-36T^5+8T^6}\right)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]=
$$\frac{T}{2-3T+2T^2}+\epsilon\left(-\frac{(-1+T)^2T(5-4T+5T^2)}{(2-3T+2T^2)^3}-\frac{4T(1+T)\,w_0}{(2-3T+2T^2)^2}\right)+\epsilon^2\left(\frac{(-1+T)^2T(18-88T+179T^2-263T^3+179T^4-88T^5+18T^6)}{2(2-3T+2T^2)^5}+\frac{4T(1+T)(5-13T+13T^2-13T^3+5T^4)\,w_0}{(2-3T+2T^2)^4}+\frac{2T(6+19T+6T^2)\,w_0^2}{(2-3T+2T^2)^3}\right)$$

In[∘]:=
```
Z61 = Z@Knot[6, 1]
Z61 // a2w0
Collect[%[[3]], {ϵ, w0}, Factor]
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ -\frac{T}{2-5T+2T^2}+\left(\frac{5T-16T^2+10T^3+4T^4-3T^5}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}+\frac{(4T-4T^3)\,a_0}{4-20T+33T^2-20T^3+4T^4}+\frac{(4T+4T^2)\,x_0\,y_0}{4-20T+33T^2-20T^3+4T^4}\right)\epsilon+$$
$$\Big((-26T+140T^2-193T^3-186T^4+722T^5-698T^6+263T^7-12T^8-10T^9)\,/\,(64-800T+4320T^2-13\,200T^3+25\,140T^4-31\,050T^5+25\,140T^6-13\,200T^7+4320T^8-800T^9+64T^{10})+$$
$$\frac{(-20T+28T^2+140T^3-320T^4+220T^5-28T^6-12T^7)\,a_0}{16-160T+664T^2-1480T^3+1921T^4-1480T^5+664T^6-160T^7+16T^8}+\frac{(-8T-20T^2+48T^3-20T^4-8T^5)\,a_0^2}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}+$$
$$\frac{(-4T+24T^2-16T^3-16T^4+24T^5-4T^6)\,x_0\,y_0}{16-160T+664T^2-1480T^3+1921T^4-1480T^5+664T^6-160T^7+16T^8}+\frac{(-16T-72T^2+48T^3+32T^4)\,a_0\,x_0\,y_0}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}+$$
$$\frac{(-12T-42T^2-12T^3)\,x_0^2\,y_0^2}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\ 0,\ -\frac{T}{2-5T+2T^2}+\left(\frac{T-6T^2+10T^3-6T^4+T^5}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}+\frac{(4T+4T^2)\,w_0}{4-20T+33T^2-20T^3+4T^4}\right)\epsilon+$$
$$\Big((6T-20T^2-67T^3+371T^4-580T^5+371T^6-67T^7-20T^8+6T^9)\,/\,(64-800T+4320T^2-13\,200T^3+25\,140T^4-31\,050T^5+25\,140T^6-13\,200T^7+4320T^8-800T^9+64T^{10})+$$
$$\frac{(-4T+24T^2-16T^3-16T^4+24T^5-4T^6)\,w_0}{16-160T+664T^2-1480T^3+1921T^4-1480T^5+664T^6-160T^7+16T^8}+\frac{(-12T-42T^2-12T^3)\,w_0^2}{8-60T+174T^2-245T^3+174T^4-60T^5+8T^6}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]=
$$-\frac{T}{(-2+T)\,(-1+2T)}+\epsilon\left(\frac{(-1+T)^2\,T\,(1-4T+T^2)}{(-2+T)^3\,(-1+2T)^3}+\frac{4T\,(1+T)\,w_0}{(-2+T)^2\,(-1+2T)^2}\right)+$$
$$\epsilon^2\left(\frac{(-1+T)^2\,T\,(3-7T+3T^2)\,(2+2T-27T^2+2T^3+2T^4)}{2\,(-2+T)^5\,(-1+2T)^5}-\frac{4T\,(1+T)\,(1-7T+11T^2-7T^3+T^4)\,w_0}{(-2+T)^4\,(-1+2T)^4}-\frac{6T\,(2+7T+2T^2)\,w_0^2}{(-2+T)^3\,(-1+2T)^3}\right)$$

```
In[]:= Z63 = Z@Knot[6, 3]
       Z63 // a2w₀
       Collect[%[[3]], {ϵ, w₀}, Factor]
```

Out[]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\,0,\,\frac{T^2}{1-3T+5T^2-3T^3+T^4}+\Big(\frac{-2T^2+3T^3-3T^5+2T^6}{1-6T+19T^2-36T^3+45T^4-36T^5+19T^6-6T^7+T^8}+\frac{(-4T^2+6T^3-6T^5+4T^6)\,a_0}{1-6T+19T^2-36T^3+45T^4-36T^5+19T^6-6T^7+T^8}+\frac{(-4T^2+2T^3+2T^4-4T^5)\,x_0\,y_0}{1-6T+19T^2-36T^3+45T^4-36T^5+19T^6-6T^7+T^8}\Big)\epsilon+\Big((4T^2-21T^3+34T^4+32T^5-244T^6+522T^7-654T^8+522T^9-244T^{10}+32T^{11}+34T^{12}-21T^{13}+4T^{14})\,/\,(2-24T+148T^2-600T^3+1766T^4-3960T^5+6952T^6-9696T^7+10826T^8-9696T^9+6952T^{10}-3960T^{11}+1766T^{12}-600T^{13}+148T^{14}-24T^{15}+2T^{16})+\frac{(8T^2-18T^3-22T^4+108T^5-156T^6+108T^7-22T^8-18T^9+8T^{10})\,a_0}{1-9T+42T^2-126T^3+267T^4-414T^5+479T^6-414T^7+267T^8-126T^9+42T^{10}-9T^{11}+T^{12}}+\frac{(8T^2-18T^3-22T^4+108T^5-156T^6+108T^7-22T^8-18T^9+8T^{10})\,a_0^2}{1-9T+42T^2-126T^3+267T^4-414T^5+479T^6-414T^7+267T^8-126T^9+42T^{10}-9T^{11}+T^{12}}+\frac{(16T^2-12T^3-84T^4+180T^5-156T^6+36T^7+40T^8-24T^9)\,a_0\,x_0\,y_0}{1-9T+42T^2-126T^3+267T^4-414T^5+479T^6-414T^7+267T^8-126T^9+42T^{10}-9T^{11}+T^{12}}+\frac{(10T^2-3T^3-33T^4+51T^5-33T^6-3T^7+10T^8)\,x_0^2\,y_0^2}{1-9T+42T^2-126T^3+267T^4-414T^5+479T^6-414T^7+267T^8-126T^9+42T^{10}-9T^{11}+T^{12}}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[]=
$$\mathbb{E}_{\{\}\to\{0\}}\Big[0,\,0,\,\frac{T^2}{1-3T+5T^2-3T^3+T^4}+\frac{(-4T^2+2T^3+2T^4-4T^5)\,w_0\,\epsilon}{1-6T+19T^2-36T^3+45T^4-36T^5+19T^6-6T^7+T^8}+\Big((-2T^2+11T^3-34T^4+44T^5+28T^6-172T^7+250T^8-172T^9+28T^{10}+44T^{11}-34T^{12}+11T^{13}-2T^{14})\,/\,(4-48T+296T^2-1200T^3+3532T^4-7920T^5+13904T^6-19392T^7+21652T^8-19392T^9+13904T^{10}-7920T^{11}+3532T^{12}-1200T^{13}+296T^{14}-48T^{15}+4T^{16})+\frac{(10T^2-3T^3-33T^4+51T^5-33T^6-3T^7+10T^8)\,w_0^2}{1-9T+42T^2-126T^3+267T^4-414T^5+479T^6-414T^7+267T^8-126T^9+42T^{10}-9T^{11}+T^{12}}\Big)\epsilon^2+O[\epsilon]^3\Big]$$

Out[]=
$$\frac{T^2}{1-3T+5T^2-3T^3+T^4}-\frac{2T^2(1+T)(2-3T+2T^2)\,\epsilon\,w_0}{(1-3T+5T^2-3T^3+T^4)^2}+\epsilon^2\left(-\frac{(-1+T)^2T^2(2-7T+18T^2-T^3-48T^4+77T^5-48T^6-T^7+18T^8-7T^9+2T^{10})}{4(1-3T+5T^2-3T^3+T^4)^4}+\frac{T^2(10-3T-33T^2+51T^3-33T^4-3T^5+10T^6)\,w_0^2}{(1-3T+5T^2-3T^3+T^4)^3}\right)$$

In[∘]:= $\frac{T - 4 T^2 - 12 T^3 + 22 T^4 - 12 T^5 - 4 T^6 + T^7}{12 - 144 T + 696 T^2 - 1728 T^3 + 2340 T^4 - 1728 T^5 + 696 T^6 - 144 T^7 + 12 T^8}$ // Factor

Out[∘]= $\frac{T \left(1 - 4 T - 12 T^2 + 22 T^3 - 12 T^4 - 4 T^5 + T^6\right)}{12 \left(1 - 3 T + T^2\right)^4}$

In[∘]:= $\left(5 + \frac{1}{t^4} - \frac{4}{t^2} - 4 t^2 + t^4\right)$ /. $t \to T^{1/2}$ // Factor

Out[∘]= $\frac{\left(1 - 3 T + T^2\right) \left(1 - T + T^2\right)}{T^2}$

In[∘]:= $1 - \frac{1}{t^4} + \frac{1}{t^2} + t^2 - t^4$ /. $t \to T^{1/2}$ // Factor

Out[∘]= $-\frac{1 - T - T^2 - T^3 + T^4}{T^2}$

In[∘]:= $-T + 2 T^2 - 2 T^3 + 2 T^4 - T^5$ // Factor

Out[∘]= $-\left(-1 + T\right)^2 T \left(1 + T^2\right)$

In[∘]:= $\frac{\left(-2 T^2 + 4 T^3 - 5 T^4 + 6 T^5 - 6 T^6 + 6 T^7 - 5 T^8 + 4 T^9 - 2 T^{10}\right)}{-\left(-1 + T\right)^2 T}$ // Factor // Expand

Out[∘]= $2 T + 3 T^3 + 3 T^5 + 2 T^7$

In[∘]:= $\left(22 T^2 - 93 T^3 + 205 T^4 - 355 T^5 + 505 T^6 - 721 T^7 + 1024 T^8 - 1330 T^9 + 1510 T^{10} - 1330 T^{11} + 1024 T^{12} - 721 T^{13} + 505 T^{14} - 355 T^{15} + 205 T^{16} - 93 T^{17} + 22 T^{18}\right)$ // Factor

Out[∘]= $T^2 \left(22 - 93 T + 205 T^2 - 355 T^3 + 505 T^4 - 721 T^5 + 1024 T^6 - 1330 T^7 + 1510 T^8 - 1330 T^9 + 1024 T^{10} - 721 T^{11} + 505 T^{12} - 355 T^{13} + 205 T^{14} - 93 T^{15} + 22 T^{16}\right)$

In[∘]:= **Knots = Z /@ AllKnots[{3, 5}]**

Out[∘]=
$$\left\{\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T}{1-T+T^2}+\left(\frac{-T+2T^2-2T^3+2T^4-T^5}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{(-2T-2T^2)w}{1-2T+3T^2-2T^3+T^4}\right)\in+\left(\frac{5T-23T^2+41T^3-74T^4+110T^5-74T^6+41T^7-23T^8+5T^9}{12-60T+180T^2-360T^3+540T^4-612T^5+540T^6-360T^7+180T^8-60T^9+12T^{10}}+\frac{(2T-2T^2-4T^3-4T^4-2T^5+2T^6)w}{1-4T+10T^2-16T^3+19T^4-16T^5+10T^6-4T^7+T^8}+\frac{(3T+9T^2+3T^3)w^2}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}\right)\in^2+O[\in]^3\right],\right.$$

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,0,-\frac{T}{1-3T+T^2}+\frac{(2T+2T^2)w\in}{1-6T+11T^2-6T^3+T^4}+\left(\frac{T-4T^2-12T^3+22T^4-12T^5-4T^6+T^7}{12-144T+696T^2-1728T^3+2340T^4-1728T^5+696T^6-144T^7+12T^8}+\frac{(-3T-11T^2-3T^3)w^2}{1-9T+30T^2-45T^3+30T^4-9T^5+T^6}\right)\in^2+O[\in]^3\right],$$

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T^2}{1-T+T^2-T^3+T^4}+\left(\frac{-2T^2+4T^3-5T^4+6T^5-6T^6+6T^7-5T^8+4T^9-2T^{10}}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}+\frac{(-4T^2-2T^3-2T^4-4T^5)w}{1-2T+3T^2-4T^3+5T^4-4T^5+3T^6-2T^7+T^8}\right)\in+\right.$$
$$\Big(\big(22T^2-93T^3+205T^4-355T^5+505T^6-721T^7+1024T^8-1330T^9+1510T^{10}-1330T^{11}+1024T^{12}-721T^{13}+505T^{14}-355T^{15}+205T^{16}-93T^{17}+22T^{18}\big)\big/\big(12-60T+180T^2-420T^3+840T^4-1452T^5+2220T^6-3060T^7+3840T^8-4380T^9+4572T^{10}-4380T^{11}+3840T^{12}-3060T^{13}+2220T^{14}-1452T^{15}+840T^{16}-420T^{17}+180T^{18}-60T^{19}+12T^{20}\big)+$$
$$\big(\big(8T^2-12T^3+12T^4-6T^5-18T^6-4T^7-4T^8-18T^9-6T^{10}+12T^{11}-12T^{12}+8T^{13}\big)w\big)\big/\big(1-4T+10T^2-20T^3+35T^4-52T^5+68T^6-80T^7+85T^8-80T^9+68T^{10}-52T^{11}+35T^{12}-20T^{13}+10T^{14}-4T^{15}+T^{16}\big)+$$
$$\left.\frac{(10T^2+15T^3+15T^4+45T^5+15T^6+15T^7+10T^8)w^2}{1-3T+6T^2-10T^3+15T^4-18T^5+19T^6-18T^7+15T^8-10T^9+6T^{10}-3T^{11}+T^{12}}\Big)\in^2+O[\in]^3\right],$$

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T}{2-3T+2T^2}+\left(\frac{-5T+14T^2-18T^3+14T^4-5T^5}{8-36T+78T^2-99T^3+78T^4-36T^5+8T^6}+\frac{(-4T-4T^2)w}{4-12T+17T^2-12T^3+4T^4}\right)\in+\right.$$
$$\Big(\big(70T-412T^2+1147T^3-2085T^4+2568T^5-2085T^6+1147T^7-412T^8+70T^9\big)\big/\big(192-1440T+5280T^2-12\,240T^3+19\,740T^4-23\,058T^5+19\,740T^6-12\,240T^7+5280T^8-1440T^9+192T^{10}\big)+$$
$$\left.\left.\frac{(20T-32T^2-32T^5+20T^6)w}{16-96T+280T^2-504T^3+609T^4-504T^5+280T^6-96T^7+16T^8}+\frac{(12T+38T^2+12T^3)w^2}{8-36T+78T^2-99T^3+78T^4-36T^5+8T^6}\Big)\in^2+O[\in]^3\right]\right\}$$

```
In[]:= Factor[12 (Normal@#[[3]] /. {w → 0, ϵ → 0})^-5 Coefficient[#[[3]] /. w → 0, ϵ^2]] & /@ Knots
```

Out[]= $\left\{\frac{5-23T+41T^2-74T^3+110T^4-74T^5+41T^6-23T^7+5T^8}{T^4},\right.$
$-\frac{(1-3T+T^2)(1-4T-12T^2+22T^3-12T^4-4T^5+T^6)}{T^4}, \frac{1}{T^8}$
$(22-93T+205T^2-355T^3+505T^4-721T^5+1024T^6-1330T^7+1510T^8-1330T^9+1024T^{10}-721T^{11}+505T^{12}-355T^{13}+205T^{14}-93T^{15}+22T^{16}),$
$\left.\frac{2(70-412T+1147T^2-2085T^3+2568T^4-2085T^5+1147T^6-412T^7+70T^8)}{T^4}\right\}$

```
In[]:= -2 T^2 + 4 T^3 - 5 T^4 + 6 T^5 - 6 T^6 + 6 T^7 - 5 T^8 + 4 T^9 - 2 T^10 // Factor
```

Out[]= $-(-1+T)^2 T^2 (1+T^2)(2+T^2+2T^4)$

```
In[]:= Analyse[EE_] := CoefficientList[Coefficient[EE[[3]], ϵ], w] // Factor
```

In[∘]:= **Knots = Analyse /@ Z /@ AllKnots[{3, 7}]**

Out[∘]=
$$\Big\{\Big\{-\frac{(-1+T)^2 T (1+T^2)}{(1-T+T^2)^3}, -\frac{2T(1+T)}{(1-T+T^2)^2}\Big\}, \Big\{0, \frac{2T(1+T)}{(1-3T+T^2)^2}\Big\},$$
$$\Big\{-\frac{(-1+T)^2 T^2 (1+T^2)(2+T^2+2T^4)}{(1-T+T^2-T^3+T^4)^3}, -\frac{2T^2(1+T)(2-T+2T^2)}{(1-T+T^2-T^3+T^4)^2}\Big\},$$
$$\Big\{-\frac{(-1+T)^2 T (5-4T+5T^2)}{(2-3T+2T^2)^3}, -\frac{4T(1+T)}{(2-3T+2T^2)^2}\Big\},$$
$$\Big\{\frac{(-1+T)^2 T (1-4T+T^2)}{(-2+T)^3(-1+2T)^3}, \frac{4T(1+T)}{(-2+T)^2(-1+2T)^2}\Big\},$$
$$\Big\{\frac{(-1+T)^2 T^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{(1-3T+3T^2-3T^3+T^4)^3}, \frac{2T^2(1+T)(2-3T+2T^2)}{(1-3T+3T^2-3T^3+T^4)^2}\Big\},$$
$$\Big\{0, -\frac{2T^2(1+T)(2-3T+2T^2)}{(1-3T+5T^2-3T^3+T^4)^2}\Big\},$$
$$\Big\{-\frac{(-1+T)^2 T^3 (1+T^2)(3+2T^2+4T^4+2T^6+3T^8)}{(1-T+T^2-T^3+T^4-T^5+T^6)^3}, -\frac{2T^3(1+T)(3-2T+4T^2-2T^3+3T^4)}{(1-T+T^2-T^3+T^4-T^5+T^6)^2}\Big\},$$
$$\Big\{-\frac{2(-1+T)^2 T (7-8T+7T^2)}{(3-5T+3T^2)^3}, -\frac{6T(1+T)}{(3-5T+3T^2)^2}\Big\},$$
$$\Big\{\frac{(-1+T)^2 T^2 (9-8T+16T^2-12T^3+16T^4-8T^5+9T^6)}{(2-3T+3T^2-3T^3+2T^4)^3}, -\frac{2T^2(1+T)(4-3T+4T^2)}{(2-3T+3T^2-3T^3+2T^4)^2}\Big\},$$
$$\Big\{\frac{8(-1+T)^2 T (3-4T+3T^2)}{(4-7T+4T^2)^3}, -\frac{8T(1+T)}{(4-7T+4T^2)^2}\Big\},$$
$$\Big\{-\frac{(-1+T)^2 T^2 (9-16T+29T^2-28T^3+29T^4-16T^5+9T^6)}{(2-4T+5T^2-4T^3+2T^4)^3}, -\frac{8T^2(1+T)(1-T+T^2)}{(2-4T+5T^2-4T^3+2T^4)^2}\Big\},$$
$$\Big\{\frac{(-1+T)^2 T^2 (1-8T+19T^2-20T^3+19T^4-8T^5+T^6)}{(1-5T+7T^2-5T^3+T^4)^3}, \frac{2(-2+T)T^2(1+T)(-1+2T)}{(1-5T+7T^2-5T^3+T^4)^2}\Big\},$$
$$\Big\{\frac{(-1+T)^2 T^4 (3-8T+3T^2)}{(1-5T+9T^2-5T^3+T^4)^3}, -\frac{2(-2+T)T^2(1+T)(-1+2T)}{(1-5T+9T^2-5T^3+T^4)^2}\Big\}\Big\}$$

In[∘]:=
```
Knots // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix}
-\frac{(-1+T)^2 T (1+T^2)}{(1-T+T^2)^3} & -\frac{2T(1+T)}{(1-T+T^2)^2} \\
0 & \frac{2T(1+T)}{(1-3T+T^2)^2} \\
-\frac{(-1+T)^2 T^2 (1+T^2)(2+T^2+2T^4)}{(1-T+T^2-T^3+T^4)^3} & -\frac{2T^2(1+T)(2-T+2T^2)}{(1-T+T^2-T^3+T^4)^2} \\
-\frac{(-1+T)^2 T (5-4T+5T^2)}{(2-3T+2T^2)^3} & -\frac{4T(1+T)}{(2-3T+2T^2)^2} \\
\frac{(-1+T)^2 T (1-4T+T^2)}{(-2+T)^3(-1+2T)^3} & \frac{4T(1+T)}{(-2+T)^2(-1+2T)^2} \\
\frac{(-1+T)^2 T^2 (1-4T+4T^2-4T^3+4T^4-4T^5+T^6)}{(1-3T+3T^2-3T^3+T^4)^3} & \frac{2T^2(1+T)(2-3T+2T^2)}{(1-3T+3T^2-3T^3+T^4)^2} \\
0 & -\frac{2T^2(1+T)(2-3T+2T^2)}{(1-3T+5T^2-3T^3+T^4)^2} \\
-\frac{(-1+T)^2 T^3 (1+T^2)(3+2T^2+4T^4+2T^6+3T^8)}{(1-T+T^2-T^3+T^4-T^5+T^6)^3} & -\frac{2T^3(1+T)(3-2T+4T^2-2T^3+3T^4)}{(1-T+T^2-T^3+T^4-T^5+T^6)^2} \\
-\frac{2(-1+T)^2 T (7-8T+7T^2)}{(3-5T+3T^2)^3} & -\frac{6T(1+T)}{(3-5T+3T^2)^2} \\
\frac{(-1+T)^2 T^2 (9-8T+16T^2-12T^3+16T^4-8T^5+9T^6)}{(2-3T+3T^2-3T^3+2T^4)^3} & -\frac{2T^2(1+T)(4-3T+4T^2)}{(2-3T+3T^2-3T^3+2T^4)^2} \\
\frac{8(-1+T)^2 T (3-4T+3T^2)}{(4-7T+4T^2)^3} & -\frac{8T(1+T)}{(4-7T+4T^2)^2} \\
-\frac{(-1+T)^2 T^2 (9-16T+29T^2-28T^3+29T^4-16T^5+9T^6)}{(2-4T+5T^2-4T^3+2T^4)^3} & -\frac{8T^2(1+T)(1-T+T^2)}{(2-4T+5T^2-4T^3+2T^4)^2} \\
\frac{(-1+T)^2 T^2 (1-8T+19T^2-20T^3+19T^4-8T^5+T^6)}{(1-5T+7T^2-5T^3+T^4)^3} & \frac{2(-2+T)T^2(1+T)(-1+2T)}{(1-5T+7T^2-5T^3+T^4)^2} \\
\frac{(-1+T)^2 T^4 (3-8T+3T^2)}{(1-5T+9T^2-5T^3+T^4)^3} & -\frac{2(-2+T)T^2(1+T)(-1+2T)}{(1-5T+9T^2-5T^3+T^4)^2}
\end{pmatrix}$$

In[∘]:=
```
Alexes[KK_] := D[(Alexander@KK)@T, T] // Factor
```

In[∘]:=
```
Alex[KK_] := ((Alexander@KK)@T)^2
```

In[∘]:=
```
Alexes /@ AllKnots[{3, 7}]
```

Out[∘]=

$$\left\{\frac{(-1+T)(1+T)}{T^2},\ -\frac{(-1+T)(1+T)}{T^2},\ \frac{(-1+T)(1+T)(2-T+2T^2)}{T^3},\ \frac{2(-1+T)(1+T)}{T^2},\ -\frac{2(-1+T)(1+T)}{T^2},\ -\frac{(-1+T)(1+T)(2-3T+2T^2)}{T^3},\ \frac{(-1+T)(1+T)(2-3T+2T^2)}{T^3},\ \frac{(-1+T)(1+T)(3-2T+4T^2-2T^3+3T^4)}{T^4},\ \frac{3(-1+T)(1+T)}{T^2},\ \frac{(-1+T)(1+T)(4-3T+4T^2)}{T^3},\ \frac{4(-1+T)(1+T)}{T^2},\ \frac{4(-1+T)(1+T)(1-T+T^2)}{T^3},\ -\frac{(-2+T)(-1+T)(1+T)(-1+2T)}{T^3},\ \frac{(-2+T)(-1+T)(1+T)(-1+2T)}{T^3}\right\}$$

In[∘]:=
```
{(Alex /@ AllKnots[{3, 7}]) Knots[[;; , 2]] / Alexes /@ AllKnots[{3, 7}] // Factor} // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} & -\frac{2T}{-1+T} \end{pmatrix}$$

The most basic tensors

In[∘]:= $\text{wkm}_{i,j\to k}$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[w_k\,\omega_i + w_k\,\omega_j,\ y_k\,\eta_i + y_k\,\eta_j + x_k\,\xi_i + (1-T)\,\eta_j\,\xi_i + x_k\,\xi_j,$
$1+\Big(-T\,\eta_j\,\xi_i - \frac{2\,T\,w_k\,\eta_j\,\xi_i}{-1+T} + \frac{(-1+3\,T)\,x_k\,y_k\,\eta_j\,\xi_i}{-1+T} +$
$\frac{1}{2}(1-3\,T)\,y_k\,\eta_j^2\,\xi_i + \frac{1}{2}(1-3\,T)\,x_k\,\eta_j\,\xi_i^2 + \frac{1}{4}(1-4\,T+3\,T^2)\,\eta_j^2\,\xi_i^2\Big)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= $(\text{wkm}_{1,2\to1}\ //\ \text{wkm}_{1,3\to1}) \equiv (\text{wkm}_{2,3\to2}\ //\ \text{wkm}_{1,2\to1})$

Out[∘]= True

In[∘]:= $\text{wkR}_{i,j}$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\Big[-\frac{t}{2} + \frac{t\,w_j}{1-T},\ x_j\,y_i - x_j\,y_j,$
$1+\Big(\frac{1}{2} + \frac{w_i}{-2+2\,T} + \frac{w_j}{-2+2\,T} + \frac{w_i\,w_j}{1-2\,T+T^2} - \frac{w_j^2}{1-2\,T+T^2} - \frac{x_i\,y_i}{-2+2\,T} - \frac{w_j\,x_i\,y_i}{1-2\,T+T^2} - \frac{1}{4}x_j^2\,y_i^2 +$
$\frac{T\,x_j\,y_j}{-2+2\,T} - \frac{T\,w_i\,x_j\,y_j}{1-2\,T+T^2} + \frac{2\,T\,w_j\,x_j\,y_j}{1-2\,T+T^2} + \frac{T\,x_i\,x_j\,y_i\,y_j}{1-2\,T+T^2} + \frac{(1-2\,T-3\,T^2)\,x_j^2\,y_j^2}{4-8\,T+4\,T^2}\Big)\,\epsilon + O[\epsilon]^2\Big]$

In[∘]:= $(\text{wkR}_{1,2}\,\text{wkR}_{4,3}\,\text{wkR}_{5,6}\ //\ \text{wkm}_{1,4\to1}\,\text{wkm}_{2,5\to2}\,\text{wkm}_{3,6\to3}) \equiv (\text{wkR}_{2,3}\,\text{wkR}_{1,6}\,\text{wkR}_{4,5}\ //\ \text{wkm}_{1,4\to1}\,\text{wkm}_{2,5\to2}\,\text{wkm}_{3,6\to3})$

Out[∘]= True

Even basicer tensors

In[∘]:= $\text{bbm}_{i\_,j\_\to k\_} :=$
$\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\,\eta_i + y_k\,\eta_j + x_k\,\xi_i + (1-T)\,\eta_j\,\xi_i + x_k\,\xi_j,\ 1+\epsilon\Big(-T\,\eta_j\,\xi_i + \frac{(-1+3\,T)\,x_k\,y_k\,\eta_j\,\xi_i}{-1+T} +$
$\frac{1}{2}(1-3\,T)\,y_k\,\eta_j^2\,\xi_i + \frac{1}{2}(1-3\,T)\,x_k\,\eta_j\,\xi_i^2 + \frac{1}{4}(1-4\,T+3\,T^2)\,\eta_j^2\,\xi_i^2\Big) + O[\epsilon]^2\Big]$

In[∘]:= $(\text{bbm}_{1,2\to1}\ //\ \text{bbm}_{1,3\to1}) \equiv (\text{bbm}_{2,3\to2}\ //\ \text{bbm}_{1,2\to1})$

Out[∘]= True

In[∘]:= $\text{bbR}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\Big[0,\ x_j\,y_i - x_j\,y_j,$
$1+\epsilon\Big(\frac{1}{2} - \frac{x_i\,y_i}{-2+2\,T} - \frac{1}{4}x_j^2\,y_i^2 + \frac{T\,x_j\,y_j}{-2+2\,T} + \frac{T\,x_i\,x_j\,y_i\,y_j}{1-2\,T+T^2} + \frac{(1-2\,T-3\,T^2)\,x_j^2\,y_j^2}{4-8\,T+4\,T^2}\Big) + O[\epsilon]^2\Big]$

In[∘]:= $(\text{bbR}_{1,2}\,\text{bbR}_{4,3}\,\text{bbR}_{5,6}\ //\ \text{bbm}_{1,4\to1}\,\text{bbm}_{2,5\to2}\,\text{bbm}_{3,6\to3}) \equiv (\text{bbR}_{2,3}\,\text{bbR}_{1,6}\,\text{bbR}_{4,5}\ //\ \text{bbm}_{1,4\to1}\,\text{bbm}_{2,5\to2}\,\text{bbm}_{3,6\to3})$

Out[∘]= True

Even basicer tensors

In[∘]:=
$$\mathrm{bbm}_{i\_,j\_\to k\_} := \mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\,\eta_i + y_k\,\eta_j + x_k\,\xi_i + (1-T)\,\eta_j\,\xi_i + x_k\,\xi_j,\ 1+\epsilon\Big(-T\,\eta_j\,\xi_i + \frac{(-1+3T)\,x_k\,y_k\,\eta_j\,\xi_i}{-1+T} + \frac{1}{2}(1-3T)\,y_k\,\eta_j^2\,\xi_i + \frac{1}{2}(1-3T)\,x_k\,\eta_j\,\xi_i^2 + \frac{1}{4}(1-4T+3T^2)\,\eta_j^2\,\xi_i^2\Big) + O[\epsilon]^2\Big]$$

In[∘]:=
$$(\mathrm{bbm}_{1,2\to1}\ //\ \mathrm{bbm}_{1,3\to1}) \equiv (\mathrm{bbm}_{2,3\to2}\ //\ \mathrm{bbm}_{1,2\to1})$$

Out[∘]= True

In[∘]:=
$$\mathrm{bbR}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\Big[0,\ x_j\,y_i - x_j\,y_j,\ 1+\epsilon\Big(\frac{1}{2} - \frac{x_i\,y_i}{-2+2T} - \frac{1}{4}x_j^2\,y_i^2 + \frac{T\,x_j\,y_j}{-2+2T} + \frac{T\,x_i\,x_j\,y_i\,y_j}{1-2T+T^2} + \frac{(1-2T-3T^2)\,x_j^2\,y_j^2}{4-8T+4T^2}\Big) + O[\epsilon]^2\Big]$$

In[∘]:=
$$(\mathrm{bbR}_{1,2}\,\mathrm{bbR}_{4,3}\,\mathrm{bbR}_{5,6}\ //\ \mathrm{bbm}_{1,4\to1}\,\mathrm{bbm}_{2,5\to2}\,\mathrm{bbm}_{3,6\to3}) \equiv (\mathrm{bbR}_{2,3}\,\mathrm{bbR}_{1,6}\,\mathrm{bbR}_{4,5}\ //\ \mathrm{bbm}_{1,4\to1}\,\mathrm{bbm}_{2,5\to2}\,\mathrm{bbm}_{3,6\to3})$$

Out[∘]= True