In[ ]:=
```
PP_ = Identity; $k = 0; γ = 1; ħ;
```

The “Speedy” Engine

Internal Utilities

Canonical Form:

In[ ]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :→ e^(x+y) /. e^x_ :→ e^CCF[x]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := Module[
   {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

In[ ]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

In[ ]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

In[ ]:=
```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

In[ ]:=
```
U2l = {B_i_^p_. :→ e^(-p ħ γ b_i), B^p_. :→ e^(-p ħ γ b), T_i_^p_. :→ e^(-p ħ t_i), T^p_. :→ e^(-p ħ t), 𝒜_i_^p_. :→ e^(p γ α_i), 𝒜^p_. :→ e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^(-c/(ħ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ħ γ)) e^d,
    e^(c_. t_i_ + d_.) :→ T_i^(-c/ħ) e^d, e^(c_. t + d_.) :→ T^(-c/ħ) e^d,
    e^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
    e^ℰ_ :→ e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

In[∘]:=
```
D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_(b_i_)[f_] := ∂_(b_i) f - ℏ γ B_i ∂_(B_i) f;
D_t[f_] := ∂_t f - ℏ T ∂_T f; D_(t_i_)[f_] := ∂_(t_i) f - ℏ T_i ∂_(T_i) f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_(α_i_)[f_] := ∂_(α_i) f + γ 𝒜_i ∂_(𝒜_i) f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] :=
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :→ (D_{ζ*,d}[f])) /. ζ* → 0 /.
   ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}|\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i}\Big|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

In[∘]:=
```
QZip_ζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_(z,ζ*) - ∂_(z,ζ) Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^Q as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

In[ ]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[K δ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule,
     zrule /. r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
     Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ]];
```

In[ ]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
     L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
     R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{is}[L, R];
```

𝔼 morphisms with domain and range.

In[∘]:=
```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{dr_}[L_, Q_, P_]_{$k_} := 𝔼_{dr} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}〚i〛;
X_{i_,j_} := 𝔼_{{}→{i,j}}[0, (y_i - y_j) x_j, 1 + ϵ (x_i y_i - x_j y_i - x_j y_j) +
    ϵ^2 (Sum[r_{If[#===i,0,1]&@k,If[#===i,0,1]&@l} y_k x_l, {k, {i, j}}, {l, {i, j}}] +
       Sum[s_{If[#===i,0,1]&@k,If[#===i,0,1]&@l,If[#===i,0,1]&@m,If[#===i,0,1]&@n} y_k x_l y_m x_n,
        {k, {i, j}}, {l, {i, j}}, {m, {i, j}}, {n, {i, j}}]) + O[ϵ]^3] /.
  {r_{1,0} → 0, r_{1,1} → 1, r_{0,0} → -1, r_{0,1} → 0, s_{0,0,0,0} → 1/2, s_{1,0,1,0} → 0, s_{1,0,0,0} → 1,
   s_{0,0,1,0} → -1, s_{1,1,1,1} → 1/2, s_{1,1,1,0} → 1, s_{1,0,1,1} → -1, s_{0,0,1,1} → 1, s_{0,1,0,1} → 1, s_{0,1,1,0} → -1,
   s_{1,0,0,1} → 1, s_{1,1,0,0} → -1, s_{0,0,0,1} → -1, s_{0,1,0,0} → -1, s_{0,1,1,1} → 1, s_{1,1,0,1} → -1/2}
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_{is__} = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_{isp}, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

In[∘]:=
```
Define[m_{i,j→k} = 𝔼_{{i,j}→{k}}[0, - ξ_i η_j + (η_i + η_j) y_k + (ξ_i + ξ_j) x_k, 1]]
(*Heisenberg multiplication*)
```

The timid R-matrix and its inverse

$X_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ (y_i - y_j)\ (T-1)\ x_j,\ 1 + \epsilon\ (x_i\, y_i - (T-1)\ x_j\, y_i - x_j\, y_j) + O[\epsilon]^2\right]$

$\overline{X}_{i\_,j\_} := \mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ (y_i - y_j)\ (T^{-1}-1)\ x_j,\ 1 - \epsilon\left(x_i\, y_i + \frac{1-T}{T^2}\, x_j\, y_i + 0\, x_i\, y_j - T^{-2}\, x_j\, y_j\right) + O[\epsilon]^2\right]$

$CC_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{T} + -\frac{-1+T}{2\sqrt{T}}\,\epsilon + O[\epsilon]^2\right]$

$\overline{CC}_{i\_} := \mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \left(\sqrt{T}\right)^{-1} + -\frac{1-T}{2\,T^{3/2}}\,\epsilon + O[\epsilon]^2\right]$

$Kink_{i\_} := CC_3\, X_{1,2}\ //\ m_{2,3\to2}\ //\ m_{2,1\to i}$

$\overline{Kink}_{i\_} := CC_3\, \overline{X}_{1,2}\ //\ m_{1,3\to1}\ //\ m_{1,2\to i}$

In[∘]:= $CC_1\, \overline{CC}_2\ //\ m_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ 1 + \left(\frac{a}{\sqrt{T}} + b\sqrt{T}\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:= $\text{Solve}\left[\left\{\frac{a}{\sqrt{T}} + b\sqrt{T} == 0,\ b == \left(\frac{-1+T}{T^{3/2}} + \frac{a}{T}\right)\right\},\ \{a, b\}\right]$

Out[∘]= $\left\{\left\{a \to -\frac{-1+T}{2\sqrt{T}},\ b \to -\frac{1-T}{2\,T^{3/2}}\right\}\right\}$

In[∘]:= $\text{Solve}\left[\left\{-f - \sqrt{T} + f\,T + T^{3/2} == 0,\ g\,T == \frac{f}{T^2}\right\},\ \{f, g\}\right]$

Out[∘]= $\left\{\left\{f \to -\sqrt{T},\ g \to -\frac{1}{T^{5/2}}\right\}\right\}$

OC fails

In[∘]:= $(X_{1,2}\, X_{4,3}\ //\ m_{1,4\to1}) \equiv (X_{1,3}\, X_{4,2}\ //\ m_{1,4\to1})\ //\ \text{Simplify}$

Out[∘]= $(-1+T)\ \epsilon\ (x_2 - x_3)\ y_1 == 0$

R2 braid-like

In[∘]:= $X_{1,2}\, \overline{X}_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}$
$\overline{X}_{1,2}\, X_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

'naked' R2 cyclic (!!):

In[∘]:= $X_{3,2}\, \overline{X}_{1,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}$
$X_{1,4}\, \overline{X}_{3,2}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

Proper R2 cyclic:

In[∘]:= $\left(CC_3\, X_{5,2}\, \overline{X}_{1,4} \,//\, m_{1,3\to1} \,//\, m_{1,5\to1} \,//\, m_{2,4\to2}\right) \equiv CC_1\, \mathbb{E}_{\{\}\to\{2\}}\left[0,\, 0,\, 1 + O[\epsilon]^2\right]$

$\left(\overline{CC}_4\, X_{1,6}\, \overline{X}_{3,2} \,//\, m_{1,3\to1} \,//\, m_{2,4\to2} \,//\, m_{2,6\to2}\right) \equiv \overline{CC}_2\, \mathbb{E}_{\{\}\to\{1\}}\left[0,\, 0,\, 1 + O[\epsilon]^2\right]$

Out[∘]= True

Out[∘]= True

R3:

In[∘]:= $(X_{1,2}\, X_{4,3}\, X_{5,6} \,//\, m_{1,4\to1} \,//\, m_{2,5\to2} \,//\, m_{3,6\to3}) \equiv (X_{2,3}\, X_{4,5}\, X_{1,6} \,//\, m_{1,4\to1} \,//\, m_{2,5\to2} \,//\, m_{3,6\to3})$

Out[∘]= True

Pairwise equality of the four kinks:

In[∘]:= $Kink_1 \equiv \left(\overline{CC}_3\, X_{1,2} \,//\, m_{1,3\to1} \,//\, m_{1,2\to1}\right)$

$\overline{Kink}_1 \equiv \left(\overline{CC}_3\, \overline{X}_{1,2} \,//\, m_{2,3\to2} \,//\, m_{2,1\to1}\right)$

Out[∘]= True

Out[∘]= True

Trefoils?:

In[∘]:= $\overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}\, \overline{CC}_7\, X_{1,4}\, X_{5,2}\, X_{3,6} \,//\, m_{1,2\to1} \,//\, m_{1,3\to1} \,//\, m_{1,7\to1} \,//\, m_{1,4\to1} \,//\, m_{1,5\to1} \,//\, m_{1,6\to1} \,//\, m_{1,8\to1} \,//\, m_{1,9\to1} \,//\, m_{1,10\to1}$

$\overline{Kink}_8\, \overline{Kink}_9\, \overline{Kink}_{10}\, CC_7\, X_{4,1}\, X_{2,5}\, X_{6,3} \,//\, m_{1,2\to1} \,//\, m_{1,3\to1} \,//\, m_{1,7\to1} \,//\, m_{1,4\to1} \,//\, m_{1,5\to1} \,//\, m_{1,6\to1} \,//\, m_{1,8\to1} \,//\, m_{1,9\to1} \,//\, m_{1,10\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\, 0,\, \frac{T}{1 - T + T^2} + \frac{\left(1 - T - T^2 + T^3\right)\epsilon}{1 - 2T + 3T^2 - 2T^3 + T^4} + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\, 0,\, \frac{T}{1 - T + T^2} + \frac{\left(1 - T - T^2 + T^3\right)\epsilon}{1 - 2T + 3T^2 - 2T^3 + T^4} + O[\epsilon]^2\right]$

Unfortunately? This timid invariant does not see the difference between the mirror trefoils. Perhaps it is actually determined by Alexander as Dror feared.

In[∘]:= $Kink_8\, Kink_9\, Kink_{10}\, CC_4\, \overline{X}_{1,5}\, \overline{X}_{6,2}\, \overline{X}_{3,7} \,//\, m_{1,2\to1} \,//\, m_{1,3\to1} \,//\, m_{1,4\to1} \,//\, m_{1,5\to1} \,//\, m_{1,6\to1} \,//\, m_{1,7\to1} \,//\, m_{1,8\to1} \,//\, m_{1,9\to1} \,//\, m_{1,10\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\, 0,\, \frac{T}{1 - T + T^2} + \frac{\left(1 - T - T^2 + T^3\right)\epsilon}{1 - 2T + 3T^2 - 2T^3 + T^4} + O[\epsilon]^2\right]$

In[∘]:= $\text{Factor}\left[\frac{\left(1 - T - T^2 + T^3\right)\epsilon}{1 - 2T + 3T^2 - 2T^3 + T^4}\right]$

Out[∘]= $\frac{(-1 + T)^2\,(1 + T)\,\epsilon}{\left(1 - T + T^2\right)^2}$