Double of the Sweedler Hopf algebra verification

The Sweedler Hopf algebra SW is a 4 dim Hopf algebra with generators s,w with relations $s^2 = 1$, $w^2 = 0$, sw+ws=0.

The Hopf structure is ϵ(s) = 1 ϵ(w)=0, $\Delta(s) = s_1 s_2$, $\Delta(w) = w_1 s_2 + w_2$, S(s) = s, S(w) = sw.

Our preferred basis is 1,s,w, sw.

Its dual is isomorphic to SW and we take it to be generated by t,m subject to the same relations where t plays the role of s and m the role of w.

The dual pairing is given by the following matrix:

<>	1	s	w	sw
1	1	1	0	0
t	1	-1	0	0
m	0	0	1	1
tm	0	0	1	-1

As the Drinfeld double we take $SW^{*\,cop}$ tensor SW with canonical basis the 16 monomials in order $t^i m^j s^k w^l$ with i,j,k,l = 1 or 0.

Hence, the R-matrix is given by $R_{i,j} = \left(\frac{1+t_i}{2} - \frac{1-t_i}{2} s_j\right)(1 + m_i w_j)$

Below we verify this is indeed a solution to Yang-Baxter and compute some knot invariants.

In[ ]:=
```
PBWRule = {t → 1, m → 2, s → 3, w → 4};

B[U@s, U@w] = - (B[U@w, U@s] = -2 U[s, w]);
B[U@t, U@m] = - (B[U@m, U@t] = -2 U[t, m]);

B[U@s, U@t] = - (B[U@t, U@s] = 0);
B[U@m, U@s] = - (B[U@s, U@m] = -2 U[m, s]);
B[U@t, U@w] = - (B[U@w, U@t] = -2 U[t, w]);
B[U@m, U@w] = - (B[U@w, U@m] = U[s] - U[t]);
```

In[ ]:=
```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[]; UU[l_, r___] := U[l] ** UU[r];
U_i_[ℰ_] := ℰ /. {u_U :> UU @@ Replace[u, x_ :> x_i, 1]};
```

In[ ]:=
```
B[x_, x_] = 0;
B[U[(x_)_i_], U[(y_)_i_]] := U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, y_] := x ** y - y ** x;
```

```
In[*]:= x_ ≤ y_ := OrderedQ[{x, y} /. PBWRule];
x_ ≺ y_ := ! OrderedQ[{y, x} /. PBWRule]; (*ordering according to PBWRule*)
Simp[ℰ_] := Collect[ℰ, _U, Together] /. {
    U[a___, s, s, b___] → U[a, b], U[a___, w, w, b___] → 0,
    U[a___, s_i_, s_i_, b___] → U[a, b], U[a___, w_i_, w_i_, b___] → 0,
    U[a___, t, t, b___] → U[a, b], U[a___, m, m, b___] → 0,
    U[a___, t_i_, t_i_, b___] → U[a, b], U[a___, m_i_, m_i_, b___] → 0};
```

```
In[*]:= Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
U[1] = U[];
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := (*U[x]**U[y] =*)
  If[x ≺ y, U[x, y] // Simp, (U[y, x] + B[U@x, U@y]) // Simp];
U[x_] ** U[y1_, yy__] := (*U[x] ** U[y1,yy] =*)
  If[x ≤ y1, U[x, y1, yy] // Simp, ((U@x ** U@y1) ** U@yy) // Simp];
U[xx__, xn_] ** U[yy__] := (*U[xx,xn]**U[yy] =*) (U@xx ** (U@xn ** U@yy)) // Simp;
```

```
In[*]:= σ[i_, j_][ℰ_] := ℰ /. {t_i → t_j, m_i → m_j, s_i → s_j, w_i → w_j};
```

```
In[*]:= mul[i_, j_][ℰ_] := Simp[ℰ /. x_U ⧴ DeleteCases[x, _j] ** U @@ Cases[x, y_j ⧴ y_i]];
m_{i_,j_→k_}[ℰ_] := ℰ // mul[i, j] // σ[i, k];
```

R-matrix and its inverse

```
In[*]:= R_{i_,j_} := (1/2 U[] + 1/2 U[t_i] + 1/2 U[s_j] - 1/2 U[t_i, s_j]) ** (U[] + U[m_i, w_j])
R̄_{i_,j_} := 1/2 U[] + 1/2 U[t_i] + 1/2 U[s_j] - 1/2 U[t_i, s_j] +
   (1/2 U[s_j] + 1/2 U[t_i, s_j] - 1/2 U[] + 1/2 U[t_i]) ** (U[m_i, w_j])
```

Check Reidemeister 3b

```
In[*]:= R_{1,2} ** R_{1,3} ** R_{2,3} - R_{2,3} ** R_{1,3} ** R_{1,2}
Out[*]= 0
```

Check Reidemeister 2b

```
In[*]:= {R_{1,2} ** R̄_{1,2}, R̄_{1,2} ** R_{1,2}}
Out[*]= {U[], U[]}
```

In[∘]:= $\left(R_{1,2} ** \overline{R}_{3,4}\right)$ // $m_{1,3\to 1}$ // $m_{2,4\to 2}$

Out[∘]= U[ ]

naive Reidemeister 2c fails because antipode has order 4.

In[∘]:= $\left(R_{1,2} ** \overline{R}_{3,4}\right)$ // $m_{3,1\to 1}$ // $m_{2,4\to 2}$

Out[∘]= U[ ] + 2 U[t_1, m_1, w_2]

Since S^2 sends s,t to themselves and m and w to -m and -w we may explicitly write the four modified R-matrices R1,R2,$\overline{\text{R1}}$ $\overline{\text{R2}}$ obtained by applying S^{+-2} on either side as follows. Note that in this case $S^{-2} = S^2$.

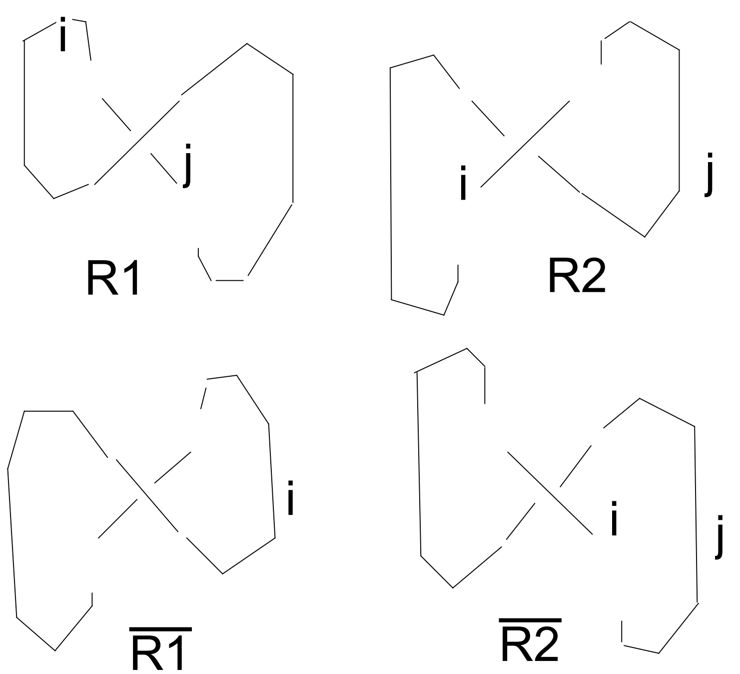
In[∘]:=

$$\text{R1}_{i\_,j\_} := \left(\frac{1}{2}U[] + \frac{1}{2}U[t_i] + \frac{1}{2}U[s_j] - \frac{1}{2}U[t_i, s_j]\right) ** \left(U[] - U[m_i, w_j]\right)$$

$$\text{R2}_{i\_,j\_} := \left(\frac{1}{2}U[] + \frac{1}{2}U[t_i] + \frac{1}{2}U[s_j] - \frac{1}{2}U[t_i, s_j]\right) ** \left(U[] - U[m_i, w_j]\right)$$

$$\overline{\text{R1}}_{i\_,j\_} := \frac{1}{2}U[] + \frac{1}{2}U[t_i] + \frac{1}{2}U[s_j] - \frac{1}{2}U[t_i, s_j] + \left(\frac{1}{2}U[s_j] + \frac{1}{2}U[t_i, s_j] - \frac{1}{2}U[] + \frac{1}{2}U[t_i]\right) ** \left(-U[m_i, w_j]\right)$$

$$\overline{\text{R2}}_{i\_,j\_} := \frac{1}{2}U[] + \frac{1}{2}U[t_i] + \frac{1}{2}U[s_j] - \frac{1}{2}U[t_i, s_j] + \left(\frac{1}{2}U[s_j] + \frac{1}{2}U[t_i, s_j] - \frac{1}{2}U[] + \frac{1}{2}U[t_i]\right) ** \left(-U[m_i, w_j]\right)$$



The point of these R-matrices is that stitching them with R, $\overline{R}$ we can produce any 0-framed 0-rotation number tangle without closed components. No Drinfeld element or ribbon element necessary.

Trefoil knot:

```
In[∘]:= ((R̄1_{1,2} // m_{1,2→0}) ** (R̄1_{1,2} // m_{2,1→0}) ** (R̄1_{1,2} // m_{1,2→0})) **
         ((R1_{1,4} ** R1_{5,2} ** R1_{3,6}) // m_{1,2→1} // m_{1,3→1} // m_{1,4→1} // m_{1,5→1} // m_{1,6→0})
Out[∘]= -U[] + 2 U[t_0, s_0]
```

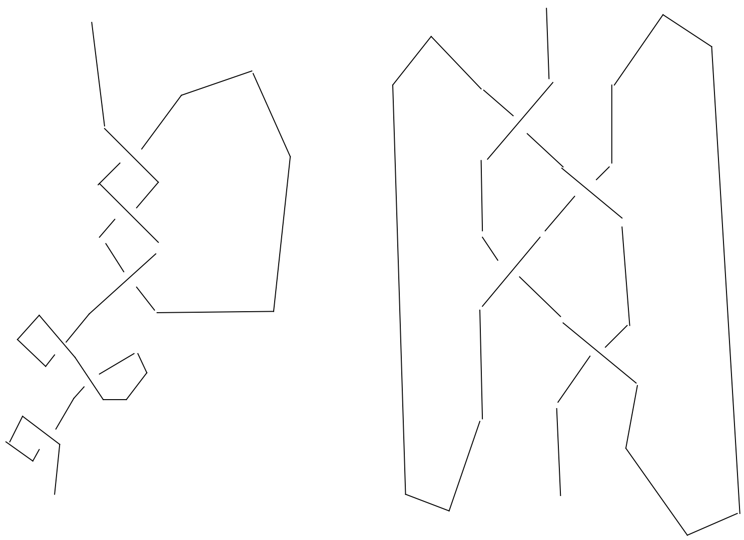


Figure eight knot:

```
In[∘]:= (R̄_{6,1} ** R̄2_{2,5} ** R1_{4,7} ** R_{8,3}) // m_{1,2→1} // m_{1,3→1} // m_{1,4→1} // m_{1,5→1} // m_{1,6→1} // m_{1,7→1} // m_{1,8→1}
Out[∘]= 3 U[] - 2 U[t_1, s_1]
```

Implementing the Hopf algebra structure (under construction)

```
In[∘]:= Δ_{i_→j_,k_}[ℰ_] := Simp[ℰ /. {
          z_. x_U :→ z NonCommutativeMultiply @@ (x /. {
               s_i → U[s_j, s_k],
               w_i → U[w_j, s_k] + U[w_k],
               t_i → U[t_j, t_k],
               m_i → U[m_k, t_j] + U[m_j],
               (*Opposite coproduct on the tm side because of double*)
               y_{_ι_} :→ U@y_ι
             })
        }]
```

In[∘]:=
```
Sᵢ_[ℰ_] := Simp[ℰ /. {z_. x_U :→ z Sᵢ[x]}];
Sᵢ_[U[]] = U[];
Sᵢ_[U[y_j_, more___]] /; i ≠ j := U[y_j] ** Sᵢ[U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
Sᵢ_[U[s_i_, more___]] := Sᵢ[U[more]] ** (U@s_i);
Sᵢ_[U[w_i_, more___]] := Sᵢ[U[more]] ** (U[s_i, w_i]);
Sᵢ_[U[t_i_, more___]] := Sᵢ[U[more]] ** (U@t_i);
Sᵢ_[U[m_i_, more___]] := Sᵢ[U[more]] ** (-U[t_i, m_i]);
(*inverted antipode on the tm side because of double.*)
```

In[∘]:=
```
Sinv[i_][ℰ_] := Simp[ℰ /. {z_. x_U :→ (z /. {c_i → -c_i, t_i → t_i^-1}) Sinv[i][x]}];
Sinv[i_][U[]] = U[];
Sinv[i_][U[y_j_, more___]] /; i ≠ j := U[y_j] ** Sinv[i][U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
Sinv[i_][U[u_i_, more___]] := Sinv[i][U[more]] ** (-U@u_i);
Sinv[i_][U[M_i_, more___]] := Sinv[i][U[more]] ** ((1 + γ) S[i][U@M_i] // Expand);
Sinv[i_][U[w_i_, more___]] := Sinv[i][U[more]] ** ((1 - γ) S[i][U@w_i] // Expand);
```

In[∘]:=
```
CoUnit[i_][ℰ_] :=
  Simp[ℰ /. {z_. x_U :→ (z /. {t → 1, c → 0, c_i → 0, t_i → 1}) CoUnit[i][x]}];
CoUnit[i_][U[]] = U[];
CoUnit[i_][U[y_j_, more___]] /; i ≠ j := U[y_j] ** CoUnit[i][U[more]];
CoUnit[i_][U[y_i_, more___]] /; y ≠ a := 0;
CoUnit[i_][U[a_i_, more___]] := CoUnit[i][U[more]];
```

In[∘]:=
```
UExp[ℰ_, n_] := Module[{t = U[], k}, U[] + Sum[(t = t ** ℰ)/k!, {k, n}]] // Simp
```

In[∘]:=
```
ToDegree[n_][ℰ_] := (Simp[ℰ] /.
    {γ → ℏ γ, c_i_ :→ ℏ c_i, t_i_ :→ ℯ^(ℏ c_i), c → ℏ c, t :→ ℯ^(ℏ c), x_U :→ ℏ^(Count[x,w|w_]+Count[x,M|M_]) x} /.
   a_. x_U :→ Normal[Series[a, {ℏ, 0, n}]] * x) /. ℏ → 1
```

Hopf algebra axioms

```
(*Sinv is the inverse of S *)
Sinv[1][S[1][U@M1]] // Simp
Sinv[1][S[1][U@u1]] // Simp
Sinv[1][S[1][U@w1]] // Simp
```

```
U[M1]

U[u1]

U[w1]
```

```
(*Coassociativity only works properly with ti instead of t!*)
(U@w1 // Δ[1, x, yy] // Δ[yy, y, z]) -
 (U@w1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@u1 // Δ[1, x, yy] // Δ[yy, y, z]) -
 (U@u1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@M1 // Δ[1, x, yy] // Δ[yy, y, z]) -
 (U@M1 // Δ[1, xx, z] // Δ[xx, x, y])
```

```
0

0

0
```

```
(*Convolution inverse*)
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[w1]]]]
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[u1]]]]
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[M1]]]]

mul[2, 3, 1][S[3][Δ[1, 2, 3][U[w1]]]]
mul[2, 3, 1][S[3][Δ[1, 2, 3][U[u1]]]]
mul[2, 3, 1][S[3][Δ[1, 2, 3][U[M1]]]]
```

```
0

0

0

0

0

0
```

Testing Yang-Baxter

```
In[]:= R[i_, j_, d_] :=
  Sum[(UU[w_i^a] ** UU[u_i^b] ** (c_j^b U[] + γ b c_j^(b-1) U[u_j]) ** (U[] + γ a U[u_j]) ** UU[M_j^a]) / (a! b!)
    (1 + γ 1/4 (a - 1) a), {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
R3[d_] := (ToDegree[d][R[1, 2, d] ** R[1, 3, d]] ** R[2, 3, d]) -
  (ToDegree[d][R[2, 3, d] ** R[1, 3, d]] ** R[1, 2, d])
```

```
(R3[2] // ToDegree[2])
```

0

```
Timing[R3[3] // ToDegree[2]]
```

{0.448, 0}

```
Timing[R3[4] // ToDegree[3]]
```

{3.492, 0}

```
Timing[R3[5] // ToDegree[4]]
```

$\left\{27.384, -\frac{1}{6} c_2^3 U[w_1, M_3]\right\}$

```
Timing[R3[6] // ToDegree[4]]
```

{206.372, 0}

```
Timing[R3[7] // ToDegree[5]]
```

$Aborted

```
(*Verifying the inverse is given by (S tensor id)(R)*)
(S[1][R[1, 2, 7]] ** R[1, 2, 7] - U[]) // ToDegree[4]
(*Verifying the inverse is given by (id tensor S^{-1})(R)  *)
```

0

```
(Sinv[2][R[1, 2, 7]] ** R[1, 2, 7] - U[]) // ToDegree[4]
```

0

```
In[]:= Rinv[i_, j_, d_] := S[i][R[i, j, d]] // Expand
```

```
Swaub[a_, b_, i_] := S[i][(UU[w_i^a] ** UU[u_i^b])] // Simp
```

```
Swaub[2, 1, i]
```

$2\times(-1+\gamma)\,U[w_i, w_i] + (-1+5\gamma)\,U[w_i, w_i, u_i] + 2\gamma\,U[w_i, w_i, u_i, u_i]$

```
S[i][UU[w_i^2]] // Simp
```

$(1-\gamma)\,U[w_i, w_i] - 2\gamma\,U[w_i, w_i, u_i]$

```
Rinv0[i_, j_, d_] :=
  Sum[(Swaub[a, b, i] ** (c_j^b U[] + γ b c_j^(b-1) U[u_j]) ** (U[] + γ a U[u_j]) ** UU[M_j^a])/(a! b!)
    (1 + γ 1/4 (a - 1) a), {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
(Rinv[1, 2, 11] - Rinv0[1, 2, 11]) // Simp // ToDegree[6]
```

0

```
TestSw[a_, b_] := -Swaub[a, b, i] + (-1)^(a+b) (1 - a (a - 1) γ/2)
    UU[w_i^a] ** (U[] - a γ U[u_i]) ** (Sum[Binomial[b, k] UU[u_i^k] a^(b-k), {k, 0, b}] // Simp) // Simp
```

```
TestSw[5, 4]
```

0

```
Rinv01[i_, j_, d_] :=
  Sum[1/(a! b!) ((-1)^(a+b) (1 - a (a - 1) γ/2) UU[w_i^a] ** (U[] - a γ U[u_i]) ** (Sum[Binomial[b, k]
          UU[u_i^k] If[a == 0 && b - k == 0, 1, a^(b-k)], {k, 0, b}] // Simp) // Simp) **
     (c_j^b U[] + γ b c_j^(b-1) U[u_j]) ** (U[] + γ a U[u_j]) ** UU[M_j^a] (1 + γ 1/4 (a - 1) a),
    {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
(Rinv[1, 2, 8] - Rinv01[1, 2, 8]) // Simp // ToDegree[4]
```

0

```
Rinv02[i_, j_, d_] :=
  Sum[(-1)^(a+b)/(a! b!) (UU[w_i^a] ** (U[] - a γ U[u_i]) ** (Sum[Binomial[b, k] UU[u_i^k] If[
          a == 0 && b - k == 0, 1, a^(b-k)], {k, 0, b}] // Simp) // Simp) **
     (c_j^b U[] + γ b c_j^(b-1) U[u_j]) ** (U[] + γ a U[u_j]) ** UU[M_j^a] (1 - γ 1/4 (a - 1) a),
    {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
(Rinv[1, 2, 8] - Rinv02[1, 2, 8]) // Simp // ToDegree[4]
```

0

```
Rinv03[i_, j_, d_] :=
 Sum[((-1)^(a+b) t_j^-a c_j^b)/(a! b!) UU[w_i^a] ** (U[] - a γ U[u_i]) ** UU[u_i^b] ** (U[] - γ UU[u_j, u_i] - γ a U[u_j]) **
    (U[] + γ a U[u_j]) ** UU[M_j^a] (1 - γ 1/4 (a - 1) a), {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
(Rinv[1, 2, 8] - Rinv03[1, 2, 8]) // Simp // ToDegree[4]
```

0

```
Rinv04[i_, j_, d_] :=
 Sum[((-1)^(a+b) t_j^-a c_j^b)/(a! b!) UU[w_i^a] ** (U[] - a γ U[u_i]) ** UU[u_i^b] ** (U[] - γ UU[u_j, u_i]) ** UU[M_j^a]
    (1 - γ 1/4 (a - 1) a), {b, 0, d}, {a, 0, d - b}] // ToDegree[d]
```

```
(Rinv[1, 2, 8] - Rinv04[1, 2, 8]) // Simp // ToDegree[4]
```

0

Quasi triangularity axioms

```
(*Check the three quasi-triangularity axioms*)
(Δ[i, k, l][R[i, j, 5]] - R[k, j, 5] ** R[l, j, 5]) // ToDegree[5]
```

0

```
(Δ[j, k, l][R[i, j, 8]] - R[i, l, 8] ** R[i, k, 8]) // ToDegree[5]
```

0

```
CheckRDR[x_, d_] :=
 (R[2, 3, d + 4] ** Δ[1, 2, 3][x] ** Rinv[2, 3, d + 4] - σ[2, 3][Δ[1, 2, 3][x]]) // ToDegree[d]
```

```
CheckRDR[U@u_i, 5]
CheckRDR[U@w_i, 5]
CheckRDR[U@M_i, 5]
```

0

0

0

Drinfeld element

```
(*Drinfeld element*)
Dr[d_] := R[1, 2, d] // S[2] // mul[2, 1, 1]
```

```
(*Check that S(Dr) and Dr commute*)
S[1][Dr[4]] ** Dr[4] - S[1][Dr[4]] ** Dr[4]
```

0

```
S[1][Dr[2]] // ToDegree[2]
```

$U[] - c_1 U[u_1] + (-1+\gamma) U[w_1, M_1] + \frac{1}{2} U[w_1, w_1, M_1, M_1]$

```
(*S(Dr) = t^-1 e^-2εc Dr *)
```

$S[1][Dr[6]] - (U[] - 2\gamma U[u_1]) ** \left(Dr[6]\, t_1^{-1} // ToDegree[6]\right) // ToDegree[3]$

0

$(U[] - \gamma U[u_1]) ** Dr[4] - Dr[4] ** (U[] - \gamma U[u_1]) // ToDegree[4]$

0

Therefore the Ribbon element v is implied by $v^2 = S(Dr)Dr = t^{-1} e^{-2\gamma u} Dr^2$ so choose $v = t^{-1/2} e^{-\gamma u} Dr$, note Dr commutes with $e^{-\gamma u}$.
According to Ohtsuki p.72 read upside down, we should set the left-moving cup and cap to 1 and the right-moving cap nr should be $v\, Dr^{-1} = t^{-1/2} e^{-\gamma u}$ and
the right-moving cup ur should be $Dr\, v^{-1} = t^{1/2} e^{\gamma u}$.

```
(*Square of antipode*)
S[1]@S[1]@U@u1
S[1]@S[1]@U@w1
S[1]@S[1]@U@M1
```

$U[u_1]$

$(1+\gamma)\, U[w_1]$

$(1-\gamma)\, U[M_1]$

```
(U[] + γ U[u]) ** U[u] ** (U[] - γ U[u]) // Simp
(U[] + γ U[u]) ** U[w] ** (U[] - γ U[u]) // Simp
(U[] + γ U[u]) ** U[M] ** (U[] - γ U[u]) // Simp
```

U[u]

$(1+\gamma)\, U[w]$

$(1-\gamma)\, U[M]$

Logos

```
B[U@M, U@w]
```

$(1-t)\, U[] - (1+t)\, \gamma U[u]$

In[ ]:=
```
η = (t + 1)/(t - 1) γ;
q = 1 + η;
qI[k_] := k (1 + η (k - 1) / 2) // Expand
qFac[n_] := n! (1 + η (n - 1) n / 4) // Expand
InvqFac[n_] := (1 - η (n - 1) n / 4) / n! // Expand
qBin[n_, k_] := Binomial[n, k] (1 + η k (n - k) / 2) // Expand
```

```
(*Checking the commutation relation for powers of a,w*)
WmAn[m_, n_] := Sum[(1 - t)^j qBin[m, j] qBin[n, j]
   qFac[j] UU[w^(n-j)] ** (U[] + j η U[u]) ** UU[M^(m-j)], {j, 0, Min[m, n]}]
TestWmAn[m_, n_] := -UU[M^m, w^n] + WmAn[m, n]
TestWmAn[5, 3] // Together
```
0

```
Mawb[a_, b_] := Sum[a! b! / ((a - j)! (b - j)! j!) (1 - t)^j
   UU[w^(b-j)] ** ((U[] - (1 + t)/(1 - t) γ (((a + b - 2)/2 U[] + U[u]) j - 3/4 j (j - 1) U[])) // Simp) **
    UU[M^(a-j)], {j, 0, Min[a, b]}]
TestMawb[a_, b_] := -UU[M^a, w^b] + Mawb[a, b] // Simp
TestMawb[7, 10]
```
0

```
TestLemma20[d_] := -Sum[Mawb[k, k] δ^k/k!, {k, 0, d}] +
  Sum[
   Sum[δ^(k-j) k! / ((k - j)! (k - j)! j!) δ^j (1 - t)^j
     UU[w^(k-j)] ** ((U[] - (1 + t)/(1 - t) γ (((k - 1) U[] + U[u]) j - 3/4 j (j - 1) U[])) // Simp) **
      UU[M^(k-j)], {j, 0, k}]
   , {k, 0, d}]
TestLemma20[4] // ToDegree[3]
```
0

```
TestLemma21[d_] := -Sum[Mawb[k, k] δ^k/k!, {k, 0, d}] +
  Sum[
   Sum[(δ^l (l + j)!)/(l! l! j!) δ^j (1 - t)^j UU[w^l] **
      ((U[] - (1 + t)/(1 - t) γ ((l U[] + U[u]) j + 1/4 j (j - 1) U[])) // Simp) ** UU[M^l], {j, 0, d}]
   , {l, 0, d}]

TestLemma21[6] // ToDegree[6]
0

TestLemma22[d_] := -Sum[Mawb[k, k] δ^k/k!, {k, 0, d}] +
  Sum[δ^l/l! UU[w^l] ** (((1 - δ (1 - t))^(-l-1) U[]
         - (1 + t)/(1 - t) γ (δ (1 - t) (1 - δ (1 - t))^(-l-2) (l + 1) (l U[] + U[u])
            + δ^2 (1 - t)^2 (1 - δ (1 - t))^(-l-3) (l + 1) (l + 2) 1/4 U[])) // Simp) ** UU[M^l]
   , {l, 0, d}]

TestLemma22[7] // ToDegree[6]
0

TestLemma23[d_] := Module[{ν},
  ν = (1 - δ (1 - t))^-1;
  -Sum[Mawb[k, k] δ^k/k!, {k, 0, d}] +
   ν Sum[(δ ν)^l/l! UU[w^l] ** ((U[]
          - (1 + t)/(1 - t) γ (δ (1 - t) ν (l + 1) (l U[] + U[u])
             + δ^2 (1 - t)^2 ν^2 (l + 1) (l + 2) 1/4 U[])) // Simp) ** UU[M^l]
     , {l, 0, d}]
  ]
TestLemma23[7] // ToDegree[6]
0
```

```
TestLemma24[d_] := Module[{ν},
  ν = (1 - δ (1 - t))^-1;
  - Sum[Mawb[k, k] δ^k/k!, {k, 0, d}] +
   ν Sum[(δ ν)^l/l! UU[w^l] ** ((U[]
          - (1 + t) δ ν ɣ (δ (1 - t) ν U[] 1/2 +
             l (2 + δ (1 - t) ν) U[] +
             (l - 1) l (1 + 1/4 δ (1 - t) ν) U[]
             + (l + 1) U[u])) // Simp) ** UU[M^l]
     , {l, 0, d}]
 ]
TestLemma24[7] // ToDegree[6]
```

0

```
(*Guess qLogos first at q=1*)
ToDegh[F_, x_] := Series[F /. {α → α h, δ → δ h, β → β h}, {h, 0, x}]
d = 8;
LHS = Sum[α^m δ^k β^n InvqFac[m] InvqFac[k] InvqFac[n] WmAn[m + k, n + k],
    {m, 0, d}, {k, 0, d - m}, {n, 0, d - m - k}] // Simp;
(*powers of nu as power series*)
nuA[z_] := Sum[Binomial[z - 1 + x, x] (1 - t)^x δ^x, {x, 0, d}];
RHS =
  Sum[(1 - t)^j nuA[m + k + n + j + 1] α^(m+j) δ^k β^(n+j) InvqFac[m] InvqFac[k] InvqFac[n] InvqFac[j]
     UU[a^(n+k), w^(m+k)], {m, 0, d}, {k, 0, d - m}, {n, 0, d - m - k}, {j, 0, d - m - n - k}] // Simp;
ToDegh[(LHS - RHS) /. {ϵ → 0} // Simp, 8]
```

$O[h]^9$

```
(*Now set up the LHS and RHS to find the Logos relating them.*)
d = 11;
```

$$LHS = Sum\left[\alpha^m\,\delta^k\,\beta^n\,\frac{WmAn[m+k,\,n+k]}{m!\,k!\,n!},\ \{m,0,d\},\ \{k,0,d-m\},\ \{n,0,d-m-k\}\right]\ //\ Simp;$$

$$RHS = Sum\left[(1-t)^j\,\mu^{-(m+k+n+j+1)}\,\frac{\alpha^{m+j}\,\delta^k\,\beta^{n+j}}{m!\,k!\,n!\,j!}\,UU\left[a^{n+k},\,w^{m+k}\right],\right.$$
$$\left.\{m,0,d\},\ \{k,0,d-m\},\ \{n,0,d-m-k\},\ \{j,0,d-m-n-k\}\right];$$

$$CRHS = Sum\left[(1-t)^j\,\mu^{-(m+k+n+j+1)}\,\frac{\alpha^{m+j}\,\delta^k\,\beta^{n+j}}{m!\,n!\,k!\,j!}\,UU\left[a^{n+k},\,c,\,w^{m+k}\right],\ \{m,0,d\},\right.$$
$$\left.\{k,0,d-m\},\ \{n,0,d-m-k\},\ \{j,0,d-m-n-k\}\right]\ //\ Simp;$$

$$\mu = 1 - (1-t)\,\delta;$$

$$\eta$$

$$\frac{(1+t)\,\epsilon}{-1+t}$$

```
(*Here we guess and verify the Logos coefficient by coefficient*)
ToDegh[Coefficient[(-LHS + RHS +
    μ^-4 (1 - t) η (
      (1 - t) × (1/2 δ^2 μ^2 + α β δ μ + (α^2 β^2)/4) RHS +
      μ^2 (δ μ + α β) CRHS +
      β (δ μ + (α β)/2) U[a] ** RHS +
      α (δ μ + (α β)/2) RHS ** U[w] +
      β δ μ^2 U[a] ** CRHS +
      α δ μ^2 CRHS ** U[w] +
      δ (1 + μ) (μ δ + α β) U[a] ** RHS ** U[w] +
      δ^2 μ^2 U[a] ** CRHS ** U[w] +
      (β^2 δ)/4 (1 + μ) U[a, a] ** RHS +
      (α^2 δ)/4 (1 + μ) RHS ** U[w, w] +
      (β δ^2)/2 (1 + 2 μ) U[a, a] ** RHS ** U[w] +
      (α δ^2)/2 (1 + 2 μ) U[a] ** RHS ** U[w, w] +
      δ^3/4 (1 + 3 μ) U[a, a] ** RHS ** U[w, w])
  ), U[]], 10]
```

$O[h]^{11}$

```
(*Final check, put back ∈.*)
ToDegh[(-LHS + RHS +
    -μ^-4 (1 + t) ∈ (
       (1 - t) × (1/2 δ^2 μ^2 + α β δ μ + α^2 β^2/4) RHS +
       μ^2 (δ μ + α β) CRHS +
       β (δ μ + α β/2) U[a] ** RHS +
       α (δ μ + α β/2) RHS ** U[w] +
       β δ μ^2 U[a] ** CRHS +
       α δ μ^2 CRHS ** U[w] +
       δ (1 + μ) (μ δ + α β) U[a] ** RHS ** U[w] +
       δ^2 μ^2 U[a] ** CRHS ** U[w] +
       β^2 δ/4 (1 + μ) U[a, a] ** RHS +
       α^2 δ/4 (1 + μ) RHS ** U[w, w] +
       β δ^2/2 (1 + 2 μ) U[a, a] ** RHS ** U[w] +
       α δ^2/2 (1 + 2 μ) U[a] ** RHS ** U[w, w] +
       δ^3/4 (1 + 3 μ) U[a, a] ** RHS ** U[w, w])
  ), 10] // Simp
```

0

Double Reverse

```
Δ[1, 2, 3][U@c1] // Simp
Δ[1, 2, 3][U@w1] // Simp
Δ[1, 2, 3][U@a1] // Simp
```

$U[c_2] + U[c_3]$

$U[w_2] + U[w_3] + \in U[c_3, w_2]$

$t_3 U[a_2] + U[a_3] - \in U[a_3, c_2]$

```
((Δ[1, 2, 3][U@w1] // Simp) ** (Δ[1, 2, 3][U@w1] // Simp) -
   (U[w2] + U[w3]) ** (U[w2] + U[w3]) - ϵ U[w2, w3] - 2 ϵ U[c3, w2, w3] - 2 ϵ U[c3, w2, w2]) // Simp
```

0

```
((Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) - (t3 U[a2] + U[a3]) **
    (t3 U[a2] + U[a3]) + ϵ t3 U[a2, a3] + 2 ϵ t3 U[a2, a3, c2] + 2 ϵ U[a3, a3, c2]) // Simp
```

0

```
((Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) -
   (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) +
   3 ϵ t3 ( t3 U[a2] + U[a3]) ** U[a2, a3] +
   3 ϵ ( t3 U[a2] + U[a3]) ** ( t3 U[a2] + U[a3]) ** U[a3, c2]) // Simp
```

0

```
((Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) **
    (Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) -
   (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) +
   6 ϵ t3 ( t3 U[a2] + U[a3]) ** ( t3 U[a2] + U[a3]) ** U[a2, a3] +
   4 ϵ ( t3 U[a2] + U[a3]) ** ( t3 U[a2] + U[a3]) ** ( t3 U[a2] + U[a3]) ** U[a3, c2]) // Simp
```

0

```
((((Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp) **
        (Δ[1, 2, 3][U@a1] // Simp) ** (Δ[1, 2, 3][U@a1] // Simp)) // Simp) **
    (((Δ[1, 2, 3][U@w1] // Simp) ** (Δ[1, 2, 3][U@w1] // Simp) **
        (Δ[1, 2, 3][U@w1] // Simp) ** (Δ[1, 2, 3][U@w1] // Simp)) // Simp) -
   (
    (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) **
      (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3])
     + ϵ (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) **
       (4 U[c3, w2]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3])
     + ϵ (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) **
       (t3 U[a2] + U[a3]) ** (6 U[w2, w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3])
     - ϵ (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (4 U[a3, c2]) **
       (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3])
     - ϵ (t3 U[a2] + U[a3]) ** (t3 U[a2] + U[a3]) ** (6 t3 U[a2, a3]) **
       (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3]) ** (U[w2] + U[w3])

   ) // Expand // Simp
```

0

```
UU[c, a]
```

U[a] + U[a, c]

S[1]@U@w_1 − (−U[w_1] + ∈ UU[w_1, c_1]) // Simp
S[1]@U@c_1 // Simp
S[1]@U@a_1 − $\left(-t_1^{-1}\,(\text{U}[a_1] + \in \text{UU}[c_1, a_1])\right)$ // Expand // Simp

0

−U[c_1]

0

Δ[1, 2, 3][U@c_1] // S[2]
Δ[1, 2, 3][U@w_1] // S[2] // Simp
((Δ[1, 2, 3][U@a_1] // S[2] // Simp) /. t_ → t) // Simp

−U[c_2] + U[c_3]

(−1 + ∈) U[w_2] + U[w_3] + ∈ U[c_2, w_2] − ∈ U[c_3, w_2]

(−1 − ∈) U[a_2] + U[a_3] − ∈ U[a_2, c_2] + ∈ U[a_3, c_2]

U[w] ** ((U[c] + U[]) ** U[w]) // Simp

2 U[w, w] + U[c, w, w]

(S[1]@U[w_1]) // Simp
(S[1]@U[w_1]) ** (S[1]@U[w_1]) // Simp
(S[1]@U[w_1]) ** (S[1]@U[w_1]) ** (S[1]@U[w_1]) // Simp
(S[1]@U[w_1]) ** (S[1]@U[w_1]) ** (S[1]@U[w_1]) ** (S[1]@U[w_1]) // Simp

(−1 + ∈) U[w_1] + ∈ U[c_1, w_1]

(1 − 3 ∈) U[w_1, w_1] − 2 ∈ U[c_1, w_1, w_1]

(−1 + 6 ∈) U[w_1, w_1, w_1] + 3 ∈ U[c_1, w_1, w_1, w_1]

(1 − 10 ∈) U[w_1, w_1, w_1, w_1] − 4 ∈ U[c_1, w_1, w_1, w_1, w_1]

(S[1]@U[a_1]) // Simp
(S[1]@U[a_1]) ** (S[1]@U[a_1]) // Simp
(S[1]@U[a_1]) ** (S[1]@U[a_1]) ** (S[1]@U[a_1]) // Simp
(S[1]@U[a_1]) ** (S[1]@U[a_1]) ** (S[1]@U[a_1]) ** (S[1]@U[a_1]) // Simp

$$\frac{(-1-\in)\,\text{U}[a_1]}{t_1} - \frac{\in\,\text{U}[a_1, c_1]}{t_1}$$

$$\frac{(1+3\in)\,\text{U}[a_1, a_1]}{t_1^2} + \frac{2\in\,\text{U}[a_1, a_1, c_1]}{t_1^2}$$

$$\frac{(-1-6\in)\,\text{U}[a_1, a_1, a_1]}{t_1^3} - \frac{3\in\,\text{U}[a_1, a_1, a_1, c_1]}{t_1^3}$$

$$\frac{(1+10\in)\,\text{U}[a_1, a_1, a_1, a_1]}{t_1^4} + \frac{4\in\,\text{U}[a_1, a_1, a_1, a_1, c_1]}{t_1^4}$$

```
(U[c] - 4 U[]) ** UU[a^4] - UU[a^4, c]
```

0