Pensieve header: The “Speedy” engine.

```
In[∘]:= PP_ = Identity; $k = 1; γ = 1; ℏ = 1;
```

# The “Speedy” Engine

## Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := PP_CCF@ExpandDenominator@ExpandNumerator@PP_Together@Together[PP_Exp[
          Expand[ℰ] //. 𝕖^x_ 𝕖^y_ :> 𝕖^(x+y) /. 𝕖^x_ :> 𝕖^CCF[x]]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := PP_CF@Module[
            {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
            Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
          ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker $\delta$:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $\mathbb{e}^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

## Zip and Bind

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
In[∘]:= U2l = {B_i_^p_. :> 𝕖^(-p ℏ γ b_i), B^p_. :> 𝕖^(-p ℏ γ b), T_i_^p_. :> 𝕖^(-p ℏ t_i), T^p_. :> 𝕖^(-p ℏ t), 𝒜_i_^p_. :> 𝕖^(p γ α_i), 𝒜^p_. :> 𝕖^(p γ α)};
        l2U = {𝕖^(c_. b_i_ + d_.) :> B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :> B^(-c/(ℏ γ)) 𝕖^d,
            𝕖^(c_. t_i_ + d_.) :> T_i^(-c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :> T^(-c/ℏ) 𝕖^d,
            𝕖^(c_. α_i_ + d_.) :> 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :> 𝒜^(c/γ) 𝕖^d,
            𝕖^ℰ_ :> 𝕖^Expand@ℰ};
```

Derivatives in the presence of exponentiated variables:

```
In[∘]:= D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_{b_i_}[f_] := ∂_{b_i} f - ℏ γ B_i ∂_{B_i} f;
D_t[f_] := ∂_t f + ℏ T ∂_T f; D_{t_i_}[f_] := ∂_{t_i} f + ℏ T_i ∂_{T_i} f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_{α_i_}[f_] := ∂_{α_i} f + γ 𝒜_i ∂_{𝒜_i} f;
D_{v_}[f_] := ∂_v f; D_{{v_,0}}[f_] := f; D_{{}}[f_] := f; D_{{v_,n_Integer}}[f_] := D_v[D_{{v,n-1}}[f]];
D_{{l_List,ls___}}[f_] := D_{{ls}}[D_l[f]];
```

Finite Zips:

```
In[∘]:= collect[sd_SeriesData, ξ_] := MapAt[collect[#, ξ] &, sd, 3];
collect[ℰ_, ξ_] := PP_Collect@Collect[ℰ, ξ];
Zip_{{}}[P_] := P;
Zip_{ξs_}[Ps_List] := Zip_{ξs} /@ Ps;
Zip_{{ξ_,ξs___}}[P_] := PP_Zip[
  (collect[P // Zip_{{ξs}}, ξ] /. f_. ξ^d_. :> (D_{{ξ*,d}}[f])) /. ξ* → 0 /.
   ((ξ* /. {b → B, t → T, α → 𝒜}) → 1)]
```

QZip implements the “$Q$-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the $L$ variables as scalars.

$$\left\langle P(z_i, \zeta^j) e^{c+\eta^i z_i + y_j \zeta^j + q^i_j z_i \zeta^j} \right\rangle = |\tilde{q}| \left\langle P(z_i, \zeta^j) e^{c+\eta^i z_i} \Big|_{z_i \to \tilde{q}^k_i (z_k + y_k)} \right\rangle$$
$$= |\tilde{q}| e^{c+\eta^i \tilde{q}^k_i y_k} \left\langle P\left(\tilde{q}^k_i (z_k + y_k), \zeta^j + \eta^i \tilde{q}^j_i\right) \right\rangle.$$

```
In[∘]:= QZip_{ξs_List}@𝔼[L_, Q_, P_] := PP_QZip@Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule, out},
    zs = Table[ξ*, {ξ, ξs}];
    c = CF[Q /. Alternatives @@ (ξs ⋃ zs) → 0];
    ys = CF@Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}];
    qt = CF@Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ξrule = Thread[ξs → ξs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ξs[P /. (zrule ⋃ ξrule)]] ];
QZipM_{ξs_List}@𝔼[L_, Q_, P_] := Module[{ξ, z, zs, c, ys, ηs, qt, zrule, ξrule, out},
   zs = Table[ξ*, {ξ, ξs}] // Echo;
   c = CF[Q /. Alternatives @@ (ξs ⋃ zs) → 0];
   ys = CF@Table[∂_ξ (Q /. Alternatives @@ zs → 0), {ξ, ξs}] // Echo;
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ξs → 0), {z, zs}] // Echo;
   (CF@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}]) // MatrixForm // Echo;
   qt = CF@Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} Q, {ξ, ξs}, {z, zs}] // Echo;
   zrule = Thread[zs → CF[qt.(zs + ys)]] // Echo;
   ξrule = Thread[ξs → ξs + ηs.qt] // Echo;
   P /. (zrule ⋃ ξrule) // Echo;
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ξs[P /. (zrule ⋃ ξrule)]] ];
```

LZip implements the “$L$-level zips” on $\mathbb{E}(L, Q, P) = P e^{L+Q}$. Such zips regard all of $P e^Q$ as a single”$P$”. Here the $z$'s are $b$ and $\alpha$ and the $\zeta$'s are $\beta$ and $a$.

In[∘]:=
```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  PP_LZip@Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
    zs = Table[ζ*, {ζ, ζs}];
    Zs = zs /. {b → B, t → T, α → 𝒜};
    c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
    ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
    lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ} L, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → lt.(zs + ys)];
    Zrule = Join[zrule,
      zrule /. r_Rule :> ((U = r[[1]] /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
    ζrule = Thread[ζs → ζs + ηs.lt];
    Q1 = Q /. (Zrule ⋃ ζrule);
    EEQ[ps___] := EEQ[ps] = PP_"EEQ"@
       (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
    CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
      Det[lt] (Zip_{ζs}[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
         Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ]];
```

In[∘]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := PP_B@Module[{n},
    Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
    ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

## 𝔼 morphisms with domain and range.

In[∘]:=
```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_{dr_}[L_, Q_, P_]$k_ := 𝔼_{dr} @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

## 𝔼[Λ]

```
In[∘]:= 𝔼dr_[Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, ℯ^(Λ-Λ0)]$k /. l2U]
```

## Exponentials as needed.

Task. Define $\text{Exp}_{m,i,k}[P]$ to compute $e^{\mathbb{O}(P)}$ to $\epsilon^k$ in the using the $m_{i,i\to i}$ multiplication, where $P$ is an $\epsilon$-dependent near-docile element, giving the answer in 𝔼-form.

Methodology. If $P_0 := P_{\epsilon=0}$ and $e^{\lambda\,\mathbb{O}(P)} = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)$, then $F(\lambda = 0) = 1$ and we have:

$\mathbb{O}\left(e^{\lambda P_0}(P_0 F(\lambda) + \partial_\lambda F)\right) = \mathbb{O}\left(\partial_\lambda e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right) = \partial_\lambda e^{\lambda\,\mathbb{O}(P)} = e^{\lambda\,\mathbb{O}(P)}\,\mathbb{O}(P) = \mathbb{O}\left(e^{\lambda P_0} F(\lambda)\right)\mathbb{O}(P)$.

This is a linear ODE for $F$. Setting inductively $F_k = F_{k-1} + \epsilon^k \varphi$ we find that $F_0 = 1$ and solve for $\varphi$.

```
In[∘]:= (* Bug: The first line is valid only if 𝕆(ℯ^P0) == ℯ^𝕆(P0). *)
Exp_m_,i_,0[P_] := Module[{LQ = Normal@P /. ϵ → 0},
   𝔼[LQ /. (x | y)i → 0, LQ /. (b | a | t)i → 0, 1] ];
```

```
In[∘]:= Exp_m_,i_,k_[P_] := Block[{$k = k},
  Module[{P0, λ, φ, φs, F, j, rhs, eqn, pows, at0, atλ},
   P0 = Normal@P /. ϵ → 0;
   F = Normal@Last@Exp_m,i,k-1[λ P];
   While[
    rhs = m_i,j→i[𝔼_{}→{i}[λ P0 /. (x | y)i → 0, λ P0 /. (b | a | t)i → 0, F]k
        sσ_i→j@𝔼_{}→{i}[0, 0, P]k] // Last // Normal;
    eqn = CF[(∂λ F) + P0 F - rhs];
    eqn =!= 0, (*do*)
    pows = First /@ CoefficientRules[eqn, {yi, bi, ai, xi}];
    F += Sum[ϵ^k φjs[λ] Times @@ {yi, bi, ai, xi}^js, {js, pows}];
    rhs = m_i,j→i[𝔼_{}→{i}[λ P0 /. (x | y)i → 0, λ P0 /. (b | a | t)i → 0, F]k
        sσ_i→j@𝔼_{}→{i}[0, 0, P]k] // Last // Normal;
    eqn = CF[(∂λ F) + P0 F - rhs];
    φs = Table[φjs[λ], {js, pows}];
    at0 = Table[φjs[0] == 0, {js, pows}];
    atλ = (# == 0) & /@ (pows /. CoefficientRules[eqn, {yi, bi, ai, xi}]);
    F = F /. DSolve[And @@ (at0 ⋃ atλ), φs, λ] [[1]]
   ];
   𝔼_{}→{i}[P0 /. (x | y)i → 0, P0 /. (b | a | t)i → 0, F + O[ϵ]^(k+1) /. λ → 1]] ]
```

## “Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
    ReleaseHold[Hold[
        SD[op_{nisp,$k_Integer}, PP_Boot@Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
        SD[op_isp, op_{{is},$k}]; SD[op_{sis__}, op_{{sis}}];
      ] /. {SD → SetDelayed,
        isp → {is} /. {i → i_, j → j_, k → k_},
        nis → {is} /. {i → ii, j → jj, k → kk},
        nisp → {is} /. {i → ii_, j → jj_, k → kk_}
      }] ]]
```

# The Objects

## Symmetric Algebra Objects

In[∘]:=
$$sm_{i\_,j\_\to k\_} := \mathbb{E}_{\{i,j\}\to\{k\}}[b_k\ (\beta_i+\beta_j) + t_k\ (\tau_i+\tau_j) + a_k\ (\alpha_i+\alpha_j) + y_k\ (\eta_i+\eta_j) + x_k\ (\xi_i+\xi_j)];$$
$$s\Delta_{i\_\to j\_,k\_} := \mathbb{E}_{\{i\}\to\{j,k\}}\left[\beta_i\ (b_j+b_k) + \tau_i\ (t_j+t_k) + \alpha_i\ (a_j+a_k) + \eta_i\ (y_j+y_k) + \xi_i\ (x_j+x_k)\right];$$
$$sS_{i\_} := \mathbb{E}_{\{i\}\to\{i\}}[-\beta_i\, b_i - \tau_i\, t_i - \alpha_i\, a_i - \eta_i\, y_i - \xi_i\, x_i];$$
$$s\epsilon_{i\_} := \mathbb{E}_{\{\}\to\{i\}}[0];$$
$$s\eta_{i\_} := \mathbb{E}_{\{i\}\to\{\}}[0];$$

In[∘]:=
$$s\sigma_{i\_\to j\_} := \mathbb{E}_{\{i\}\to\{j\}}[\beta_i\, b_j + \tau_i\, t_j + \alpha_i\, a_j + \eta_i\, y_j + \xi_i\, x_j];$$
$$s\Upsilon_{i\_\to j\_,k\_,l\_,m\_} := \mathbb{E}_{\{i\}\to\{j,k,l,m\}}[\beta_i\, b_k + \tau_i\, t_k + \alpha_i\, a_l + \eta_i\, y_j + \xi_i\, x_m];$$

## The CU Definitions

In[∘]:=
$$c\Lambda = \left(\eta_i + \frac{e^{-\gamma\,\alpha_i-\epsilon\,\beta_i}\,\eta_j}{1+\gamma\,\epsilon\,\eta_j\,\xi_i}\right) y_k + \left(\beta_i+\beta_j+\frac{\text{Log}[1+\gamma\,\epsilon\,\eta_j\,\xi_i]}{\epsilon}\right) b_k + \left(\alpha_i+\alpha_j+\frac{\text{Log}[1+\gamma\,\epsilon\,\eta_j\,\xi_i]}{\gamma}\right) a_k + \left(\frac{e^{-\gamma\,\alpha_j-\epsilon\,\beta_j}\,\xi_i}{1+\gamma\,\epsilon\,\eta_j\,\xi_i}+\xi_j\right) x_k;$$
$$\text{Define}[cm_{i,j\to k} = \mathbb{E}_{\{i,j\}\to\{k\}}[c\Lambda]]$$

In[∘]:=
$$\text{Define}[c\sigma_{i\to j} = s\sigma_{i,j}\ /.\ \tau_i\to 0,\ c\epsilon_i = s\epsilon_i,\ c\eta_i = s\eta_i,\ c\Delta_{i\to j,k} = s\Delta_{i\to j,k},$$
$$cS_i = sS_i\ //\ s\Upsilon_{i\to 1,2,3,4}\ //\ cm_{4,3\to i}\ //\ cm_{i,2\to i}\ //\ cm_{i,1\to i}];$$

## Booting Up QU

In[∘]:=
$$\text{Define}[a\sigma_{i\to j} = \mathbb{E}_{\{i\}\to\{j\}}[a_j\,\alpha_i + x_j\,\xi_i],\ b\sigma_{i\to j} = \mathbb{E}_{\{i\}\to\{j\}}[b_j\,\beta_i + y_j\,\eta_i]]$$

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + ⅇ^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of $R$ in the algebra $\mathbb{B}\otimes\mathbb{A}$.

$P$ is the inverse of $R$ as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$.

$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=
```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Sum_{k=1}^{$k+1} ((1 - ⅇ^{γ ϵ ℏ})^k (ℏ y_i x_j)^k) / (k (1 - ⅇ^{k γ ϵ ℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
      (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_i α_j / ℏ, η_i ξ_j / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
      (R_{1,2} // ((P_{{1,j},0})_{$k} (P_{{i,2},$k-1})_{$k}))[3]]]]
```

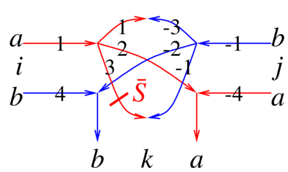
In[∘]:=
```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{1,2},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
      ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

(was $\mathrm{aS}_j = \overline{R}_{i,j} \sim B_i \sim P_{i,j}$).

In[∘]:=
```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{1,2},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{1,2},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{3,i},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{i,3}]
```

(was $\mathrm{bS}_i = R_{i,1} \sim B_1 \sim \mathrm{aS}_1 \sim B_1 \sim P_{i,1}$, $\overline{\mathrm{bS}}_i = R_{i,1} \sim B_1 \sim \overline{\mathrm{aS}}_1 \sim B_1 \sim P_{i,1}$).

The Drinfel'd double:




In[∘]:=
```
Define[
  dm_{i,j→k} = ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3) (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{-1,3} P_{-3,1} am_{2,-4→k} bm_{4,-2→k})]
```

In[∘]:=
```
Define[dσi→j = aσi→j bσi→j,
  dϵi = sϵi, dηi = sηi,
  dSi = sϒi→1,1,2,2 // (b̄S̄1 aS2) // dm2,1→i,
  d̄S̄i = sϒi→1,1,2,2 // (bS1 āS̄2) // dm2,1→i,
  dΔi→j,k = (bΔi→3,1 aΔi→2,4) // (dm3,4→k dm1,2→j)]
```

In[∘]:=
```
Define[Ci = 𝔼{}→{i}[0, 0, Bi^(1/2) ℯ^(-ℏ ϵ ai/2)]$k,
  C̄i = 𝔼{}→{i}[0, 0, Bi^(-1/2) ℯ^(ℏ ϵ ai/2)]$k,
  ci = 𝔼{}→{i}[0, 0, Bi^(1/4) ℯ^(-ℏ ϵ ai/4)]$k,
  c̄i = 𝔼{}→{i}[0, 0, Bi^(-1/4) ℯ^(ℏ ϵ ai/4)]$k,
  Kinki = (R1,3 C̄2) // dm1,2→1 // dm1,3→i,
  K̄īn̄k̄i = (R̄1,3 C2) // dm1,2→1 // dm1,3→i,
  ρi = (c1 c̄3 dSi) // dm1,i→i // dmi,3→i] (*ρ reverses a strand*)
```

Note. $t == -\epsilon a + \gamma b$ and $b == t/\gamma + \epsilon a/\gamma$ previously:$t == \epsilon a - \gamma b$ and $b == -t/\gamma + \epsilon a/\gamma$.

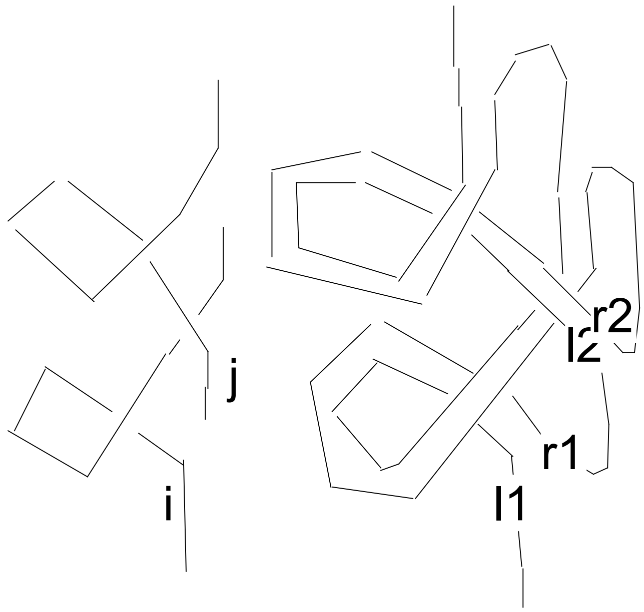
In[∘]:=
```
Define[b2ti = 𝔼{i}→{i}[αi ai + βi (ϵ ai + ti) / γ + ξi xi + ηi yi],
  t2bi = 𝔼{i}→{i}[αi ai + τi (-ϵ ai + γ bi) + ξi xi + ηi yi]]
```

## The Knot Tensors

In[∘]:=
```
Define[kRi,j = Ri,j // (b2ti b2tj) /. ti|j → t,
  k̄R̄i,j = R̄i,j // (b2ti b2tj) /. {ti|j → t, Ti|j → T},
  kmi,j→k = ((t2bi t2bj) // dmi,j→k // b2tk) /. {tk → t, Tk → T, τi|j → 0},
  kCi = (Ci // b2ti) /. Ti → T,
  k̄C̄i = (C̄i // b2ti) /. Ti → T,
  kKinki = Kinki // b2ti /. {ti → t, Ti → T},
  k̄K̄īn̄k̄i = K̄īn̄k̄i // b2ti /. {ti → t, Ti → T}]
```

In[∘]:=
```
Define[BSi,j→k =
   C3 C4 dΔi→l1,r1 dΔj→l2,r2 // d̄S̄r1 // dSr2 // dml1,3→k // dmk,r2→k // dmk,r1→k // dmk,4→k // dmk,l2→k]
Define[tBSi,j→k = (t2bi t2bj) // C3 C4 dΔi→l1,r1 dΔj→l2,r2 // d̄S̄r1 // dSr2 // dml1,3→k // dmk,r2→k //
     dmk,r1→k // dmk,4→k // dmk,l2→k // b2tk]
Define[tmi,j→k = t2bi // t2bj // dmi,j→k // b2tk]
Define[tRi,j = Ri,j // b2ti // b2tj, t̄R̄i,j = R̄i,j // b2ti // b2tj, tSi = t2bi // dSi // b2ti]
Define[tCi = Ci // b2ti , t̄C̄i = C̄i // b2ti ]
Define[tKinki = Kinki // b2ti , t̄K̄īn̄k̄i = K̄īn̄k̄i // b2ti]
```

Example tangle G is two positive kinks.




In[∘]:= $G = \left(\text{tKink}_1\,\text{tKink}_4\,\overline{\text{tR}}_{2,3}\right) \mathbin{//} \text{tm}_{1,3\to i} \mathbin{//} \text{tm}_{2,4\to j}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[a_i t_i - a_i t_j + a_j t_j,\ \frac{x_i y_i}{T_j} - \frac{x_i y_j}{T_j} + x_j y_j,$$
$$\frac{1}{\sqrt{T_i}\sqrt{T_j}} + \Bigg(\frac{a_i}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_i^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j}{\sqrt{T_i}\sqrt{T_j}} - \frac{a_i a_j}{\sqrt{T_i}\sqrt{T_j}} +$$
$$\frac{a_j^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j x_i y_i}{\sqrt{T_i}\,T_j^{3/2}} - \frac{x_i^2 y_i^2}{4\sqrt{T_i}\,T_j^{5/2}} - \frac{a_i x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} - \frac{a_j x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} + \frac{a_i x_j y_j}{\sqrt{T_i}\sqrt{T_j}} +$$
$$\frac{x_i^2 y_i y_j}{\sqrt{T_i}\,T_j^{5/2}} - \frac{x_i x_j y_i y_j}{\sqrt{T_i}\,T_j^{3/2}} - \frac{3 x_i^2 y_j^2}{4\sqrt{T_i}\,T_j^{5/2}} + \frac{x_i x_j y_j^2}{\sqrt{T_i}\,T_j^{3/2}} - \frac{x_j^2 y_j^2}{4\sqrt{T_i}\sqrt{T_j}}\Bigg)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:= $G2 = \left(\text{tR}_{5,1}\,\text{tC}_3\,\overline{\text{tR}}_{2,7}\,\text{tR}_{8,4}\,\text{tC}_6\right) \mathbin{//} \text{tm}_{1,3\to i} \mathbin{//} \text{tm}_{i,5\to i} \mathbin{//} \text{tm}_{i,7\to i} \mathbin{//} \text{tm}_{2,4\to j} \mathbin{//} \text{tm}_{j,6\to j} \mathbin{//} \text{tm}_{j,8\to j}$

Out[∘]=
$$\mathbb{E}_{\{\}\to\{i,j\}}\Big[a_i t_i - a_i t_j + a_j t_j,\ \frac{x_i y_i}{T_j} - \frac{x_i y_j}{T_j} + x_j y_j,$$
$$\frac{1}{\sqrt{T_i}\sqrt{T_j}} + \Bigg(\frac{a_i}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_i^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j}{\sqrt{T_i}\sqrt{T_j}} - \frac{a_i a_j}{\sqrt{T_i}\sqrt{T_j}} +$$
$$\frac{a_j^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j x_i y_i}{\sqrt{T_i}\,T_j^{3/2}} - \frac{x_i^2 y_i^2}{4\sqrt{T_i}\,T_j^{5/2}} - \frac{a_i x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} - \frac{a_j x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} + \frac{a_i x_j y_j}{\sqrt{T_i}\sqrt{T_j}} +$$
$$\frac{x_i^2 y_i y_j}{\sqrt{T_i}\,T_j^{5/2}} - \frac{x_i x_j y_i y_j}{\sqrt{T_i}\,T_j^{3/2}} - \frac{3 x_i^2 y_j^2}{4\sqrt{T_i}\,T_j^{5/2}} + \frac{x_i x_j y_j^2}{\sqrt{T_i}\,T_j^{3/2}} - \frac{x_j^2 y_j^2}{4\sqrt{T_i}\sqrt{T_j}}\Bigg)\,\epsilon + O[\epsilon]^2\Big]$$

In[∘]:=
```
tm_{1,2→k} /. {t_k → 0, T_k → 1}
(tm_{1,2→k} // tm_{k,3→k}) /. {t_k → 0, T_k → 1}
(tm_{1,2→k} // tm_{k,3→k} // tm_{k,4→k}) /. {t_k → 0, T_k → 1}
(tm_{1,2→k} // tm_{k,3→k} // tm_{k,4→k} // tm_{k,5→k}) /. {t_k → 0, T_k → 1}
(tm_{1,2→k} // tm_{k,3→k} // tm_{k,4→k} // tm_{k,5→k}) /. {t_k → 0, T_k → 1, 𝒜_ → 1}
```

In[∘]:=
```
tR_{i,j} /. {t_{i|j} → 0, T_{i|j} → 1}
t̄R̄_{i,j} /. {t_{i|j} → 0, T_{i|j} → 1}
tC_i /. {t_{i|j} → 0, T_{i|j} → 1}
```

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ x_j\,y_i,\ 1+\left(a_i\,a_j-\frac{1}{4}x_j^2\,y_i^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[0,\ -x_j\,y_i,\ 1+\left(-a_i\,a_j-a_i\,x_j\,y_i-a_j\,x_j\,y_i-\frac{3}{4}x_j^2\,y_i^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ 1-a_i\,\epsilon+O[\epsilon]^2\right]$

In[∘]:=
```
G0 = G /. ∈ → 0
G1 = G0; G1〚3〛 = ∈ Coefficient[G〚3〛, ∈]; G1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[a_i\,t_i-a_i\,t_j+a_j\,t_j,\ \frac{x_i\,y_i}{T_j}-\frac{x_i\,y_j}{T_j}+x_j\,y_j,\ \frac{1}{\sqrt{T_i}\,\sqrt{T_j}}\right]$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i,j\}}\Bigg[a_i\,t_i-a_i\,t_j+a_j\,t_j,\ \frac{x_i\,y_i}{T_j}-\frac{x_i\,y_j}{T_j}+x_j\,y_j,\ \epsilon\Bigg(\frac{a_i}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_i^2}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_j}{\sqrt{T_i}\,\sqrt{T_j}}-\frac{a_i\,a_j}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_j^2}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_j\,x_i\,y_i}{\sqrt{T_i}\,T_j^{3/2}}-\frac{x_i^2\,y_i^2}{4\sqrt{T_i}\,T_j^{5/2}}-\frac{a_i\,x_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}-\frac{a_j\,x_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}+\frac{a_i\,x_j\,y_j}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{x_i^2\,y_i\,y_j}{\sqrt{T_i}\,T_j^{5/2}}-\frac{x_i\,x_j\,y_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}-\frac{3\,x_i^2\,y_j^2}{4\sqrt{T_i}\,T_j^{5/2}}+\frac{x_i\,x_j\,y_j^2}{\sqrt{T_i}\,T_j^{3/2}}-\frac{x_j^2\,y_j^2}{4\sqrt{T_i}\,\sqrt{T_j}}\Bigg)\Bigg]$$

In[∘]:=
```
tBS_{i,j→k} /. ∈ → 0
```

Out[∘]= $$\mathbb{E}_{\{i,j\}\to\{k\}}\Bigg[0,\ y_k\left(1-\mathcal{A}_j\right)\eta_i+\frac{y_k\left(-\mathcal{A}_j+\mathcal{A}_i\,\mathcal{A}_j\right)\eta_j}{\mathcal{A}_i}+\frac{x_k\left(-\mathcal{A}_i+\mathcal{A}_i\,\mathcal{A}_j\right)\xi_i}{\mathcal{A}_j}+\left(\mathcal{A}_i-T_k\,\mathcal{A}_i-\mathcal{A}_i\,\mathcal{A}_j+T_k\,\mathcal{A}_i\,\mathcal{A}_j\right)\eta_i\,\xi_i+\left(-\mathcal{A}_i+T_k\,\mathcal{A}_i-\mathcal{A}_j+T_k\,\mathcal{A}_j+\mathcal{A}_i\,\mathcal{A}_j-T_k\,\mathcal{A}_i\,\mathcal{A}_j\right)\eta_j\,\xi_i+x_k\left(1-\mathcal{A}_i\right)\xi_j+\left(\mathcal{A}_i\,\mathcal{A}_j-T_k\,\mathcal{A}_i\,\mathcal{A}_j\right)\eta_i\,\xi_j+\left(\mathcal{A}_j-T_k\,\mathcal{A}_j-\mathcal{A}_i\,\mathcal{A}_j+T_k\,\mathcal{A}_i\,\mathcal{A}_j\right)\eta_j\,\xi_j,\ T_k\Bigg]$$

In[∘]:= $\left(\text{G0 // tBS}_{i,j\to k}\right)[[3]]$
$\left(\text{G1 // tBS}_{i,j\to k}\right)[[3]]$

Out[∘]= $\frac{T_k}{1-T_k+T_k^2}+\left(\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4}+\frac{-5T_k+28T_k^2-37T_k^3+12T_k^4+2T_k^5}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6}+\frac{\left(T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\in+O[\in]^2$

Out[∘]= $\left(\frac{T_k-22T_k^2+33T_k^3-10T_k^4-2T_k^5}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6}-\frac{3T_k x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\in+\left(\frac{a_k\left(-T_k+42T_k^2-72T_k^3-48T_k^4+99T_k^5-24T_k^6-2T_k^7\right)}{1-4T_k+10T_k^2-16T_k^3+19T_k^4-16T_k^5+10T_k^6-4T_k^7+T_k^8}+\frac{117T_k-782T_k^2-358T_k^3+4680T_k^4-6004T_k^5+2638T_k^6-219T_k^7-68T_k^8-4T_k^9}{4-20T_k+60T_k^2-120T_k^3+180T_k^4-204T_k^5+180T_k^6-120T_k^7+60T_k^8-20T_k^9+4T_k^{10}}+\frac{a_k\left(6T_k+6T_k^2-18T_k^3\right)x_k y_k}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6}+\frac{\left(39T_k-336T_k^2+637T_k^3-368T_k^4+59T_k^5+2T_k^6\right)x_k y_k}{1-4T_k+10T_k^2-16T_k^3+19T_k^4-16T_k^5+10T_k^6-4T_k^7+T_k^8}+\frac{\left(-6T_k+12T_k^2\right)x_k^2 y_k^2}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6}\right)\in^2+O[\in]^3$

In[∘]:= $\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4}+\frac{-5T_k+28T_k^2-37T_k^3+12T_k^4+2T_k^5}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6}$ /. $a_k\to-1/2$ // Together

Out[∘]= $\frac{-3T_k+26T_k^2-37T_k^3+14T_k^4}{2\left(1-T_k+T_k^2\right)^3}$

In[∘]:= $\left(\left(\text{G0 /. }\{t_{i|j}\to0, T_{i|j}\to1\}\right)\text{ // }\left(\text{tBS}_{i,j\to k}\text{ /. }\mathcal{A}\_\to1\right)\right)[[3]]+$
$\left(\text{G1 // }\left(\text{tBS}_{i,j\to k}\text{ /. }\in\to0\right)\right)[[3]]-\left(\text{G // tBS}_{i,j\to k}\right)[[3]]$

Out[∘]= $O[\in]^2$

In[∘]:= $\left(\text{G0 /. }\{t_{i|j}\to0, T_{i|j}\to1\}\right)$
$\left(\text{tBS}_{i,j\to k}\text{ /. }\{\mathcal{A}\_\to1, x\_\to0, y\_\to0\}\right)$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}[0, x_i y_i-x_i y_j+x_j y_j, 1]$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0, \left(-1+T_k\right)\eta_j\xi_i+\left(1-T_k\right)\eta_i\xi_j, T_k+\Big(-2a_kT_k+\left(-2T_k+2T_k^2\right)\eta_i\xi_i-2a_kT_k^2\eta_j\xi_i+\left(2T_k-2T_k^2\right)\eta_j\xi_i+\frac{1}{2}\left(-2T_k+2T_k^2\right)\eta_i\eta_j\xi_i^2+\frac{1}{4}\left(3T_k-4T_k^2+T_k^3\right)\eta_j^2\xi_i^2+2a_kT_k^2\eta_i\xi_j+\left(-2T_k+2T_k^2\right)\eta_j\xi_j+\left(3T_k-4T_k^2+T_k^3\right)\eta_i\eta_j\xi_i\xi_j+\frac{1}{2}\left(-2T_k+2T_k^2\right)\eta_j^2\xi_i\xi_j+\frac{1}{4}\left(-3T_k+4T_k^2-T_k^3\right)\eta_i^2\xi_j^2\Big)\in+O[\in]^2\Big]$

In[ ]:= $\text{S1} = \mathbb{E}\Big[0,\ x_i\,y_i - x_i\,y_j + x_j\,y_j + (-1 + T_k)\,\eta_j\,\xi_i + (1 - T_k)\,\eta_i\,\xi_j,$
$T_k + \Big((-2\,T_k + 2\,T_k^2)\,\eta_i\,\xi_i + (2\,T_k - 2\,T_k^2)\,\eta_j\,\xi_i + \frac{1}{2}\,(-2\,T_k + 2\,T_k^2)\,\eta_i\,\eta_j\,\xi_i^2 +$
$\frac{1}{4}\,(3\,T_k - 4\,T_k^2 + T_k^3)\,\eta_j^2\,\xi_i^2 + (-2\,T_k + 2\,T_k^2)\,\eta_j\,\xi_j + (3\,T_k - 4\,T_k^2 + T_k^3)\,\eta_i\,\eta_j\,\xi_i\,\xi_j +$
$\frac{1}{2}\,(-2\,T_k + 2\,T_k^2)\,\eta_j^2\,\xi_i\,\xi_j + \frac{1}{4}\,(-3\,T_k + 4\,T_k^2 - T_k^3)\,\eta_i^2\,\xi_j^2\Big)\,\epsilon + O[\epsilon]^2\Big];$

In[ ]:= $\text{QZipM}_{\{\xi_i,\xi_j,y_i,y_j\}}[\text{S1}]$

» $\{x_i, x_j, \eta_i, \eta_j\}$

» $\{0, 0, 0, 0\}$

» $\{0, 0, 0, 0\}$

» $$\begin{pmatrix} 1 & 0 & 0 & 1-T_k \\ 0 & 1 & -1+T_k & 0 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

» $$\left\{\left\{\frac{1}{1-T_k+T_k^2}, \frac{-1+T_k}{1-T_k+T_k^2}, \frac{-1+2T_k-T_k^2}{1-T_k+T_k^2}, \frac{-1+T_k}{1-T_k+T_k^2}\right\}, \left\{\frac{1-T_k}{1-T_k+T_k^2}, \frac{T_k}{1-T_k+T_k^2}, \frac{T_k-T_k^2}{1-T_k+T_k^2}, \frac{-1+2T_k-T_k^2}{1-T_k+T_k^2}\right\}, \left\{\frac{1}{1-T_k+T_k^2}, \frac{-1+T_k}{1-T_k+T_k^2}, \frac{T_k}{1-T_k+T_k^2}, \frac{-1+T_k}{1-T_k+T_k^2}\right\}, \left\{-\frac{T_k}{1-T_k+T_k^2}, \frac{1}{1-T_k+T_k^2}, \frac{1-T_k}{1-T_k+T_k^2}, \frac{1}{1-T_k+T_k^2}\right\}\right\}$$

» $$\left\{x_i \to \frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{(-1+2T_k-T_k^2)\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2},\right.$$
$$x_j \to \frac{(1-T_k)\,x_i}{1-T_k+T_k^2} + \frac{T_k\,x_j}{1-T_k+T_k^2} + \frac{(T_k-T_k^2)\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+2T_k-T_k^2)\,\eta_j}{1-T_k+T_k^2},$$
$$\eta_i \to \frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{T_k\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2},$$
$$\left.\eta_j \to -\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right\}$$

» $\{\xi_i \to \xi_i, \xi_j \to \xi_j, y_i \to y_i, y_j \to y_j\}$

» $$T_k + \left((2T_k-2T_k^2)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)\xi_i + \right.$$
$$(-2T_k+2T_k^2)\left(\frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{T_k\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2}\right)\xi_i +$$
$$\frac{1}{4}(3T_k-4T_k^2+T_k^3)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)^2 \xi_i^2 +$$
$$\frac{1}{2}(-2T_k+2T_k^2)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)$$
$$\left(\frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{T_k\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2}\right)\xi_i^2 +$$
$$(-2T_k+2T_k^2)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)\xi_j +$$
$$\frac{1}{2}(-2T_k+2T_k^2)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)^2 \xi_i\,\xi_j +$$
$$(3T_k-4T_k^2+T_k^3)\left(-\frac{T_k\,x_i}{1-T_k+T_k^2} + \frac{x_j}{1-T_k+T_k^2} + \frac{(1-T_k)\,\eta_i}{1-T_k+T_k^2} + \frac{\eta_j}{1-T_k+T_k^2}\right)$$
$$\left(\frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{T_k\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2}\right)\xi_i\,\xi_j +$$
$$\left.\frac{1}{4}(-3T_k+4T_k^2-T_k^3)\left(\frac{x_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,x_j}{1-T_k+T_k^2} + \frac{T_k\,\eta_i}{1-T_k+T_k^2} + \frac{(-1+T_k)\,\eta_j}{1-T_k+T_k^2}\right)^2 \xi_j^2\right)\epsilon + O[\epsilon]^2$$

Out[ ]= $$\mathbb{E}\left[0, 0, \frac{T_k}{1-T_k+T_k^2} + \frac{(-5T_k+28T_k^2-37T_k^3+12T_k^4+2T_k^5)\,\epsilon}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6} + O[\epsilon]^2\right]$$

In[∘]:= $\left(\left(\mathbf{G0}\ /.\ \{t_{i|j}\to 0,\ T_{i|j}\to 1\}\right)\ //\ \left(\mathbf{tBS}_{i,j\to k}\ /.\ \mathcal{A}\_\to 1\right)\right)[[3]]$

Out[∘]= $\frac{T_k}{1-T_k+T_k^2}+\left(\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4}+\frac{-5T_k+28T_k^2-37T_k^3+12T_k^4+2T_k^5}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6}+\frac{\left(T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\epsilon+O[\epsilon]^2$

In[∘]:= $\left(\mathbf{tBS}_{i,j\to k}\ /.\ \epsilon\to 0\right)$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k\left(1-\mathcal{A}_j\right)\eta_i+\frac{y_k\left(-\mathcal{A}_j+\mathcal{A}_i\mathcal{A}_j\right)\eta_j}{\mathcal{A}_i}+\frac{x_k\left(-\mathcal{A}_i+\mathcal{A}_i\mathcal{A}_j\right)\xi_i}{\mathcal{A}_j}+\left(\mathcal{A}_i-T_k\mathcal{A}_i-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\xi_i+\left(-\mathcal{A}_i+T_k\mathcal{A}_i-\mathcal{A}_j+T_k\mathcal{A}_j+\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_j\xi_i+x_k\left(1-\mathcal{A}_i\right)\xi_j+\left(\mathcal{A}_i\mathcal{A}_j-T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_i\xi_j+\left(\mathcal{A}_j-T_k\mathcal{A}_j-\mathcal{A}_i\mathcal{A}_j+T_k\mathcal{A}_i\mathcal{A}_j\right)\eta_j\xi_j,\ T_k\Big]$

In[∘]:= $\mathbf{G1}\ //\ \left(\mathbf{tBS}_{i,j\to k}\ /.\ \epsilon\to 0\right)$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\Big[0,\ 0,\ \frac{\epsilon T_k-22\epsilon T_k^2+33\epsilon T_k^3-10\epsilon T_k^4-2\epsilon T_k^5}{2-6T_k+12T_k^2-14T_k^3+12T_k^4-6T_k^5+2T_k^6}-\frac{3\epsilon T_k x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\Big]$

In[∘]:= $\mathbf{tC}_1$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \sqrt{T_1}-a_1\sqrt{T_1}\ \epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbf{tR}_{1,2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\Big[a_2 t_1,\ x_2 y_1,\ 1+\left(a_1 a_2-\frac{1}{4}x_2^2 y_1^2\right)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbf{G}\ //\ \mathbf{tBS}_{i,j\to k}$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\Big[0,\ 0,\ \frac{T_k}{1-T_k+T_k^2}+\left(\frac{a_k\left(-2T_k+2T_k^3\right)}{1-2T_k+3T_k^2-2T_k^3+T_k^4}+\frac{-2T_k+3T_k^2-2T_k^3+T_k^4}{1-3T_k+6T_k^2-7T_k^3+6T_k^4-3T_k^5+T_k^6}+\frac{\left(-2T_k-2T_k^2\right)x_k y_k}{1-2T_k+3T_k^2-2T_k^3+T_k^4}\right)\epsilon+O[\epsilon]^2\Big]$

Compare with trefoils:

In[∘]:= $\mathbf{tR}_{5,1}\,\mathbf{tR}_{2,6}\,\mathbf{tR}_{7,3}\,\mathbf{tC}_4\,\overline{\mathbf{tKink}_8}\,\overline{\mathbf{tKink}_9}\,\overline{\mathbf{tKink}_{10}}\ //\ \mathbf{tm}_{1,2\to1}\ //\ \mathbf{tm}_{1,3\to1}\ //\ \mathbf{tm}_{1,4\to1}\ //\ \mathbf{tm}_{1,5\to1}\ //\ \mathbf{tm}_{1,6\to1}\ //\ \mathbf{tm}_{1,7\to1}\ //\ \mathbf{tm}_{1,8\to1}\ //\ \mathbf{tm}_{1,9\to1}\ //\ \mathbf{tm}_{1,10\to1}$

$\overline{\mathbf{tR}}_{1,5}\,\overline{\mathbf{tR}}_{6,2}\,\overline{\mathbf{tR}}_{3,7}\,\mathbf{tC}_4\,\mathbf{tKink}_8\,\mathbf{tKink}_9\,\mathbf{tKink}_{10}\ //\ \mathbf{tm}_{1,2\to1}\ //\ \mathbf{tm}_{1,3\to1}\ //\ \mathbf{tm}_{1,4\to1}\ //\ \mathbf{tm}_{1,5\to1}\ //\ \mathbf{tm}_{1,6\to1}\ //\ \mathbf{tm}_{1,7\to1}\ //\ \mathbf{tm}_{1,8\to1}\ //\ \mathbf{tm}_{1,9\to1}\ //\ \mathbf{tm}_{1,10\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{T_1}{1-T_1+T_1^2}+\left(\frac{a_1\left(-2T_1+2T_1^3\right)}{1-2T_1+3T_1^2-2T_1^3+T_1^4}+\frac{-T_1^2+2T_1^3-3T_1^4+2T_1^5}{1-3T_1+6T_1^2-7T_1^3+6T_1^4-3T_1^5+T_1^6}+\frac{\left(-2T_1-2T_1^2\right)x_1 y_1}{1-2T_1+3T_1^2-2T_1^3+T_1^4}\right)\epsilon+O[\epsilon]^2\Big]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{T_1}{1-T_1+T_1^2}+\left(\frac{a_1\left(-2T_1+2T_1^3\right)}{1-2T_1+3T_1^2-2T_1^3+T_1^4}+\frac{-2T_1+3T_1^2-2T_1^3+T_1^4}{1-3T_1+6T_1^2-7T_1^3+6T_1^4-3T_1^5+T_1^6}+\frac{\left(-2T_1-2T_1^2\right)x_1 y_1}{1-2T_1+3T_1^2-2T_1^3+T_1^4}\right)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbf{tBS_{i,j\to k}}$ /. {$\mathcal{A}_{i|j}\to 1$}

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (-1+T_k)\,\eta_j\,\xi_i+(1-T_k)\,\eta_i\,\xi_j,$
$T_k+\Big(-2\,a_k\,T_k-2\,T_k\,y_k\,\eta_i+2\,T_k\,y_k\,\eta_j+2\,T_k\,x_k\,\xi_i+(-2\,T_k+2\,T_k^2)\,\eta_i\,\xi_i-2\,a_k\,T_k^2\,\eta_j\,\xi_i+$
$(2\,T_k-2\,T_k^2)\,\eta_j\,\xi_i+T_k\,x_k\,y_k\,\eta_j\,\xi_i-2\,T_k\,y_k\,\eta_i\,\eta_j\,\xi_i+T_k\,y_k\,\eta_j^2\,\xi_i+T_k\,x_k\,\eta_j\,\xi_i^2+$
$\frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_i\,\eta_j\,\xi_i^2+\frac{1}{4}\,(3\,T_k-4\,T_k^2+T_k^3)\,\eta_j^2\,\xi_i^2-2\,T_k\,x_k\,\xi_j+2\,a_k\,T_k^2\,\eta_i\,\xi_j-T_k\,x_k\,y_k\,\eta_i\,\xi_j+$
$T_k\,y_k\,\eta_i^2\,\xi_j+(-2\,T_k+2\,T_k^2)\,\eta_j\,\xi_j-2\,T_k\,x_k\,\eta_j\,\xi_i\,\xi_j+(3\,T_k-4\,T_k^2+T_k^3)\,\eta_i\,\eta_j\,\xi_i\,\xi_j+$
$\frac{1}{2}\,(-2\,T_k+2\,T_k^2)\,\eta_j^2\,\xi_i\,\xi_j+T_k\,x_k\,\eta_i\,\xi_j^2+\frac{1}{4}\,(-3\,T_k+4\,T_k^2-T_k^3)\,\eta_i^2\,\xi_j^2\Big)\,\epsilon+O[\epsilon]^2\Big]$

**Define**$\Big[\mathbf{BB_{i,j\to k}}=\mathbb{E}_{\{i,j\}\to\{\}}\big[0,\ (1-T_k)\ (\eta_i\,\xi_j-\eta_j\,\xi_i),\ T_k\big]\Big]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}[0,\ y_i\,x_i-y_j\,x_i+y_j\,x_j,\ 1]$ // $\mathbf{BB_{i,j\to k}}$

Out[∘]= $\mathbb{E}_{\{\}\to\{\}}\Big[0,\ 0,\ \frac{T_k}{1-T_k+T_k^2}\Big]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}[0,\ y_i\,x_i-y_j\,x_i+y_j\,x_j,\ 1]$ // ($\mathbf{tBS_{i,j\to k}}$ /. {$\mathcal{A}_{i|j}\to 1$})

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\Big[0,\ 0,\ \frac{T_k}{1-T_k+T_k^2}+$
$\Big(\frac{a_k\,(-2\,T_k+2\,T_k^3)}{1-2\,T_k+3\,T_k^2-2\,T_k^3+T_k^4}+\frac{-5\,T_k+28\,T_k^2-37\,T_k^3+12\,T_k^4+2\,T_k^5}{2-6\,T_k+12\,T_k^2-14\,T_k^3+12\,T_k^4-6\,T_k^5+2\,T_k^6}+\frac{(T_k-2\,T_k^2)\,x_k\,y_k}{1-2\,T_k+3\,T_k^2-2\,T_k^3+T_k^4}\Big)\,\epsilon+$
$O[\epsilon]^2\Big]$

In[∘]:= **G**

$\mathbb{E}_{\{\}\to\{i,j\}}\Big[a_i\,t_i-a_i\,t_j+a_j\,t_j,\ \frac{x_i\,y_i}{T_j}-\frac{x_i\,y_j}{T_j}+x_j\,y_j,$
$\frac{1}{\sqrt{T_i}\,\sqrt{T_j}}+\Big(\frac{a_i}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_i^2}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_j}{\sqrt{T_i}\,\sqrt{T_j}}-\frac{a_i\,a_j}{\sqrt{T_i}\,\sqrt{T_j}}+$
$\frac{a_j^2}{\sqrt{T_i}\,\sqrt{T_j}}+\frac{a_j\,x_i\,y_i}{\sqrt{T_i}\,T_j^{3/2}}-\frac{x_i^2\,y_i^2}{4\,\sqrt{T_i}\,T_j^{5/2}}-\frac{a_i\,x_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}-\frac{a_j\,x_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}+\frac{a_i\,x_j\,y_j}{\sqrt{T_i}\,\sqrt{T_j}}+$
$\frac{x_i^2\,y_i\,y_j}{\sqrt{T_i}\,T_j^{5/2}}-\frac{x_i\,x_j\,y_i\,y_j}{\sqrt{T_i}\,T_j^{3/2}}-\frac{3\,x_i^2\,y_j^2}{4\,\sqrt{T_i}\,T_j^{5/2}}+\frac{x_i\,x_j\,y_j^2}{\sqrt{T_i}\,T_j^{3/2}}-\frac{x_j^2\,y_j^2}{4\,\sqrt{T_i}\,\sqrt{T_j}}\Big)\,\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbb{E}_{\{\}\to\{i,j\}}\Big[a_i t_i - a_i t_j + a_j t_j,\ \frac{x_i y_i}{T_j} - \frac{x_i y_j}{T_j} + x_j y_j,$
$\Big(\frac{a_i}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_i^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j}{\sqrt{T_i}\sqrt{T_j}} - \frac{a_i a_j}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j^2}{\sqrt{T_i}\sqrt{T_j}} + \frac{a_j x_i y_i}{\sqrt{T_i}\,T_j^{3/2}} -$
$\frac{x_i^2 y_i^2}{4\sqrt{T_i}\,T_j^{5/2}} - \frac{a_i x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} - \frac{a_j x_i y_j}{\sqrt{T_i}\,T_j^{3/2}} + \frac{a_i x_j y_j}{\sqrt{T_i}\sqrt{T_j}} + \frac{x_i^2 y_i y_j}{\sqrt{T_i}\,T_j^{5/2}} - \frac{x_i x_j y_i y_j}{\sqrt{T_i}\,T_j^{3/2}} -$
$\frac{3 x_i^2 y_j^2}{4\sqrt{T_i}\,T_j^{5/2}} + \frac{x_i x_j y_j^2}{\sqrt{T_i}\,T_j^{3/2}} - \frac{x_j^2 y_j^2}{4\sqrt{T_i}\sqrt{T_j}}\Big)\,\epsilon\Big]\ //\ \big(tBS_{i,j\to k}\ /.\ \{\mathcal{A}_{i|j}\to 1,\ \epsilon\to 0\}\big)$

Out[∘]= $\mathbb{E}_{\{\}\to\{k\}}\Big[0,\ 0,\ \frac{\epsilon T_k - 7\epsilon T_k^3 + 10\epsilon T_k^4 - 4\epsilon T_k^5}{2 - 6T_k + 12T_k^2 - 14T_k^3 + 12T_k^4 - 6T_k^5 + 2T_k^6}\Big]$

In[∘]:= $\frac{-5T_k + 28T_k^2 - 37T_k^3 + 12T_k^4 + 2T_k^5}{2 - 6T_k + 12T_k^2 - 14T_k^3 + 12T_k^4 - 6T_k^5 + 2T_k^6} + \frac{T_k - 7\,T_k^3 + 10\,T_k^4 - 4\,T_k^5}{2 - 6T_k + 12T_k^2 - 14T_k^3 + 12T_k^4 - 6T_k^5 + 2T_k^6}$ // Together

Out[∘]= $\frac{-2T_k + 14T_k^2 - 22T_k^3 + 11T_k^4 - T_k^5}{(1 - T_k + T_k^2)^3}$

$y_k(1-\mathcal{A}_j)\eta_i + \frac{y_k(-\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j)\eta_j}{\mathcal{A}_i} + \frac{x_k(-\mathcal{A}_i + \mathcal{A}_i\mathcal{A}_j)\xi_i}{\mathcal{A}_j} + x_k(1-\mathcal{A}_i)\xi_j +$
$(\mathcal{A}_i - T_k\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_i + (-\mathcal{A}_i + T_k\mathcal{A}_i - \mathcal{A}_j + T_k\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_i +$
$(\mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_j + (\mathcal{A}_j - T_k\mathcal{A}_j - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_j$

In[∘]:= $y_k(1-\mathcal{A}_j)\eta_i + \frac{y_k(-\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j)\eta_j}{\mathcal{A}_i} + \frac{x_k(-\mathcal{A}_i + \mathcal{A}_i\mathcal{A}_j)\xi_i}{\mathcal{A}_j} + x_k(1-\mathcal{A}_i)\xi_j +$
$(\mathcal{A}_i - T_k\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_i + (-\mathcal{A}_i + T_k\mathcal{A}_i - \mathcal{A}_j + T_k\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_i +$
$(\mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_j + (\mathcal{A}_j - T_k\mathcal{A}_j - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_j$ // Simplify

Out[∘]= $-\frac{y_k\mathcal{A}_j\eta_j}{\mathcal{A}_i} + y_k\big(-(-1+\mathcal{A}_j)\eta_i + \mathcal{A}_j\eta_j\big) + x_k\Big(\mathcal{A}_i\Big(\Big(1 - \frac{1}{\mathcal{A}_j}\Big)\xi_i - \xi_j\Big) + \xi_j\Big) +$
$(-1+T_k)\big(\mathcal{A}_j\eta_j(\xi_i - \xi_j) + \mathcal{A}_i(\eta_i - \eta_j)\big((-1+\mathcal{A}_j)\xi_i - \mathcal{A}_j\xi_j\big)\big)$

In[∘]:= $y_k(1-\mathcal{A}_j)\eta_i + \frac{y_k(-\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j)\eta_j}{\mathcal{A}_i}$ // Simplify

Out[∘]= $y_k\Big(-(-1+\mathcal{A}_j)\eta_i + \frac{(-1+\mathcal{A}_i)\mathcal{A}_j\eta_j}{\mathcal{A}_i}\Big)$

In[∘]:= $tBS_{i,j\to k}$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ y_k(1-\mathcal{A}_j)\eta_i + \frac{y_k(-\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j)\eta_j}{\mathcal{A}_i} + \frac{x_k(-\mathcal{A}_i + \mathcal{A}_i\mathcal{A}_j)\xi_i}{\mathcal{A}_j} +$
$(\mathcal{A}_i - T_k\mathcal{A}_i - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_i + (-\mathcal{A}_i + T_k\mathcal{A}_i - \mathcal{A}_j + T_k\mathcal{A}_j + \mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_i +$
$x_k(1-\mathcal{A}_i)\xi_j + (\mathcal{A}_i\mathcal{A}_j - T_k\mathcal{A}_i\mathcal{A}_j)\eta_i\xi_j + (\mathcal{A}_j - T_k\mathcal{A}_j - \mathcal{A}_i\mathcal{A}_j + T_k\mathcal{A}_i\mathcal{A}_j)\eta_j\xi_j,$
$T_k + \Big(-2a_kT_k - 3T_k y_k\mathcal{A}_j\eta_i + \frac{1}{2}y_k^2(-T_k\mathcal{A}_j + T_k\mathcal{A}_j^2)\eta_i^2 + \frac{y_k(T_k\mathcal{A}_j + 2T_k\mathcal{A}_i\mathcal{A}_j)\eta_j}{\mathcal{A}_i} +$

$$\frac{y_k^2 \left(-T_k \mathcal{A}_j^2 + T_k \mathcal{A}_i \mathcal{A}_j^2\right) \eta_j^2}{2 \mathcal{A}_i^2} + \frac{x_k \left(T_k \mathcal{A}_i + 2 T_k \mathcal{A}_i \mathcal{A}_j\right) \xi_i}{\mathcal{A}_j} +$$

$$a_k \left(2 T_k^2 \mathcal{A}_i - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_i \xi_i + \left(-T_k \mathcal{A}_i + T_k^2 \mathcal{A}_i - 2 T_k \mathcal{A}_i \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_i \xi_i +$$

$$\frac{1}{2} y_k \left(T_k \mathcal{A}_i - T_k^2 \mathcal{A}_i - 2 T_k \mathcal{A}_i \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j + T_k \mathcal{A}_i \mathcal{A}_j^2 - 3 T_k^2 \mathcal{A}_i \mathcal{A}_j^2\right) \eta_i^2 \xi_i +$$

$$x_k y_k \left(2 T_k - T_k \mathcal{A}_i - T_k \mathcal{A}_j + T_k \mathcal{A}_i \mathcal{A}_j\right) \eta_j \xi_i +$$

$$\left(T_k \mathcal{A}_i - T_k^2 \mathcal{A}_i + T_k \mathcal{A}_j - T_k^2 \mathcal{A}_j + 2 T_k \mathcal{A}_i \mathcal{A}_j - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_j \xi_i + a_k \left(-2 T_k^2 \mathcal{A}_i - 2 T_k^2 \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right)$$

$$\eta_j \xi_i + y_k \left(-2 T_k \mathcal{A}_j + 2 T_k^2 \mathcal{A}_j - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j - 2 T_k^2 \mathcal{A}_j^2 + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j^2\right) \eta_i \eta_j \xi_i + \frac{1}{2 \mathcal{A}_i}$$

$$y_k \left(3 T_k \mathcal{A}_i \mathcal{A}_j - 3 T_k^2 \mathcal{A}_i \mathcal{A}_j - T_k \mathcal{A}_i^2 \mathcal{A}_j + 3 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - T_k \mathcal{A}_j^2 - T_k^2 \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - 3 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2\right)$$

$$\eta_j^2 \xi_i + \frac{x_k^2 \left(-T_k \mathcal{A}_i^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j\right) \xi_i^2}{2 \mathcal{A}_j^2} + \frac{x_k \left(3 T_k \mathcal{A}_i^2 - T_k^2 \mathcal{A}_i^2 - 4 T_k \mathcal{A}_i^2 \mathcal{A}_j + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i \xi_i^2}{2 \mathcal{A}_j} +$$

$$\frac{1}{4} \left(-3 T_k \mathcal{A}_i^2 + 4 T_k^2 \mathcal{A}_i^2 - T_k^3 \mathcal{A}_i^2 + 4 T_k \mathcal{A}_i^2 \mathcal{A}_j - 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i^2 \xi_i^2 + \frac{1}{2 \mathcal{A}_j}$$

$$x_k \left(-T_k \mathcal{A}_i^2 - T_k^2 \mathcal{A}_i^2 + 3 T_k \mathcal{A}_i \mathcal{A}_j - 3 T_k^2 \mathcal{A}_i \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - T_k \mathcal{A}_i \mathcal{A}_j^2 + 3 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - 3 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2\right)$$

$$\eta_j \xi_i^2 + \frac{1}{2} \left(3 T_k \mathcal{A}_i^2 - 4 T_k^2 \mathcal{A}_i^2 + T_k^3 \mathcal{A}_i^2 - T_k \mathcal{A}_i \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j - T_k^3 \mathcal{A}_i \mathcal{A}_j -\right.$$

$$\left.4 T_k \mathcal{A}_i^2 \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - T_k \mathcal{A}_i \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i \eta_j \xi_i^2 +$$

$$\frac{1}{4} \left(-T_k \mathcal{A}_i^2 + T_k^3 \mathcal{A}_i^2 + 4 T_k \mathcal{A}_i \mathcal{A}_j - 8 T_k^2 \mathcal{A}_i \mathcal{A}_j + 4 T_k^3 \mathcal{A}_i \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - 4 T_k^3 \mathcal{A}_i^2 \mathcal{A}_j - T_k \mathcal{A}_j^2 +\right.$$

$$\left.T_k^3 \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 - 4 T_k^3 \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 + 3 T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_j^2 \xi_i^2 -$$

$$3 T_k x_k \mathcal{A}_i \xi_j + 2 a_k T_k^2 \mathcal{A}_i \mathcal{A}_j \eta_i \xi_j - T_k x_k y_k \mathcal{A}_i \mathcal{A}_j \eta_i \xi_j + \left(2 T_k \mathcal{A}_i \mathcal{A}_j - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_i \xi_j +$$

$$\frac{1}{2} y_k \left(T_k \mathcal{A}_i \mathcal{A}_j - T_k^2 \mathcal{A}_i \mathcal{A}_j + T_k \mathcal{A}_i \mathcal{A}_j^2 + T_k^2 \mathcal{A}_i \mathcal{A}_j^2\right) \eta_i^2 \xi_j + a_k \left(2 T_k^2 \mathcal{A}_j - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_j \xi_j +$$

$$\left(-T_k \mathcal{A}_j + T_k^2 \mathcal{A}_j - 2 T_k \mathcal{A}_i \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j\right) \eta_j \xi_j + y_k \left(2 T_k \mathcal{A}_j^2 - 2 T_k \mathcal{A}_i \mathcal{A}_j^2\right) \eta_i \eta_j \xi_j +$$

$$\frac{y_k \left(3 T_k \mathcal{A}_j^2 - T_k^2 \mathcal{A}_j^2 - 4 T_k \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_j^2 \xi_j}{2 \mathcal{A}_i} + x_k \left(2 T_k \mathcal{A}_i^2 - 2 T_k \mathcal{A}_i^2 \mathcal{A}_j\right) \eta_i \xi_i \xi_j +$$

$$\frac{1}{2} \left(-3 T_k \mathcal{A}_i^2 \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j - T_k^3 \mathcal{A}_i^2 \mathcal{A}_j + 3 T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i^2 \xi_i \xi_j +$$

$$x_k \left(-2 T_k \mathcal{A}_i + 2 T_k^2 \mathcal{A}_i - 2 T_k^2 \mathcal{A}_i^2 - 2 T_k^2 \mathcal{A}_i \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j\right) \eta_j \xi_i \xi_j +$$

$$\left(3 T_k \mathcal{A}_i^2 \mathcal{A}_j - 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j + 3 T_k \mathcal{A}_i \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i \mathcal{A}_j^2 - 3 T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 - T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right)$$

$$\eta_i \eta_j \xi_i \xi_j + \frac{1}{2} \left(-T_k \mathcal{A}_i \mathcal{A}_j + 2 T_k^2 \mathcal{A}_i \mathcal{A}_j - T_k^3 \mathcal{A}_i \mathcal{A}_j - T_k \mathcal{A}_i^2 \mathcal{A}_j + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j + 3 T_k \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_j^2 +\right.$$

$$\left.T_k^3 \mathcal{A}_j^2 - 4 T_k \mathcal{A}_i \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 + T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_j^2 \xi_i \xi_j + \frac{1}{2} x_k^2 \left(-T_k \mathcal{A}_i + T_k \mathcal{A}_i^2\right) \xi_j^2 +$$

$$\frac{1}{2} x_k \left(T_k \mathcal{A}_i \mathcal{A}_j - T_k^2 \mathcal{A}_i \mathcal{A}_j + T_k \mathcal{A}_i^2 \mathcal{A}_j + T_k^2 \mathcal{A}_i^2 \mathcal{A}_j\right) \eta_i \xi_j^2 + \frac{1}{4} \left(-3 T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 - T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i^2 \xi_j^2 +$$

$$\frac{1}{2} x_k \left(T_k \mathcal{A}_j - T_k^2 \mathcal{A}_j - 2 T_k \mathcal{A}_i \mathcal{A}_j + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j + T_k \mathcal{A}_i^2 \mathcal{A}_j - 3 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j\right) \eta_j \xi_j^2 +$$

$$\frac{1}{2} \left(-3 T_k \mathcal{A}_i \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 - T_k^3 \mathcal{A}_i \mathcal{A}_j^2 + 3 T_k \mathcal{A}_i^2 \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_i \eta_j \xi_j^2 +$$

$$\frac{1}{4} \left(-3 T_k \mathcal{A}_j^2 + 4 T_k^2 \mathcal{A}_j^2 - T_k^3 \mathcal{A}_j^2 + 4 T_k \mathcal{A}_i \mathcal{A}_j^2 - 4 T_k^2 \mathcal{A}_i \mathcal{A}_j^2 - T_k \mathcal{A}_i^2 \mathcal{A}_j^2 + T_k^3 \mathcal{A}_i^2 \mathcal{A}_j^2\right) \eta_j^2 \xi_j^2 \Bigg) \epsilon + O[\epsilon]^2 \Bigg]$$

*In[ ]:=* **D[tBS$_{i,j\to k}$[[3]], $\tau_j$]**

Out[∘]= 0

In[∘]:= $\mathbf{tBS_{i,j\to k}}$ /. {$\mathcal{A}_{i|j}\to 1$, $x_k\to 0$, $y_k\to 0$, $a_k\to 0$}

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[0,\ (-1+T_k)\,\eta_j\,\xi_i+(1-T_k)\,\eta_i\,\xi_j,$
$T_k+\Big(\big(-3T_k+3T_k^2\big)\eta_i\xi_i+\big(4T_k-4T_k^2\big)\eta_j\xi_i+\frac{1}{2}\big(-2T_k+2T_k^2\big)\eta_i\eta_j\xi_i^2+\frac{1}{4}\big(3T_k-4T_k^2+T_k^3\big)\eta_j^2\xi_i^2+\big(2T_k-2T_k^2\big)\eta_i\xi_j+\big(-3T_k+3T_k^2\big)\eta_j\xi_j+\big(3T_k-4T_k^2+T_k^3\big)\eta_i\eta_j\xi_i\xi_j+\frac{1}{2}\big(-2T_k+2T_k^2\big)\eta_j^2\xi_i\xi_j+\frac{1}{4}\big(-3T_k+4T_k^2-T_k^3\big)\eta_i^2\xi_j^2\Big)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbf{tBS_{i,j\to k}}[\![3]\!]$ /. {$\mathcal{A}_{i|j}\to 1$, $x_k\to 0$, $y_k\to 0$, $a_k\to 0$}

Out[∘]= $T_k+\Big(\big(-3T_k+3T_k^2\big)\eta_i\xi_i+\big(4T_k-4T_k^2\big)\eta_j\xi_i+\frac{1}{2}\big(-2T_k+2T_k^2\big)\eta_i\eta_j\xi_i^2+\frac{1}{4}\big(3T_k-4T_k^2+T_k^3\big)\eta_j^2\xi_i^2+\big(2T_k-2T_k^2\big)\eta_i\xi_j+\big(-3T_k+3T_k^2\big)\eta_j\xi_j+\big(3T_k-4T_k^2+T_k^3\big)\eta_i\eta_j\xi_i\xi_j+\frac{1}{2}\big(-2T_k+2T_k^2\big)\eta_j^2\xi_i\xi_j+\frac{1}{4}\big(-3T_k+4T_k^2-T_k^3\big)\eta_i^2\xi_j^2\Big)\epsilon+O[\epsilon]^2$

In[∘]:= $\mathbf{tm_{i,j\to k}}$

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\alpha_i+a_k\alpha_j+t_k\tau_i+t_k\tau_j,\ y_k\eta_i+\frac{y_k\eta_j}{\mathcal{A}_i}+\frac{x_k\xi_i}{\mathcal{A}_j}+(1-T_k)\eta_j\xi_i+x_k\xi_j,$
$1+\Big(2a_kT_k\eta_j\xi_i+\frac{x_ky_k\eta_j\xi_i}{\mathcal{A}_i\mathcal{A}_j}+\frac{(1-3T_k)y_k\eta_j^2\xi_i}{2\mathcal{A}_i}+\frac{(1-3T_k)x_k\eta_j\xi_i^2}{2\mathcal{A}_j}+\frac{1}{4}\big(1-4T_k+3T_k^2\big)\eta_j^2\xi_i^2\Big)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\mathbf{tm_{i,j\to k}}$ /. {$t_k\to 0$, $T_k\to 1$}

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\Big[a_k\alpha_i+a_k\alpha_j,\ y_k\eta_i+\frac{y_k\eta_j}{\mathcal{A}_i}+\frac{x_k\xi_i}{\mathcal{A}_j}+x_k\xi_j,$
$1+\Big(2a_k\eta_j\xi_i+\frac{x_ky_k\eta_j\xi_i}{\mathcal{A}_i\mathcal{A}_j}-\frac{y_k\eta_j^2\xi_i}{\mathcal{A}_i}-\frac{x_k\eta_j\xi_i^2}{\mathcal{A}_j}\Big)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= $\big(\mathbf{tm_{1,2\to k}}$ // $\mathbf{tm_{k,3\to k}}\big)$ /. {$t_k\to 0$, $T_k\to 1$}

Out[∘]= $\mathbb{E}_{\{1,2,3\}\to\{k\}}\Big[a_k\alpha_1+a_k\alpha_2+a_k\alpha_3,\ y_k\eta_1+\frac{y_k\eta_2}{\mathcal{A}_1}+\frac{y_k\eta_3}{\mathcal{A}_1\mathcal{A}_2}+\frac{x_k\xi_1}{\mathcal{A}_2\mathcal{A}_3}+\frac{x_k\xi_2}{\mathcal{A}_3}+x_k\xi_3,$
$1+\Big(2a_k\eta_2\xi_1+\frac{x_ky_k\eta_2\xi_1}{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}-\frac{y_k\eta_2^2\xi_1}{\mathcal{A}_1}+\frac{2a_k\eta_3\xi_1}{\mathcal{A}_2}+\frac{x_ky_k\eta_3\xi_1}{\mathcal{A}_1\mathcal{A}_2^2\mathcal{A}_3}-\frac{2y_k\eta_2\eta_3\xi_1}{\mathcal{A}_1\mathcal{A}_2}-\frac{y_k\eta_3^2\xi_1}{\mathcal{A}_1\mathcal{A}_2^2}-\frac{x_k\eta_2\xi_1^2}{\mathcal{A}_2\mathcal{A}_3}-\frac{x_k\eta_3\xi_1^2}{\mathcal{A}_2^2\mathcal{A}_3}+2a_k\eta_3\xi_2+\frac{x_ky_k\eta_3\xi_2}{\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3}-\frac{y_k\eta_3^2\xi_2}{\mathcal{A}_1\mathcal{A}_2}-\frac{2x_k\eta_3\xi_1\xi_2}{\mathcal{A}_2\mathcal{A}_3}-\frac{x_k\eta_3\xi_2^2}{\mathcal{A}_3}\Big)\epsilon+O[\epsilon]^2\Big]$

In[∘]:= **tS_k**

Out[∘]= $\mathbb{E}_{\{k\}\to\{k\}}\Big[-a_k\,\alpha_k - t_k\,\tau_k,\ -\frac{y_k\,\mathcal{A}_k\,\eta_k}{T_k} - x_k\,\mathcal{A}_k\,\xi_k + \frac{(\mathcal{A}_k - T_k\,\mathcal{A}_k)\,\eta_k\,\xi_k}{T_k},$

$1 + \Big(\frac{y_k\,\mathcal{A}_k\,\eta_k}{T_k} - \frac{a_k\,y_k\,\mathcal{A}_k\,\eta_k}{T_k} - \frac{y_k^2\,\mathcal{A}_k^2\,\eta_k^2}{2\,T_k^2} - a_k\,x_k\,\mathcal{A}_k\,\xi_k + \frac{2\,a_k\,\mathcal{A}_k\,\eta_k\,\xi_k}{T_k} -$

$\frac{x_k\,y_k\,\mathcal{A}_k^2\,\eta_k\,\xi_k}{T_k} + \frac{(-\mathcal{A}_k + T_k\,\mathcal{A}_k)\,\eta_k\,\xi_k}{T_k} + \frac{y_k\,\big(3\,\mathcal{A}_k^2 - T_k\,\mathcal{A}_k^2\big)\,\eta_k^2\,\xi_k}{2\,T_k^2} - \frac{1}{2}\,x_k^2\,\mathcal{A}_k^2\,\xi_k^2 +$

$\frac{x_k\,\big(3\,\mathcal{A}_k^2 - T_k\,\mathcal{A}_k^2\big)\,\eta_k\,\xi_k^2}{2\,T_k} + \frac{\big(-3\,\mathcal{A}_k^2 + 4\,T_k\,\mathcal{A}_k^2 - T_k^2\,\mathcal{A}_k^2\big)\,\eta_k^2\,\xi_k^2}{4\,T_k^2}\Big)\,\epsilon + O[\epsilon]^2\Big]$

```
((tBS_{i,j→k} // tS_k)[[2]] -
    (tBS_{i,j→k})[[2]] // Together) /. {x_ → 0, y_ → 0}
```

Out[∘]= $\frac{1}{T_k\,\mathcal{A}_i\,\mathcal{A}_j}$

$\big(-\mathcal{A}_i^2\,\eta_i\,\xi_i + T_k\,\mathcal{A}_i^2\,\eta_i\,\xi_i + \mathcal{A}_i^2\,\mathcal{A}_j\,\eta_i\,\xi_i - 2\,T_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_i\,\xi_i + T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_i\,\xi_i + T_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_i\,\xi_i - T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_i\,\xi_i +$

$\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i - T_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_j\,\xi_i + T_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j\,\xi_i - T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_j\,\xi_i + T_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j\,\xi_i - T_k^2\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j\,\xi_i -$

$T_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j\,\xi_i + T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j\,\xi_i + \mathcal{A}_i\,\mathcal{A}_j\,\eta_i\,\xi_j - T_k\,\mathcal{A}_i\,\mathcal{A}_j\,\eta_i\,\xi_j - \mathcal{A}_i^2\,\mathcal{A}_j\,\eta_i\,\xi_j + T_k\,\mathcal{A}_i^2\,\mathcal{A}_j\,\eta_i\,\xi_j -$

$\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_i\,\xi_j + T_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_i\,\xi_j - T_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_i\,\xi_j + T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_i\,\xi_j - \mathcal{A}_j^2\,\eta_j\,\xi_j + T_k\,\mathcal{A}_j^2\,\eta_j\,\xi_j +$

$\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j\,\xi_j - 2\,T_k\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j\,\xi_j + T_k^2\,\mathcal{A}_i\,\mathcal{A}_j^2\,\eta_j\,\xi_j + T_k\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j\,\xi_j - T_k^2\,\mathcal{A}_i^2\,\mathcal{A}_j^2\,\eta_j\,\xi_j\big)$