Normally ordered variables in the q-variant of the 1-co sl_2 algebra.

For pragmatic reasons, $\mathbb{E}[\omega, L, Q, P]$ means $\omega^{-1}(1+\epsilon\,\omega^{-4}P)\,\mathrm{Exp}[L+\omega^{-1}Q]$, where ω is an ϵ-free scalar, L is linear and contains only c's and b's, Q is a balanced quadratic in the u's and the w's and contains no c's and b's, and P is a balanced quartic polynomial in the c's, u's, and w's.

```
(*The R-matrix and its inverse*)
R⁺[i_,j_] := 𝔼[1, b_i c_j, u_i w_j, c_i c_j - c_j u_i w_j - 1/4 u_i^2 w_j^2]
R⁻[i_,j_] := 𝔼[1, -b_i c_j, -t_i^-1 u_i w_j, -c_i c_j + t_i^-1 c_i u_i w_j - t_i^-1 u_i w_j + t_i^-2/4 u_i^2 w_j^2]
```

```
(*The ribbon element is Drinfeld*t^-1/2
 e^-ϵc. We assign suitable values to the cups and caps,
following Ohtsuki p.72. Only the left-moving cups and caps need to be recorded.*)
nr[i_] := 𝔼[t_i^1/2, 0, 0, -t_i^2 c_i]
nl[i_] := 𝔼[1, 0, 0, 0]
ur[i_] := 𝔼[t_i^-1/2, 0, 0, t_i^-2 c_i]
ul[i_] := 𝔼[1, 0, 0, 0]
```

```
(*The q-Logos*)
Λ[k_] := 1/4 (α^2 β^2 + 4 α β δ μ + 2 δ^2 μ^2) (1 - 3 t_k) (1 - t_k) - 2 μ^2 (α β + δ μ) c_k t_k -
  2 β δ μ^2 c_k t_k u_k + 1/2 β ((α β + 2 δ μ) (1 - 3 t_k) + 4 α β μ t_k + 8 δ μ^2 t_k) u_k +
  1/4 β^2 δ ((1 + μ) (1 - 3 t_k) + 8 μ t_k) u_k^2 + 1/2 α (α β + 2 δ μ) (1 - 3 t_k) w_k - 2 α δ μ^2 c_k t_k w_k -
  2 δ^2 μ^2 c_k t_k u_k w_k + (α β + δ μ) (1 - 2 δ^2 (1 - t_k) t_k) u_k w_k + 1/2 β δ (1 + μ - 2 δ t_k + 4 δ μ t_k) u_k^2 w_k +
  1/4 α^2 δ (1 + μ) (1 - 3 t_k) w_k^2 + 1/2 α δ (1 + μ - 2 δ t_k) u_k w_k^2 + 1/4 δ^2 (4 μ + δ^2 (1 - 3 t_k) (1 - t_k)) u_k^2 w_k^2
```

```
DP[x_→D_α_, y_→D_β_][P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) :> c D[f, {α, m}, {β, n}]]
```

```
CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
    𝔼[ω /. b_l_ :> Log[t_l], L, Q /. b_l_ :> Log[t_l], P /. b_l_ :> Log[t_l]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] ≡ 𝔼[ω2_, L2_, Q2_, P2_] :=
  (ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2);
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
```

```
N_{u_{i_},c_{j_},k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ⅇ^{-γ} β u_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω ⅇ^{-γ} β u_k + (Q /. u_i → 0), ⅇ^{-q} DP_{c_j→D_γ, u_i→D_β}[P][ⅇ^q]] /.
     {γ → ∂_{c_j} L, β → ω^{-1} ∂_{u_i} Q}]];
N_{w_{i_},c_{j_},k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ⅇ^γ α w_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω ⅇ^γ α w_k + (Q /. w_i → 0), ⅇ^{-q} DP_{c_j→D_γ, w_i→D_α}[P][ⅇ^q]] /.
     {γ → ∂_{c_j} L, α → ω^{-1} ∂_{w_i} Q}]];
```

```
N_{w_{i_},u_{j_},k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = (1 - t_k) μ^{-1} α β + μ^{-1} β u_k + μ^{-1} δ u_k w_k + μ^{-1} α w_k}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | u_j → 0), μ^4 ⅇ^{-q} DP_{w_i→D_α, u_j→D_β}[P][ⅇ^q] + ω^4 Λ[k]] /. μ →
       1 + (t_k - 1) δ /. {α → ω^{-1} (∂_{w_i} Q /. u_j → 0), β → ω^{-1} (∂_{u_j} Q /. w_i → 0), δ → ω^{-1} ∂_{w_i,u_j} Q}]];
```

```
m_{i_,j_→k_}[Z_] := Module[{x, y},
  Z // ReplaceAll[{b_{i|j} → b_k, t_{i|j} → t_k}] // N_{w_i,c_j,x} // N_{w_x,u_j,y} //
     ReplaceAll[{c_{x|y} → c_x, w_j → w_y}] // N_{u_i,c_x,x} //
    ReplaceAll[{c_{i|x} → c_k, u_{x|y} → u_k, w_y → w_k, b_{x|y} → b_k, t_{x|y} → t_k}] // CF]
```

Testing routines.

```
(*Test meta-associativity*)
Q0 = 𝔼[x, Sum[RandomInteger[{-2, 2}] b_i c_j, {i, 3}, {j, 3}],
   Sum[RandomInteger[{-2, 2}] u_i w_j, {i, 3}, {j, 3}],
   Sum[RandomInteger[{-2, 2}] u_i w_j, {i, 3}, {j, 3}]];
t1 = Q0 // m_{1,2→1} // m_{1,3→1};
t2 = Q0 // m_{2,3→2} // m_{1,2→1};
t1 ≡ t2
```

True

```
(*Check Reidemeister 2,3*)
(R^+_{2,1} R^+_{4,3} R^+_{6,5} // m_{2,4→x} // m_{1,6→y} // m_{3,5→z}) ≡ (R^+_{2,1} R^+_{4,3} R^+_{6,5} // m_{4,6→x} // m_{2,5→y} // m_{1,3→z})
(R^-_{1,2} R^-_{3,4} R^-_{5,6} // m_{2,4→x} // m_{1,6→y} // m_{3,5→z}) ≡ (R^-_{1,2} R^-_{3,4} R^-_{5,6} // m_{4,6→x} // m_{2,5→y} // m_{1,3→z})
R^+_{1,2} R^-_{3,4} // m_{4,2→x} // m_{3,1→y}
```

True

True

$\mathbb{E}[1, 0, 0, 0]$

```
(*rotated crossings*)
t2 = ur_1 R^-_{2,5} nr_3 ur_4 nr_6 // m_{1,2→1} // m_{1,3→1} // m_{4,5→4} // m_{4,6→4}
t3 = (ul_1 R^-_{2,5} nl_3 ul_4 nl_6 // m_{1,2→1} // m_{1,3→1} // m_{4,5→4} // m_{4,6→4})
t2 ≡ t3
```

$$\mathbb{E}\left[1, -b_1 c_4, -\frac{u_1 w_4}{t_1}, -c_1 c_4 - \frac{u_1 w_4}{t_1} + \frac{c_1 u_1 w_4}{t_1} + \frac{u_1^2 w_4^2}{4 t_1^2}\right]$$

$$\mathbb{E}\left[1, -b_1 c_4, -\frac{u_1 w_4}{t_1}, -c_1 c_4 - \frac{u_1 w_4}{t_1} + \frac{c_1 u_1 w_4}{t_1} + \frac{u_1^2 w_4^2}{4 t_1^2}\right]$$

```
True
```

$ul_1\, nl_2$ // $m_{1,2\to1}$
$nl_1\, ul_2$ // $m_{1,2\to1}$
$nr_1\, ur_2$ // $m_{1,2\to1}$
$ur_1\, nr_2$ // $m_{1,2\to1}$

$\mathbb{E}[1, 0, 0, 0]$

$\mathbb{E}[1, 0, 0, 0]$

$\mathbb{E}[1, 0, 0, 0]$

$\mathbb{E}[1, 0, 0, 0]$

(*the oppositely oriented RII *)
$R^-_{1,2}\, R^+_{3,4}\, ur_5\, nr_6$ // $m_{1,3\to1}$ // $m_{4,5\to4}$ // $m_{4,2\to4}$ // $m_{4,6\to4}$

$\mathbb{E}[1, 0, 0, 0]$

Computations

Kinks

$R^+_{1,2}\, nr_3$ // $m_{1,3\to1}$ // $m_{1,2\to1}$
$R^+_{1,2}\, ur_3$ // $m_{2,3\to2}$ // $m_{2,1\to1}$
$R^-_{2,1}\, nr_3$ // $m_{1,3\to1}$ // $m_{1,2\to1}$
$R^-_{2,1}\, ur_3$ // $m_{2,3\to2}$ // $m_{2,1\to1}$

$$\mathbb{E}\left[\sqrt{t_1},\, b_1 c_1,\, \frac{u_1 w_1}{\sqrt{t_1}},\, -c_1 t_1^2 + c_1^2 t_1^2 + 2\, t_1 u_1 w_1 - 2\, c_1 t_1 u_1 w_1 + \frac{3}{4} u_1^2 w_1^2\right]$$

$$\mathbb{E}\left[\sqrt{t_1},\, b_1 c_1,\, \frac{u_1 w_1}{\sqrt{t_1}},\, -c_1 t_1^2 + c_1^2 t_1^2 + 2\, t_1 u_1 w_1 - 2\, c_1 t_1 u_1 w_1 + \frac{3}{4} u_1^2 w_1^2\right]$$

$$\mathbb{E}\left[\frac{1}{\sqrt{t_1}},\, -b_1 c_1,\, -\frac{u_1 w_1}{\sqrt{t_1}},\, \frac{c_1}{t_1^2} - \frac{c_1^2}{t_1^2} + \frac{u_1^2 w_1^2}{4\, t_1^2}\right]$$

$$\mathbb{E}\left[\frac{1}{\sqrt{t_1}},\, -b_1 c_1,\, -\frac{u_1 w_1}{\sqrt{t_1}},\, \frac{c_1}{t_1^2} - \frac{c_1^2}{t_1^2} + \frac{u_1^2 w_1^2}{4\, t_1^2}\right]$$

(*Kinks should be central elements!*)

$k = R^+_{1,2}\, ur_3$ // $m_{2,3\to2}$ // $m_{2,1\to4}$;

$$X = \mathbb{E}\left[1,\, -b_1 c_1,\, -\frac{u_1 w_1}{t_1},\, -\frac{u_1 w_1}{t_1} + \frac{c_1 u_1 w_1}{t_1} + \frac{u_1^2 w_1^2}{4\, t_1^2}\right];$$

$(k\, X$ // $m_{1,4\to1}) \equiv (k\, X$ // $m_{4,1\to1})$

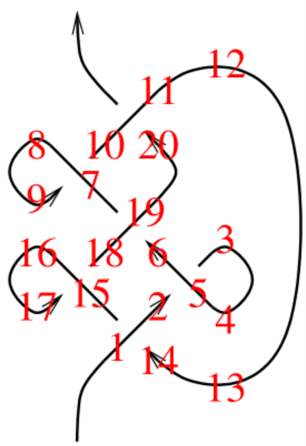
```
True
```

```
R⁺_{1,2} nr_3 R⁺_{4,5} nr_6 // m_{1,3→1} // m_{1,2→1} // m_{4,6→4} // m_{4,5→4} // m_{1,4→1}
(*Kinks are inverses*)
R⁺_{1,2} nr_3 R⁻_{5,4} nr_6 // m_{1,3→1} // m_{1,2→1} // m_{4,6→4} // m_{4,5→4} // m_{1,4→1}
```

$$\mathbb{E}\left[t_1,\ 2 b_1 c_1,\ u_1 w_1 + \frac{u_1 w_1}{t_1},\ -2 c_1 t_1^4 + 2 c_1^2 t_1^4 + 4 t_1^2 u_1 w_1 - 4 c_1 t_1^2 u_1 w_1 + 2 t_1^3 u_1 w_1 - 2 c_1 t_1^3 u_1 w_1 + \frac{7}{4} u_1^2 w_1^2 + 2 t_1 u_1^2 w_1^2 + \frac{3}{4} t_1^2 u_1^2 w_1^2\right]$$

$$\mathbb{E}[1, 0, 0, 0]$$

Compute Trefoil




```
z31 = R⁺_{1,14} R⁻_{5,2} nr_3 ul_4 R⁺_{19,6} R⁻_{7,10} nl_8 ur_9 R⁺_{11,20} nr_12 ul_13 R⁻_{15,18} nl_16 ur_17;
(Do[z31 = z31 // m_{1,k→1}, {k, 2, 20}]; z31 = z31 /. a_1 :→ a)
FromCoefficientRules[
 CoefficientRules[z31[[4]], {c, u, w}] /. {(e_ → a_) :→ (e → Simplify[a] // Factor)}, {c, u, w}]
```

$$\mathbb{E}\left[-1 + \frac{1}{t} + t,\ 0,\ 0,\ 16 + \frac{2c}{t^4} - \frac{1}{t^3} - \frac{6c}{t^3} + \frac{4}{t^2} + \frac{10c}{t^2} - \frac{10}{t} - \frac{8c}{t} - 18t + 8ct + 14t^2 - 10ct^2 - 7t^3 + 6ct^3 + 2t^4 - 2ct^4 + 2uw - \frac{2uw}{t^4} + \frac{4uw}{t^3} - \frac{6uw}{t^2} + \frac{2uw}{t} - 6tuw + 4t^2uw - 2t^3uw\right]$$

$$-\frac{2c(-1+t)(1+t)(1-t+t^2)^3}{t^4} + \frac{(-1+t)(1-t+t^2)^2(1-t+2t^2)}{t^3} - \frac{2(1+t)(1-t+t^2)^3 uw}{t^4}$$

```
(*Compare with Dror's Trefoil*)
```

$$-\frac{(1-t+t^2)^2(-1+2t-3t^2+2t^3)}{t^3} - \frac{c(1-t+t^2)^3(4+t-5t^2-t^3+t^4)}{2t^4} - \frac{(1-t+t^2)^3(-4-5t+t^3)\,uw}{2t^4}$$

$$\{(-1+2t-3t^2+2t^3),\ (4+t-5t^2-t^3+t^4),\ (-4-5t+t^3)\} \text{ // Factor}$$

$$\{(-1+t)(1-t+2t^2),\ (-1+t)(1+t)(-4-t+t^2),\ (1+t)(-4-t+t^2)\}$$

```
(*Mirror trefoil, *)
z31m = R⁻14,1 R⁺2,5 nr3 ul4 R⁻6,19 R⁺10,7 nl8 ur9 R⁻20,11 nr12 ul13 R⁺18,15 nl16 ur17;
(Do[z31m = z31m // m1,k→1, {k, 2, 20}]; z31m = z31m /. a_1 :→ a)
FromCoefficientRules[CoefficientRules[z31m[[4]], {c, u, w}] /.
  {(e_ → a_) :→ (e → Simplify[a] // Factor)}, {c, u, w}]
```

$$\mathbb{E}\left[-1+\frac{1}{t}+t,\ 0,\ 0,\ -16-\frac{2}{t^4}+\frac{2c}{t^4}+\frac{7}{t^3}-\frac{6c}{t^3}-\frac{14}{t^2}+\frac{10c}{t^2}+\frac{18}{t}-\frac{8c}{t}+10t+8ct-4t^2-10ct^2+t^3+6ct^3-2ct^4+2uw-\frac{2uw}{t^4}+\frac{4uw}{t^3}-\frac{6uw}{t^2}+\frac{2uw}{t}-6tuw+4t^2uw-2t^3uw\right]$$

$$-\frac{2c(-1+t)(1+t)(1-t+t^2)^3}{t^4}+\frac{(-1+t)(1-t+t^2)^2(2-t+t^2)}{t^4}-\frac{2(1+t)(1-t+t^2)^3uw}{t^4}$$

```
(*the 1-co invariant DOES distinguish it
  from the usual trefoil. So it is STRONGER than Alexander.*)
(z31m ≡ z31) /. t → 2 // Simplify
```

False

```
(*The figure eight knot*)

z41 = R⁺1,8 R⁻2,6 R⁺10,3 R⁻5,9 ur4 nr7;
(Do[z41 = z41 // m1,k→1, {k, 2, 10}]; z41 = z41 /. a_1 :→ a)
FromCoefficientRules[
 CoefficientRules[z41[[4]], {c, u, w}] /. {(e_ → a_) :→ (e → Simplify[a] // Factor)}, {c, u, w}]
```

$$\mathbb{E}\left[t,\ 0,\ 0,\ -2t^2+2t^3-2ct^4-2t^2uw-2t^3uw\right]$$

$$2(-1+t)t^2-2ct^4-2t^2(1+t)uw$$