Pensieve header: Normally Ordered Exponentials at 1-Co in the t variables.

For pragmatic reasons, $\mathbb{E}[\omega, L, Q, P]$ means $\omega^{-1}(1+\epsilon\,\omega^{-4}P)\,\mathrm{Exp}[L+\omega^{-1}Q]$, where ω is an ϵ-free scalar, L is linear and contains only c's and b's, Q is a balanced quadratic in the u's and the w's and contains no c's and b's, and P is a balanced quartic polynomial in the c's, u's, and w's.

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(*Dror's Logos for the algebra g_1.
  Λ[k_]:=(1-t_k) (α^2β^2+4α β δ μ+2δ^2μ^2)/2+2μ^2(α β+δ μ)c_k-β(2μ-1) (α β+2δ μ)u_k+
  2β δ μ^2c_k u_k-β^2δ(3μ-1)u_k^2/2+α(α β+2 δ μ)w_k+2α δ μ^2c_k w_k-2(t_k-1)δ^2(α β+δ μ)u_k w_k+
  2δ^2μ^2c_k u_k w_k-β δ^2(2μ-1)u_k^2 w_k+α^2δ(1+μ)w_k^2/2+α δ^2u_k w_k^2-(t_k-1)δ^4u_k^2w_k^2/2;
*)
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(*The q-Logos*)
Λ[k_] := (1/4 (α^2 β^2 + 4 α β δ μ + 2 δ^2 μ^2) (1 - 3 t_k) (1 - t_k) - 2 μ^2 (α β + δ μ) c_k t_k -
   2 β δ μ^2 c_k t_k u_k + 1/2 β ((α β + 2 δ μ) (1 - 3 t_k) + 4 α β μ t_k + 8 δ μ^2 t_k) u_k +
   1/4 β^2 δ ((1 + μ) (1 - 3 t_k) + 8 μ t_k) u_k^2 + 1/2 α (α β + 2 δ μ) (1 - 3 t_k) w_k - 2 α δ μ^2 c_k t_k w_k -
   2 δ^2 μ^2 c_k t_k u_k w_k + (α β + δ μ) (1 - 2 δ^2 (1 - t_k) t_k) u_k w_k + 1/2 β δ (1 + μ - 2 δ t_k + 4 δ μ t_k) u_k^2 w_k +
   1/4 α^2 δ (1 + μ) (1 - 3 t_k) w_k^2 + 1/2 α δ (1 + μ - 2 δ t_k) u_k w_k^2 + 1/4 δ^2 (4 μ + δ^2 (1 - 3 t_k) (1 - t_k)) u_k^2 w_k^2)
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DP_{x_→D_α_, y_→D_β_}[P_][f_] :=
 Total[CoefficientRules[P, {x, y}] /. ({m_, n_} → c_) ⧴ c D[f, {α, m}, {β, n}]]
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CF[𝔼[ω_, L_, Q_, P_]] := Expand /@ Together /@
    𝔼[ω /. b_l_ ⧴ Log[t_l], L, Q /. b_l_ ⧴ Log[t_l], P /. b_l_ ⧴ Log[t_l]];
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] ≡ 𝔼[ω2_, L2_, Q2_, P2_] :=
  (ω1 == ω2 ∧ L1 == L2 ∧ Q1 == Q2 ∧ P1 == P2);
𝔼 /: 𝔼[ω1_, L1_, Q1_, P1_] 𝔼[ω2_, L2_, Q2_, P2_] :=
  CF@𝔼[ω1 ω2, L1 + L2, ω2 Q1 + ω1 Q2, ω2^4 P1 + ω1^4 P2];
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N_{u_i_, c_j_, k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ⅇ^-γ β u_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω ⅇ^-γ β u_k + (Q /. u_i → 0), ⅇ^-q DP_{c_j→D_γ, u_i→D_β}[P][ⅇ^q]] /.
     {γ → ∂_{c_j} L, β → ω^-1 ∂_{u_i} Q}]];
N_{w_i_, c_j_, k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = ⅇ^γ α w_k + γ c_k}, CF[
    𝔼[ω, γ c_k + (L /. c_j → 0), ω ⅇ^γ α w_k + (Q /. w_i → 0), ⅇ^-q DP_{c_j→D_γ, w_i→D_α}[P][ⅇ^q]] /.
     {γ → ∂_{c_j} L, α → ω^-1 ∂_{w_i} Q}]];
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N_{w_i_, u_j_, k_}[𝔼[ω_, L_, Q_, P_]] := With[{q = (1 - t_k) μ^-1 α β + μ^-1 β u_k + μ^-1 δ u_k w_k + μ^-1 α w_k}, CF[
    𝔼[μ ω, L, μ ω q + μ (Q /. w_i | u_j → 0), μ^4 ⅇ^-q DP_{w_i→D_α, u_j→D_β}[P][ⅇ^q] + ω^4 Λ[k]] /. μ →
       1 + (t_k - 1) δ /. {α → ω^-1 (∂_{w_i} Q /. u_j → 0), β → ω^-1 (∂_{u_j} Q /. w_i → 0), δ → ω^-1 ∂_{w_i,u_j} Q}]];
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m_{i_,j_→k_}[Z_] := Module[{x, y},
  Z // ReplaceAll[{b_{i|j} → b_k, t_{i|j} → t_k}] // N_{w_i,c_j,x} // N_{w_x,u_j,y} //
      ReplaceAll[{c_{x|y} → c_x, w_j → w_y}] // N_{u_i,c_x,x} //
    ReplaceAll[{c_{i|x} → c_k, u_{x|y} → u_k, w_y → w_k, b_{x|y} → b_k, t_{x|y} → t_k}] // CF]
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(*Test meta-associativity*)
Q0 = 𝔼[1, Sum[RandomInteger[{-2, 2}] b_i c_j, {i, 3}, {j, 3}],
  Sum[RandomInteger[{-2, 2}] u_i w_j, {i, 3}, {j, 3}], 0];
t1 = Q0 // m_{1,2→1} // m_{1,3→1};
t2 = Q0 // m_{2,3→2} // m_{1,2→1};
t1 ≡ t2
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True

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(*The R-matrix*)
R⁺_{i_,j_} := 𝔼[1, b_i c_j, u_i w_j, c_i c_j - c_j u_i w_j - 1/4 u_i^2 w_j^2]
R⁻_{i_,j_} := 𝔼[1, -b_i c_j, -t_i^-1 u_i w_j, c_i c_j - c_i u_i w_j + u_i w_j + 1/4 u_i^2 w_j^2]
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(*Check Reidemeister 2,3*)
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(*Compute Trefoil*)
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(*Since the ribbon element is Drinfeld*t^{1/2}
 e^{-εc}. We assign suitable values to the cups and caps,
following Ohtsuki p.72. Only the left-moving cups and caps need to be recorded.*)
ul_{i_} := 𝔼[t_i^{-1/2}, 0, 0, -c_i]
ur_{i_} := 𝔼[1, 0, 0, 0]
nl_{i_} := 𝔼[t_i^{1/2}, 0, 0, c_i]
nr_{i_} := 𝔼[1, 0, 0, 0]
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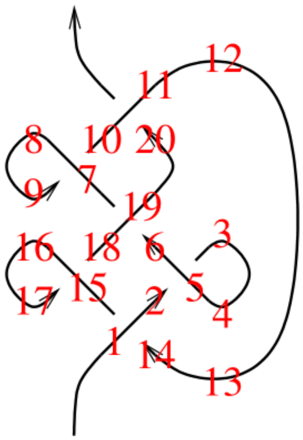
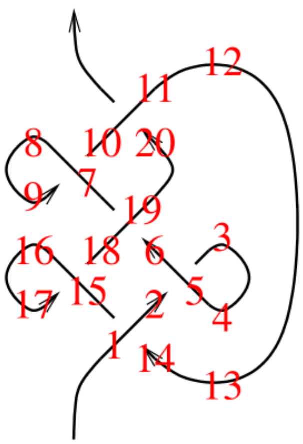


$\mathbf{z2} = R^+_{1,14}\, R^-_{5,2}\, nr_3\, ul_4\, R^+_{19,6}\, R^-_{7,10}\, nl_8\, ur_9\, R^+_{11,20}\, nr_{12}\, ul_{13}\, R^-_{15,18}\, nl_{16}\, ur_{17};$

$\left(\mathbf{Do}[\mathbf{z2} = \mathbf{z2} \;//\; m_{1,k\to 1},\ \{k, 2, 20\}];\ \mathbf{z2} = \mathbf{z2} \;/.\; a\_{}_1 :\!\to a\right)$



$$\mathbb{E}\Big[-1+\frac{1}{t}+t,\ 0,\ 0,\ -34-87\,c+114\,c^2+\frac{2}{t^6}+\frac{2\,c}{t^6}-\frac{8}{t^5}-\frac{8\,c}{t^5}+\frac{39}{2\,t^4}+\frac{21\,c}{t^4}+\frac{6\,c^2}{t^4}-\frac{26}{t^3}-\frac{28\,c}{t^3}-\frac{24\,c^2}{t^3}+\frac{16}{t^2}+\frac{13\,c}{t^2}+\frac{60\,c^2}{t^2}+\frac{10}{t}+\frac{34\,c}{t}-\frac{96\,c^2}{t}+36\,t+110\,c\,t-96\,c^2\,t-20\,t^2-91\,c\,t^2+60\,c^2\,t^2+50\,c\,t^3-24\,c^2\,t^3+10\,t^4-19\,c\,t^4+6\,c^2\,t^4-10\,t^5+4\,c\,t^5+6\,t^6-c\,t^6-2\,t^7+\frac{t^8}{2}+216\,u\,w-108\,c\,u\,w+\frac{2\,u\,w}{t^7}-\frac{8\,u\,w}{t^6}+\frac{18\,u\,w}{t^5}+\frac{2\,c\,u\,w}{t^5}-\frac{13\,u\,w}{t^4}-\frac{14\,c\,u\,w}{t^4}-\frac{24\,u\,w}{t^3}+\frac{41\,c\,u\,w}{t^3}+\frac{102\,u\,w}{t^2}-\frac{81\,c\,u\,w}{t^2}-\frac{178\,u\,w}{t}+\frac{108\,c\,u\,w}{t}-190\,t\,u\,w+75\,c\,t\,u\,w+134\,t^2\,u\,w-39\,c\,t^2\,u\,w-74\,t^3\,u\,w+12\,c\,t^3\,u\,w+34\,t^4\,u\,w-4\,c\,t^4\,u\,w-10\,t^5\,u\,w+c\,t^5\,u\,w-c\,t^6\,u\,w+2\,t^7\,u\,w-t^8\,u\,w+\frac{33\,u^2\,w^2}{4}-\frac{u^2\,w^2}{2\,t^6}+\frac{2\,u^2\,w^2}{t^5}-\frac{5\,u^2\,w^2}{2\,t^4}+\frac{u^2\,w^2}{t^3}+\frac{21\,u^2\,w^2}{4\,t^2}-\frac{8\,u^2\,w^2}{t}-\frac{15}{4}\,t^2\,u^2\,w^2+7\,t^3\,u^2\,w^2-\frac{13}{4}\,t^4\,u^2\,w^2+2\,t^5\,u^2\,w^2+\frac{1}{4}\,t^6\,u^2\,w^2+\frac{1}{4}\,t^8\,u^2\,w^2\Big]$$

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FromCoefficientRules[
 CoefficientRules[z2[[4]], {c, u, w}] /. {(e_ → a_) :> (e → Simplify[a])}, {c, u, w}]
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$$\frac{6 c^2 (1-t+t^2)^4}{t^4} - \frac{1}{t^6} c (1-t+t^2)^3 (-2+2t-3t^2-7t^3+10t^4-t^5+t^6) + \frac{1}{2t^6}(1-t+t^2)^2 (4-8t+11t^2+2t^3-17t^4+10t^5-4t^6-2t^7+5t^8-2t^9+t^{10}) - \frac{c (1-t+t^2)^4 (-2+6t+3t^2+t^3) u w}{t^5} - \frac{1}{t^7} (1-t+t^2)^3 (-2+2t-13t^3+11t^4-9t^5+2t^6-3t^7+t^8+t^9) u w + \frac{1}{4t^6}(1-t+t^2)^4 (-2+10t^2+12t^3+7t^4+4t^5+t^6) u^2 w^2$$

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(*Compare with Dror's Trefoil*)
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$$-\frac{(1-t+t^2)^2 (-1+2t-3t^2+2t^3)}{t^3} - \frac{c (1-t+t^2)^3 (4+t-5t^2-t^3+t^4)}{2t^4} - \frac{(1-t+t^2)^3 (-4-5t+t^3) u w}{2t^4}$$