

C in the middle,  $t_i$  becomes  $t$ . And  $a = ue^{-\epsilon c}$ . And finally  $\eta = -\epsilon \frac{1+t}{1-t}$

```

 $\epsilon$  /:  $\epsilon^n$  /;  $n > 1$  := 0;
PBWBasis = {a, c, w};

B[U@c, U@u] = - (B[U@u, U@c] = -U@u);
B[U@u, U@w] = - (B[U@w, U@u] = (1-t) U[] +  $\epsilon$  U[u, w] - 2 $\epsilon$ t U[c]);

B[U@c, U@w] = - (B[U@w, U@c] = U@w);
B[U@c, U@a] = - (B[U@a, U@c] = -U@a);
B[U@a, U@w] = - (B[U@w, U@a] = (1-t) U[] -  $\epsilon$  (1+t) U[c]);

(*
B[U@a, U@w] = - (B[U@w, U@a] = (1-t) U[] +  $\eta$  (1-t) U[c]);
*)

```

(\*Setting  $a = ue^{-\epsilon c}$  gives the bracket as above\*)

```

U@w ** U@u ** (U[] -  $\epsilon$  U[c]) -
  U@u ** (U[] -  $\epsilon$  U[c]) ** U[w] - ((1-t) U[] -  $\epsilon$  (1+t) U[c]) // Simp
0

```

```

UU[L___, x_n_, r___] := UU[L, Sequence@@Table[x, {n}], r];
UU[L___, 1, r___] := UU[L, r];
UU[] = U[]; UU[L_, r___] := U[L] ** UU[r];
U_i[_] :=  $\mathcal{E}$  /. {t -> t_i, u_U -> UU@@Replace[u, x_ -> x_i, 1]};

```

```

B[x_, x_] = 0;
B[U[(x_)i_], U[(y_)i_]] := U_i[B[U@x, U@y]];
B[U[(x_)i_], U[(y_)j_]] /; i != j := 0;
B[x_, y_] := x ** y - y ** x;

```

```

x_ <= y_ := OrderedQ[{x, y}]; x_ < y_ := ! OrderedQ[{y, x}];
Simp[_] := Collect[_U, Together];

```

```

Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[ab === 0, 0, Simp[ab (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := (*U[x]**U[y] =*) If[x < y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy_] := (*U[x] ** U[y1,yy] =*)
  If[x <= y1, U[x, y1, yy], (U@x ** U@y1) ** U@yy];
U[xx_, xn_] ** U[yy_] := (*U[xx,xn]**U[yy] =*) U@xx ** (U@xn ** U@yy);

```

```

 $\sigma$ [i_, j_] [_] :=  $\mathcal{E}$  /. {b_i -> b_j, t_i -> t_j, a_i -> a_j, c_i -> c_j, u_i -> u_j, w_i -> w_j};

```

```

mul[i_, j_][ε_] :=
  Simp[ε /. x_U => DeleteCases[x, _j] ** U@@ Cases[x, y_j => y_i] /. {b_j -> b_i, t_j -> t_i}];
mul[i_, j_, k_][ε_] := ε // mul[i, j] // σ[i, k];

```

```

Δ[i_, j_, k_][ε_] := Simp[ε /. {
  z_. U[] => (z /. {b_i -> b_j + b_k, t_i -> t_j t_k}) U[],
  z_. x_U => (z /. {b_i -> b_j + b_k, t_i -> t_j t_k}) NonCommutativeMultiply@@ (x /. {
    c_i -> U@c_j + U@c_k,
    u_i -> t_k U@u_j + ε t_k UU[c_k, u_j] + U@u_k,
    w_i -> U@w_j + ε UU[c_k, w_j] + U@w_k,
    a_i -> t_k U[a_j] + U[a_k] - ε U[a_k, c_j],
    y_l -> U@y_l
  })
}]

```

```

S[i_][ε_] := Simp[ε /. {z_. x_U => (z /. {b_i -> -b_i, t_i -> t_i^-1}) S[i][x]}];
S[i_][U[]] = U[];
S[i_][U[y_j, more___]] /; i ≠ j := U[y_j] ** S[i][U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
S[i_][U[c_i, more___]] := S[i][U[more]] ** (-U@c_i);
S[i_][U[u_i, more___]] := S[i][U[more]] ** (-t_i^-1 U@u_i + Expand[ε t_i^-1 UU[u_i, c_i]]);
S[i_][U[w_i, more___]] := S[i][U[more]] ** (-U@w_i + Expand[ε UU[w_i, c_i]]);
S[i_][U[a_i, more___]] := S[i][U[more]] ** (Expand[-t_i^-1 ((1 + ε) U[a_i] + ε U[a_i, c_i])]);

```

```

Sinv[i_][ε_] := Simp[ε /. {z_. x_U => (z /. {b_i -> -b_i, t_i -> t_i^-1}) Sinv[i][x]}];
Sinv[i_][U[]] = U[];
Sinv[i_][U[y_j, more___]] /; i ≠ j := U[y_j] ** Sinv[i][U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
Sinv[i_][U[c_i, more___]] := Sinv[i][U[more]] ** (-U@c_i);
Sinv[i_][U[u_i, more___]] :=
  Sinv[i][U[more]] ** (-t_i^-1 U@u_i + Expand[ε t_i^-1 UU[c_i, u_i]]);
Sinv[i_][U[w_i, more___]] := Sinv[i][U[more]] ** (-U@w_i + Expand[ε UU[c_i, w_i]]);
Sinv[i_][U[a_i, more___]] := Sinv[i][U[more]] ** (Expand[-t_i^-1 (U[a_i] + ε U[a_i, c_i])]);

```

```

CoUnit[i_][ε_] :=
  Simp[ε /. {z_. x_U => (z /. {t -> 1, b -> 0, b_i -> 0, t_i -> 1}) CoUnit[i][x]}];
CoUnit[i_][U[]] = U[];
CoUnit[i_][U[y_j, more___]] /; i ≠ j := U[y_j] ** CoUnit[i][U[more]];
CoUnit[i_][U[y_i, more___]] /; y ≠ a := 0;
CoUnit[i_][U[a_i, more___]] := CoUnit[i][U[more]];

```

```

LBasis[n_Integer] := LBasis[Range[n]];
LBasis[S_] := DeleteCases[0]@
Module[{i, j, k, l}, SortBy[({# /. {e -> 2, c_ -> 2, u_ -> 2, w_ -> 2, U -> Times}) &][
  Union@Flatten[{{U[], e U[]},
    Table[{U@c_i, U@u_i, U@w_i, e U@c_i, e U@u_i, e U@w_i}, {i, S}],
    Table[{U@u_i, w_j}, e U[u_i, w_j],
      e U@@Sort@{c_i, c_j}, e U[c_i, u_j], e U[c_i, w_j]}, {i, S}, {j, S}],
    Table[{e U[c_i, u_j, w_k], e U@@Sort@{u_i, u_j, w_k}, e U@@Sort@{u_i, w_j, w_k}},
      {i, S}, {j, S}, {k, S}],
    Table[e U@@Sort@{u_i, u_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}]]]]
]

```

```

BLBasis[n_Integer] := BLBasis[Range[n]];
BLBasis[S_] := DeleteCases[0]@
Module[{i, j, k, l}, SortBy[({# /. {e -> 2, c_ -> 2, u_ -> 2, w_ -> 2, U -> Times}) &][
  Union@Flatten[{{U[], e U[]},
    Table[{U@c_i, e U@c_i}, {i, S}],
    Table[{U@u_i, w_j}, e U[u_i, w_j], e U@@Sort@{c_i, c_j}}, {i, S}, {j, S}],
    Table[{e U[c_i, u_j, w_k]}, {i, S}, {j, S}, {k, S}],
    Table[e U@@Sort@{u_i, u_j, w_k, w_l}, {i, S}, {j, S}, {k, S}, {l, S}]]]]
]

```

```

UExp[ε_, n_] := Module[{t = U[], k}, U[] + Sum[ $\frac{t = t ** \epsilon}{k!}$ , {k, n}]] // Simp

```

```

ToDegree[n_][ε_] :=
(Simp[ε] /. {e -> ħ e, η -> ħ η, b_i -> ħ b_i, t_i -> eħ b_i, b -> ħ b, t -> eħ b, x_U ->
  ħCount[x, u|u_] + Count[x, a|a_] x} /. a_. x_U -> Normal[Series[a, {ħ, 0, n}]] * x) /. ħ -> 1

```

## Hopf algebra axioms

(\*Sinv is the inverse of S \*)

```
Sinv[1][S[1][U@a_1]] // Simp
```

```
Sinv[1][S[1][U@c_1]] // Simp
```

```
Sinv[1][S[1][U@w_1]] // Simp
```

```
Sinv[1][S[1][U@u_1]] // Simp
```

U[a<sub>1</sub>]

U[c<sub>1</sub>]

U[w<sub>1</sub>]

U[u<sub>1</sub>]

(\*Coassociativity only works properly with  $t_i$  instead of  $t!$ \*)

```
(U@w1 // Δ[1, x, yy] // Δ[yy, y, z]) -
(U@w1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@c1 // Δ[1, x, yy] // Δ[yy, y, z]) -
(U@c1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@a1 // Δ[1, x, yy] // Δ[yy, y, z]) -
(U@a1 // Δ[1, xx, z] // Δ[xx, x, y]) // Simp
(U@u1 // Δ[1, x, yy] // Δ[yy, y, z] // Simp) -
(U@u1 // Δ[1, xx, z] // Δ[xx, x, y] // Simp) // Simp
```

0

0

0

0

(\*Convolution inverse\*)

```
mul[2, 3, 1] [S[2] [Δ[1, 2, 3] [U[w1]]]]
mul[2, 3, 1] [S[2] [Δ[1, 2, 3] [U[c1]]]]
mul[2, 3, 1] [S[2] [Δ[1, 2, 3] [U[u1]]]]
mul[2, 3, 1] [S[2] [Δ[1, 2, 3] [U[a1]]]]
```

```
mul[2, 3, 1] [S[3] [Δ[1, 2, 3] [U[w1]]]]
mul[2, 3, 1] [S[3] [Δ[1, 2, 3] [U[c1]]]]
mul[2, 3, 1] [S[3] [Δ[1, 2, 3] [U[u1]]]]
mul[2, 3, 1] [S[3] [Δ[1, 2, 3] [U[a1]]]]
```

0

0

0

0

0

0

0

0

## Testing Yang-Baxter

```

R[i_, j_, d_] :=
  Sum[ $\frac{1}{m! n!} UU[a_i^n] ** (U[] + \epsilon n U[c_i]) ** (b_i^m U[] + \epsilon m b_i^{m-1} U[c_i]) ** UU[c_j^m, w_j^n]$ ,
    {m, 0, d}, {n, 0, d - m}] // ToDegree[d]

RU[i_, j_, d_] := Sum[ $\frac{UU[u_i^n] ** (b_i^m U[] + \epsilon m b_i^{m-1} U[c_i]) ** UU[c_j^m, w_j^n]}{m! n!} (1 - \frac{\epsilon}{4} (n-1) n)$ ,
  {m, 0, d}, {n, 0, d - m}] // ToDegree[d]

RU3[d_] := (ToDegree[d][RU[1, 2, d] ** RU[1, 3, d]] ** RU[2, 3, d]) -
  (ToDegree[d][RU[2, 3, d] ** RU[1, 3, d]] ** RU[1, 2, d])

Timing[RU3[2] // ToDegree[2]]
{0.335015, 0}

Timing[RU3[3] // ToDegree[3]]
{5.76008, 0}

Timing[RU3[4] // ToDegree[4]]
{125.708, 0}

R3[d_] := (ToDegree[d][R[1, 2, d] ** R[1, 3, d]] ** R[2, 3, d]) -
  (ToDegree[d][R[2, 3, d] ** R[1, 3, d]] ** R[1, 2, d])

R3[2] // ToDegree[2]
0

Timing[R3[3] // ToDegree[3]]
{4.55927, 0}

Timing[R3[4] // ToDegree[4]]
{49.7652, 0}

(*Verifying the inverse is given by (S tensor id)(R)*)
(S[1][R[1, 2, 4]] ** R[1, 2, 4] - U[]) // ToDegree[4]
(*Verifying the inverse is given by (id tensor S^{-1})(R) *)
0

(Sinv[2][R[1, 2, 4]] ** R[1, 2, 4] - U[]) // ToDegree[4]
0

```

```
Rinv[i_, j_, d_] := S[i][R[i, j, d]] // Expand
```

(\*Alternative formulas for Rinv, Rinv4 is the definitive one\*)

```
TestS[m_, n_] := -S[i][UU[a_i^n] ** (U[] + ε n U[c_i]) ** (b_i^m U[] + ε m b_i^{m-1} U[c_i])] +
  (-1)^m (b_i^m U[] + ε m b_i^{m-1} U[c_i]) ** (U[] - ε n U[c_i]) ** San[n, i]
```

```
TestS[4, 3] // Together
```

0

```
San[n_, i_] := (-1)^n t_i^{-n} UU[a_i^n] ** (U[] + \frac{n(n+1)}{2} ε U[] + n ε U[c_i]) // Expand
```

```
TestSan[n_] := -S[i][UU[a_i^n]] + San[n, i]
```

```
(TestSan[6] // Together)
```

0

```
Rinv2[i_, j_, d_] :=
```

```
Sum[\frac{1}{m! n!} (-1)^m (b_i^m U[] + ε m b_i^{m-1} U[c_i]) ** (U[] - ε n U[c_i]) ** San[n, i] ** UU[c_j^m, w_j^n]
  (1 + ε \frac{1}{4} (n-1) n), {m, 0, d}, {n, 0, d-m}] // ToDegree[d]
```

```
Rinv3[i_, j_, d_] :=
```

```
Sum[\frac{1}{m! n!} (-1)^m San[n, i] ** (b_i^m U[] + ε m b_i^{m-1} U[c_i] + ε m n b_i^{m-1} U[]) ** (U[] - ε n U[c_i] -
  ε n^2 U[]) ** UU[c_j^m, w_j^n] (1 + ε \frac{1}{4} (n-1) n), {m, 0, d}, {n, 0, d-m}] // ToDegree[d]
```

```
Rinv4[i_, j_, d_] :=
```

```
Sum[\frac{1}{m! n!} (-1)^{m+n} t_i^{-n} UU[a_i^n] ** (b_i^m U[] + ε m b_i^{m-1} U[c_i] + ε m n b_i^{m-1} U[]) ** UU[c_j^m, w_j^n]
  (1 - ε \frac{1}{4} (n-1) n), {m, 0, d}, {n, 0, d-m}] // ToDegree[d]
```

```
(Rinv[1, 2, 6] - Rinv2[1, 2, 6]) // Simp // ToDegree[6]
```

```
Rinv[1, 2, 6] - Rinv3[1, 2, 6] // ToDegree[6]
```

```
Rinv[1, 2, 6] - Rinv4[1, 2, 6] // ToDegree[6]
```

0

0

0

## Quasi triangularity axioms

(\*Check the three quasi-triangularity axioms\*)

```
(Δ[i, k, 1][R[i, j, 5]] - R[k, j, 5] ** R[1, j, 5]) // ToDegree[5]
```

0

```
(Δ[j, k, 1][R[i, j, 5]] - R[i, 1, 5] ** R[i, k, 5]) // ToDegree[5]
```

0

```
CheckRDR[x_, d_] :=
  (R[2, 3, d] ** Δ[1, 2, 3][x] ** Rinv[2, 3, d] - σ[2, 3][Δ[1, 2, 3][x]]) // ToDegree[d]
```

```
CheckRDR[U@c1, 4]
CheckRDR[U@w1, 4]
CheckRDR[U@a1, 4]
```

0

0

0

## Drinfeld element

(\*Drinfeld element\*)

```
Dr[d_] := R[1, 2, d] // S[2] // mul[2, 1, 1]
```

(\*Check that S(Dr) and Dr commute\*)

```
S[1][Dr[4]] ** Dr[4] - S[1][Dr[4]] ** Dr[4]
```

0

```
S[1][Dr[2]] // ToDegree[2]
```

$$U[] - b_1 U[c_1] + (-1 + \epsilon + b_1) U[a_1, w_1] + \left(-\epsilon + \frac{b_1^2}{2}\right) U[c_1, c_1] +$$

$$(\epsilon + b_1) U[a_1, c_1, w_1] + \epsilon b_1 U[c_1, c_1, c_1] + \frac{1}{2} U[a_1, a_1, w_1, w_1] + \epsilon U[a_1, c_1, c_1, w_1]$$

(\*S(Dr) = t^{-1} e^{-2\epsilon c} Dr \*)

```
S[1][Dr[5]] - (U[] - 2 \epsilon U[c1]) ** (Dr[5] t1^{-1} // ToDegree[5]) // ToDegree[5]
```

0

```
(U[] - \epsilon U[c1]) ** Dr[4] - Dr[4] ** (U[] - \epsilon U[c1]) // ToDegree[4]
```

0

Therefore the Ribbon element  $v$  is implied by  $v^2 = S(Dr)Dr = t^{-1} e^{-2\epsilon c} Dr^2$  so choose  $v = t^{-1/2} e^{-\epsilon c} Dr$ , note  $Dr$  commutes with  $e^{-\epsilon c}$ .

According to Ohtsuki p.72 read upside down, we should set the left-moving cup and cap to 1 and the right-moving cap  $nr$  should be  $vDr^{-1} = t^{-1/2} e^{-\epsilon c}$  and the right-moving cup  $ur$  should be  $Dr v^{-1} = t^{1/2} e^{\epsilon c}$ .

(\*Square of antipode\*)

```
S[1]@S[1]@U@c1
```

```
S[1]@S[1]@U@w1
```

```
S[1]@S[1]@U@a1
```

```
U[c1]
```

```
(1 - \epsilon) U[w1]
```

```
(1 + \epsilon) U[a1]
```

```
(U[] + ε U[c]) ** U[c] ** (U[] - ε U[c]) // Simp
(U[] + ε U[c]) ** U[w] ** (U[] - ε U[c]) // Simp
(U[] + ε U[c]) ** U[a] ** (U[] - ε U[c]) // Simp
U[c]
(1 - ε) U[w]
(1 + ε) U[a]
```

## Logos

**B[U@w, U@a]**

```
(1 - t) U[] - (1 + t) ε U[c]
```

```
(*η /: η^n- /; n>1 :=0; *)
η = (t+1)/(t-1) ε;
q = 1+η;
qI[k_] := k (1+η (k-1)/2) // Expand
qFac[n_] := n! (1+η (n-1)/4) // Expand
InvqFac[n_] := (1-η (n-1)/4)/n! // Expand
qBin[n_, k_] := Binomial[n, k] (1+η k (n-k)/2) // Expand
```

```
(*Checking the commutation relation for powers of a,w*)
WmAn[m_, n_] := Sum[(1-t)^j qBin[m, j] qBin[n, j]
  qFac[j] UU[a^n-j] ** (U[] + j η U[c]) ** UU[w^m-j], {j, 0, Min[m, n]}]
TestWmAn[m_, n_] := -UU[w^m, a^n] + WmAn[m, n]
TestWmAn[5, 3] // Together
0
```

```
(*Guess qLogos first at q=1*)
ToDegh[F_, x_] := Series[F /. {α -> α h, δ -> δ h, β -> β h}, {h, 0, x}]
d = 8;
LHS = Sum[α^m δ^k β^n InvqFac[m] InvqFac[k] InvqFac[n] WmAn[m+k, n+k],
  {m, 0, d}, {k, 0, d-m}, {n, 0, d-m-k}] // Simp;
(*powers of nu as power series*)
nuA[z_] := Sum[Binomial[z-1+x, x] (1-t)^x δ^x, {x, 0, d}];
RHS =
  Sum[(1-t)^j nuA[m+k+n+j+1] α^{m+j} δ^k β^{n+j} InvqFac[m] InvqFac[k] InvqFac[n] InvqFac[j]
    UU[a^{n+k}, w^{m+k}], {m, 0, d}, {k, 0, d-m}, {n, 0, d-m-k}, {j, 0, d-m-n-k}] // Simp;
ToDegh[(LHS - RHS) /. {ε -> 0} // Simp, 8]
0[h]^9
```



(\*Now set up the LHS and RHS to find the Logos relating them.\*)

d = 11;

$$\text{LHS} = \text{Sum} \left[ \alpha^m \delta^k \beta^n \frac{\text{WmAn}[m+k, n+k]}{m! k! n!}, \{m, \theta, d\}, \{k, \theta, d-m\}, \{n, \theta, d-m-k\} \right] // \text{Simp};$$

$$\text{RHS} = \text{Sum} \left[ (1-t)^j \mu^{-(m+k+n+j+1)} \frac{\alpha^{m+j} \delta^k \beta^{n+j}}{m! k! n! j!} \text{UU}[a^{n+k}, w^{m+k}], \{m, \theta, d\}, \{k, \theta, d-m\}, \{n, \theta, d-m-k\}, \{j, \theta, d-m-n-k\} \right];$$

$$\text{CRHS} = \text{Sum} \left[ (1-t)^j \mu^{-(m+k+n+j+1)} \frac{\alpha^{m+j} \delta^k \beta^{n+j}}{m! n! k! j!} \text{UU}[a^{n+k}, c, w^{m+k}], \{m, \theta, d\}, \{k, \theta, d-m\}, \{n, \theta, d-m-k\}, \{j, \theta, d-m-n-k\} \right] // \text{Simp};$$

$$\mu = \frac{1-t}{\delta};$$

$$\eta = \frac{(1+t)}{-1+t}$$

(\*Here we guess and verify the Logos coefficient by coefficient\*)

$$\begin{aligned} & \text{ToDegh}[\text{Coefficient} \left[ \left( -\text{LHS} + \text{RHS} + \right. \right. \\ & \quad \mu^{-4} (1-t) \eta \left( \right. \\ & \quad \quad (1-t) \left( \frac{1}{2} \delta^2 \mu^2 + \alpha \beta \delta \mu + \frac{\alpha^2 \beta^2}{4} \right) \text{RHS} + \\ & \quad \quad \mu^2 (\delta \mu + \alpha \beta) \text{CRHS} + \\ & \quad \quad \beta \left( \delta \mu + \frac{\alpha \beta}{2} \right) \text{U}[a] ** \text{RHS} + \\ & \quad \quad \alpha \left( \delta \mu + \frac{\alpha \beta}{2} \right) \text{RHS} ** \text{U}[w] + \\ & \quad \quad \beta \delta \mu^2 \text{U}[a] ** \text{CRHS} + \\ & \quad \quad \alpha \delta \mu^2 \text{CRHS} ** \text{U}[w] + \\ & \quad \quad \delta (1+\mu) (\mu \delta + \alpha \beta) \text{U}[a] ** \text{RHS} ** \text{U}[w] + \\ & \quad \quad \delta^2 \mu^2 \text{U}[a] ** \text{CRHS} ** \text{U}[w] + \\ & \quad \quad \frac{\beta^2 \delta}{4} (1+\mu) \text{U}[a, a] ** \text{RHS} + \\ & \quad \quad \frac{\alpha^2 \delta}{4} (1+\mu) \text{RHS} ** \text{U}[w, w] + \\ & \quad \quad \frac{\beta \delta^2}{2} (1+2\mu) \text{U}[a, a] ** \text{RHS} ** \text{U}[w] + \\ & \quad \quad \frac{\alpha \delta^2}{2} (1+2\mu) \text{U}[a] ** \text{RHS} ** \text{U}[w, w] + \\ & \quad \quad \left. \left. \frac{\delta^3}{4} (1+3\mu) \text{U}[a, a] ** \text{RHS} ** \text{U}[w, w] \right) \right], \text{U}[], 10] \\ & \text{O[h]}^{11} \end{aligned}$$

(\*Final check, put back  $\epsilon$ .)

```
ToDegh[[-LHS + RHS +
  -\mu^{-4} (1 + t) \epsilon (
    (1 - t) \left( \frac{1}{2} \delta^2 \mu^2 + \alpha \beta \delta \mu + \frac{\alpha^2 \beta^2}{4} \right) RHS +
    \mu^2 (\delta \mu + \alpha \beta) CRHS +
    \beta \left( \delta \mu + \frac{\alpha \beta}{2} \right) U[a] ** RHS +
    \alpha \left( \delta \mu + \frac{\alpha \beta}{2} \right) RHS ** U[w] +
    \beta \delta \mu^2 U[a] ** CRHS +
    \alpha \delta \mu^2 CRHS ** U[w] +
    \delta (1 + \mu) (\mu \delta + \alpha \beta) U[a] ** RHS ** U[w] +
    \delta^2 \mu^2 U[a] ** CRHS ** U[w] +
    \frac{\beta^2 \delta}{4} (1 + \mu) U[a, a] ** RHS +
    \frac{\alpha^2 \delta}{4} (1 + \mu) RHS ** U[w, w] +
    \frac{\beta \delta^2}{2} (1 + 2 \mu) U[a, a] ** RHS ** U[w] +
    \frac{\alpha \delta^2}{2} (1 + 2 \mu) U[a] ** RHS ** U[w, w] +
    \frac{\delta^3}{4} (1 + 3 \mu) U[a, a] ** RHS ** U[w, w]
  )
], 10] // Simp
0
```

## Double Reverse

$\Delta[1, 2, 3][U@c_1]$  // Simp

$\Delta[1, 2, 3][U@w_1]$  // Simp

$\Delta[1, 2, 3][U@a_1]$  // Simp

$U[c_2] + U[c_3]$

$U[w_2] + U[w_3] + \epsilon U[c_3, w_2]$

$t_3 U[a_2] + U[a_3] - \epsilon U[a_3, c_2]$

$(\Delta[1, 2, 3][U@w_1] // Simp) ** (\Delta[1, 2, 3][U@w_1] // Simp) -$   
 $(U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) - \epsilon U[w_2, w_3] - 2 \epsilon U[c_3, w_2, w_3] - 2 \epsilon U[c_3, w_2, w_2]) // Simp$

0

$(\Delta[1, 2, 3][U@a_1] // Simp) ** (\Delta[1, 2, 3][U@a_1] // Simp) - (t_3 U[a_2] + U[a_3]) **$   
 $(t_3 U[a_2] + U[a_3]) + \epsilon t_3 U[a_2, a_3] + 2 \epsilon t_3 U[a_2, a_3, c_2] + 2 \epsilon U[a_3, a_3, c_2]) // Simp$

0

$$\begin{aligned} & ((\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp}) - \\ & \quad (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) + \\ & \quad 3 \in t_3 (t_3 U[a_2] + U[a_3]) ** U[a_2, a_3] + \\ & \quad 3 \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** U[a_3, c_2]) // \text{Simp} \end{aligned}$$

0

$$\begin{aligned} & ((\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** \\ & \quad (\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp}) - \\ & \quad (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) + \\ & \quad 6 \in t_3 (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** U[a_2, a_3] + \\ & \quad 4 \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** U[a_3, c_2]) // \text{Simp} \end{aligned}$$

0

$$\begin{aligned} & (((\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** \\ & \quad (\Delta[1, 2, 3][U@a_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@a_1] // \text{Simp})) // \text{Simp}) ** \\ & \quad (((\Delta[1, 2, 3][U@w_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@w_1] // \text{Simp}) ** \\ & \quad (\Delta[1, 2, 3][U@w_1] // \text{Simp}) ** (\Delta[1, 2, 3][U@w_1] // \text{Simp})) // \text{Simp}) - \\ & \quad ( \\ & \quad (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** \\ & \quad (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) \\ & \quad + \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** \\ & \quad (4 U[c_3, w_2]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) \\ & \quad + \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** \\ & \quad (t_3 U[a_2] + U[a_3]) ** (6 U[w_2, w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) \\ & \quad - \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (4 U[a_3, c_2]) ** \\ & \quad (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) \\ & \quad - \in (t_3 U[a_2] + U[a_3]) ** (t_3 U[a_2] + U[a_3]) ** (6 t_3 U[a_2, a_3]) ** \\ & \quad (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) ** (U[w_2] + U[w_3]) \\ & \quad ) // \text{Expand} // \text{Simp} \end{aligned}$$

0

UU[c, a]

U[a] + U[a, c]

S[1]@U@w<sub>1</sub> - (-U[w<sub>1</sub>] + \in UU[w<sub>1</sub>, c<sub>1</sub>]) // SimpS[1]@U@c<sub>1</sub> // SimpS[1]@U@a<sub>1</sub> - (-t<sub>1</sub><sup>-1</sup> (U[a<sub>1</sub>] + \in UU[c<sub>1</sub>, a<sub>1</sub>])) // Expand // Simp

0

-U[c<sub>1</sub>]

0

```

Δ[1, 2, 3][U@c1] // S[2]
Δ[1, 2, 3][U@w1] // S[2] // Simp
((Δ[1, 2, 3][U@a1] // S[2] // Simp) /. t_ -> t) // Simp
-U[c2] + U[c3]

(-1 + ε) U[w2] + U[w3] + ε U[c2, w2] - ε U[c3, w2]
(-1 - ε) U[a2] + U[a3] - ε U[a2, c2] + ε U[a3, c2]

U[w] ** ((U[c] + U[]) ** U[w]) // Simp
2 U[w, w] + U[c, w, w]

(S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
(-1 + ε) U[w1] + ε U[c1, w1]
(1 - 3 ε) U[w1, w1] - 2 ε U[c1, w1, w1]
(-1 + 6 ε) U[w1, w1, w1] + 3 ε U[c1, w1, w1, w1]
(1 - 10 ε) U[w1, w1, w1, w1] - 4 ε U[c1, w1, w1, w1, w1]

(S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
(-1 - ε) U[a1] - ε U[a1, c1]
t1
(1 + 3 ε) U[a1, a1] + 2 ε U[a1, a1, c1]
t12
(-1 - 6 ε) U[a1, a1, a1] - 3 ε U[a1, a1, a1, c1]
t13
(1 + 10 ε) U[a1, a1, a1, a1] + 4 ε U[a1, a1, a1, a1, c1]
t14

(U[c] - 4 U[]) ** UU[a4] - UU[a4, c]
0

```