## KnotTableData

### The program

The building blocks

In[]:=
```
Define[
  X_{i,j} = 𝔼_{{}→{i,j}}[0, x_j (y_i - y_j),
    (1 + 1/(4 (-1 + T)^2) (2 - 4 T + 2 T^2 + 2 x_i y_i - 2 T x_i y_i - x_j^2 y_i^2 + 2 T x_j^2 y_i^2 - T^2 x_j^2 y_i^2 -
         2 T x_j y_j + 2 T^2 x_j y_j + 4 T x_i x_j y_i y_j + x_j^2 y_j^2 - 2 T x_j^2 y_j^2 - 3 T^2 x_j^2 y_j^2) ϵ + O[ϵ]^2)],
  X̄_{i,j} = 𝔼_{{}→{i,j}}[0, x_j (-y_i + y_j)/T, (1 + 1/(4 (-1 + T)^2 T^2)
      (-(-1 + T)^2 (2 T^2 + x_j y_i (-4 T + 3 x_j y_i)) - 2 (-1 + T) x_j (T (-1 + 2 T) + 2 x_j y_i) y_j +
        (1 + T) (-1 + 3 T) x_j^2 y_j^2 + 2 T x_i y_i ((-1 + T) T - 2 x_j ((-1 + T) y_i + y_j))) ϵ + O[ϵ]^2)],
  C_i = 𝔼_{{}→{i}}[0, 0, (1 + ϵ (1/2 - x_i y_i/(-1 + T))) + O[ϵ]^2],
  C̄_i = 𝔼_{{}→{i}}[0, 0, (1 - ϵ (1/2 - x_i y_i/(-1 + T))) + O[ϵ]^2],
  m_{i,j→k} = 𝔼_{{i,j}→{k}}[0, y_k (η_i + η_j) - (-1 + T) η_j ξ_i + x_k (ξ_i + ξ_j),
    1 + 1/(4 (-1 + T)^3) (2 (-1 + T)^2 (-1 + 3 T) x_k η_j ξ_i (2 y_k - (-1 + T) ξ_i) +
       (-1 + T)^3 η_j ξ_i (-4 T + 2 (1 - 3 T) y_k η_j + (1 - 4 T + 3 T^2) η_j ξ_i)) ϵ + O[ϵ]^2]

]
```

In[]:=
```
(2 - 4 T + 2 T^2 + 2 x_i y_i - 2 T x_i y_i - x_j^2 y_i^2 + 2 T x_j^2 y_i^2 - T^2 x_j^2 y_i^2 -
   2 T x_j y_j + 2 T^2 x_j y_j + 4 T x_i x_j y_i y_j + x_j^2 y_j^2 - 2 T x_j^2 y_j^2 - 3 T^2 x_j^2 y_j^2) // Simplify
```

Out[]=
$$2\left(-1+T\right)^2+2\left(-1+T\right)T\,x_j\,y_j+2\,x_i\,y_i\left(1-T+2\,T\,x_j\,y_j\right)-x_j^2\left(\left(-1+T\right)^2y_i^2+\left(-1+2\,T+3\,T^2\right)y_j^2\right)$$

## The knot invariant Rho1

Both Rho1 and Alexander are palindromic $f(T^{-1})$=f(T) so we only list the monomials with non-negative exponents.

Computing Rho1 of the trefoil knot or together with Alexander:

In[ ]:=
```
Rho1@Knot[3, 1]
Rho1AndAlex@Knot[3, 1]
Rho1@KnotsUpTo12[[1]]
```

Out[ ]= T

Out[ ]= {T, -1 + T}

Out[ ]= T

A small table of the first 30 knots

In[ ]:=
```
Flatten@{#, Rho1AndAlex@#} & /@ KnotsUpTo12[[ ;; 30]] // MatrixForm
```

Out[ ]//MatrixForm=

| | | |
|---|---|---|
| Knot[3, 1] | $T$ | $-1+T$ |
| Knot[4, 1] | $0$ | $-3+T$ |
| Knot[5, 1] | $3T+2T^3$ | $1-T+T^2$ |
| Knot[5, 2] | $-4+5T$ | $-3+2T$ |
| Knot[6, 1] | $4-T$ | $-5+2T$ |
| Knot[6, 2] | $4-4T+4T^2-T^3$ | $3-3T+T^2$ |
| Knot[6, 3] | $0$ | $5-3T+T^2$ |
| Knot[7, 1] | $6T+5T^3+3T^5$ | $-1+T-T^2+T^3$ |
| Knot[7, 2] | $-16+14T$ | $-5+3T$ |
| Knot[7, 3] | $12-16T+8T^2-9T^3$ | $3-3T+2T^2$ |
| Knot[7, 4] | $32-24T$ | $-7+4T$ |
| Knot[7, 5] | $-28+29T-16T^2+9T^3$ | $5-4T+2T^2$ |
| Knot[7, 6] | $20-19T+8T^2-T^3$ | $7-5T+T^2$ |
| Knot[7, 7] | $8-3T$ | $9-5T+T^2$ |
| Knot[8, 1] | $16-5T$ | $-7+3T$ |
| Knot[8, 2] | $12-13T+12T^2-10T^3+8T^4-2T^5$ | $-3+3T-3T^2+T^3$ |
| Knot[8, 3] | $0$ | $-9+4T$ |
| Knot[8, 4] | $4-6T+8T^2-3T^3$ | $5-5T+2T^2$ |
| Knot[8, 5] | $-24+22T-20T^2+13T^3-8T^4+2T^5$ | $-5+4T-3T^2+T^3$ |
| Knot[8, 6] | $32-28T+20T^2-5T^3$ | $7-6T+2T^2$ |
| Knot[8, 7] | $12-13T+12T^2-10T^3+4T^4-T^5$ | $-5+5T-3T^2+T^3$ |
| Knot[8, 8] | $16-12T+4T^2-T^3$ | $9-6T+2T^2$ |
| Knot[8, 9] | $0$ | $-7+5T-3T^2+T^3$ |
| Knot[8, 10] | $-4T+2T^2-T^3$ | $-7+6T-3T^2+T^3$ |
| Knot[8, 11] | $44-39T+24T^2-5T^3$ | $9-7T+2T^2$ |
| Knot[8, 12] | $0$ | $13-7T+T^2$ |
| Knot[8, 13] | $20-14T+4T^2-T^3$ | $11-7T+2T^2$ |
| Knot[8, 14] | $68-57T+28T^2-5T^3$ | $11-8T+2T^2$ |
| Knot[8, 15] | $-140+120T-64T^2+21T^3$ | $11-8T+3T^2$ |
| Knot[8, 16] | $-36+35T-28T^2+17T^3-6T^4+T^5$ | $-9+8T-4T^2+T^3$ |

Up to 10 crossings the pair (Alexander,Rho1) distinguishes all prime knots. However as expected the Kinoshita-Terasaka knot and its mutant the Conway knot yield the same value.

```
Rho1@Knot[11, NonAlternating, 42]
Rho1@Knot[11, NonAlternating, 34]
```

Out[ ]= $-2T^2$

Out[ ]= $-2T^2$

## More details on the knot invariant Z (computed up to first order in $\epsilon$)

Checking Reidemeister 1: (it is satisfied up to an overall factor of $T$ or $T^{-1}$)

In[∘]:=
```
X1,2 C̄3 // m1,3→1 // m1,2→1
X̄1,2 C3 // m1,3→1 // m1,2→1
X̄1,2 C̄3 // m2,3→2 // m2,1→1
X1,2 C3 // m2,3→2 // m2,1→1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

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Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,T+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{1}{T}+O[\epsilon]^2\right]$

Checking Reidemeister 2:

In[∘]:=
```
X1,2 X̄3,4 // m1,3->1 // m2,4→2
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Checking Reidemeister 3:

In[∘]:=
```
(X1,2 X4,3 X5,6 // m1,4→1 // m2,5→2 // m3,6→3) ≡
  (X2,3 X1,6 X4,5 // m1,4→1 // m2,5→2 // m3,6→3)
```

Out[∘]= True

Trefoil knot

In[∘]:=
```
(X5,1 X2,6 X7,3 C4 // m1,2→1 // m1,3→1 // m1,4→1 // m1,5→1 // m1,6→1 // m1,7→1)
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{1}{T-T^2+T^3}+\frac{\left(1-2T+2T^2-2T^3+T^4\right)\epsilon}{T-3T^2+6T^3-7T^4+6T^5-3T^6+T^7}+O[\epsilon]^2\right]$

The same can also be computed using the knot table name of the trefoil:

In[∘]:=
```
Z@Knot[3, 1]
Collect[#, ϵ, Factor] & /@ Z@Knot[3, 1]
```

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T^3}{1-T+T^2}+\frac{\left(-T^3+2T^4-2T^5+2T^6-T^7\right)\epsilon}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T^3}{1-T+T^2}-\frac{(-1+T)^2\,T^3\,\left(1+T^2\right)\epsilon}{\left(1-T+T^2\right)^3}\right]$

The interesting bit Rho1 is the numerator of the coefficient of $\epsilon$. The general form of this coefficient is $\frac{-(1-T)^2\,\mathrm{Rho1}}{\mathrm{Alexander}^3}$.