The Kerler algebra $\mathcal{N}$ with basis kef, where e,f span $\mathbb{E}$.

In[∘]:=

MT =

□	e	f	ef	k	ke	kf	kef
e	0	ef	0	-ke	0	-kef	0
f	-ef	0	0	-kf	kef	0	0
ef	0	0	0	kef	0	0	0
k	ke	kf	kef	1	e	f	ef
ke	0	kef	0	-e	0	-ef	0
kf	-kef	0	0	-f	ef	0	0
kef	0	0	0	ef	0	0	0

;

$$\mathcal{E}\_ \;//\; m_{i\_,j\_\to s\_} := \text{Expand}[\mathcal{E}] \;/.\; \text{Flatten@Table}\big[\text{MT}[\![\alpha,1]\!]_i\, \text{MT}[\![1,\beta]\!]_j \to (\text{MT}[\![\alpha,\beta]\!] \;/.\; v:(e\,|\,f\,|\,ef\,|\,k\,|\,ke\,|\,kf\,|\,kef) \mapsto v_s),\; \{\alpha,2,8\},\{\beta,2,8\}\big] \;/.\; \{u\_{}_i \to u_s,\; u\_{}_j \to u_s\};$$

$$\mathcal{E}\_ \;//\; S_{i\_} := \text{Expand}[\mathcal{E}] \;/.\; \{e_i \to -ke_i,\; f_i \to -kf_i,\; ef_i \to ef_i,\; k_i \to k_i,\; ke_i \to e_i,\; kf_i \to f_i,\; kef_i \to kef_i\}$$

$$\mathcal{E}\_ \;//\; \overline{S}_{i\_} := \text{Expand}[\mathcal{E}] \;/.\; \{e_i \to ke_i,\; f_i \to kf_i,\; ef_i \to ef_i,\; k_i \to k_i,\; ke_i \to -e_i,\; kf_i \to -f_i,\; kef_i \to kef_i\}$$

$$\mathcal{E}\_ \;//\; \Delta_{i\_\to r\_,s\_} := \text{Expand}[\mathcal{E}] \;/.\; \{e_i \to e_r + k_r\, e_s,\; f_i \to f_r + k_r\, f_s,\; ef_i \to ef_r + ef_s - ke_r\, f_s + kf_r\, e_s,\; k_i \to k_s\, k_r,\; ke_i \to ke_r\, k_s + ke_s,\; kf_i \to kf_r\, k_s + kf_s,\; kef_i \to kef_r\, k_s + k_r\, kef_s - e_r\, kf_s + f_r\, ke_s\}$$

$$\mathcal{E}\_ \;//\; \epsilon_{i\_} := \text{Expand}[\mathcal{E}] \;/.\; \{e_i \to 0,\; f_i \to 0,\; ef_i \to 0,\; k_i \to 1,\; ke_i \to 0,\; kf_i \to 0,\; kef_i \to 0\}$$

In[∘]:=

$$\text{KBasis}[\{i\_\}] := \{e_i, f_i, ef_i, k_i, ke_i, kf_i, kef_i\};$$
$$\text{KBasis}[\{i\_, is\_\_\}] := \text{Flatten@Outer}[\text{Times}, \text{KBasis}[\{i\}], \text{KBasis}[\{is\}]]$$

In[∘]:=

$$\mathcal{Z}_{r\_,s\_} := \frac{1}{2} \times (1 + k_r + k_s - k_r\, k_s)$$
$$R_{r\_,s\_} := (1 + e_{\$1}\, kf_{\$2})\; \mathcal{Z}_{\$3,\$4} \;//\; m_{\$1,\$3\to r} \;//\; m_{\$2,\$4\to s}$$
$$\overline{R}_{r\_,s\_} := R_{r,s} \;//\; S_r$$
$$C_{i\_} := k_i$$
$$\overline{C}_{i\_} := k_i$$
$$\text{Kink}_{i\_} := R_{1,3}\, \overline{C}_2 \;//\; m_{1,2\to 1} \;//\; m_{1,3\to i}$$
$$\overline{\text{Kink}}_{i\_} := \overline{R}_{1,3}\, C_2 \;//\; m_{1,2\to 1} \;//\; m_{1,3\to i}$$

In[∘]:= $R_{1,2}$

Out[∘]= $$\frac{1}{2} - \frac{e_1\, f_2}{2} + \frac{k_1}{2} + \frac{k_2}{2} - \frac{k_1\, k_2}{2} - \frac{f_2\, ke_1}{2} + \frac{e_1\, kf_2}{2} - \frac{ke_1\, kf_2}{2}$$

In[∘]:= $\mathcal{Z}_{1,2}\, \mathcal{Z}_{3,4} \;//\; m_{1,3\to 1} \;//\; m_{2,4\to 2}$

Out[∘]= 1

In[∘]:= $(R_{1,2}\, R_{4,3}\, R_{5,6}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3}) - (R_{1,6}\, R_{2,3}\, R_{4,5}\ //\ m_{1,4\to1}\ //\ m_{2,5\to2}\ //\ m_{3,6\to3})$

Out[∘]= 0

In[∘]:= $\overline{R}_{1,2}\, R_{3,4}\ //\ m_{1,3\to1}\ //\ m_{2,4\to2}$

$\left(\overline{R}_{1,2}\, R_{3,4}\, C_0\ //\ m_{1,3\to1}\ //\ m_{4,0\to4}\ //\ m_{4,2\to2}\right) - C_2$

Out[∘]= 1

Out[∘]= 0

In[∘]:= $\text{Table}\left[\left(BB\ //\ S_1\ //\ \overline{S}_1\right) - BB,\ \{BB,\ \text{KBasis}[\{1\}]\}\right]$

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

In[∘]:= $\text{Table}[(BB\ //\ \Delta_{1\to2,3}\ //\ S_2\ //\ m_{2,3\to1}) - (BB\ //\ \epsilon_1),\ \{BB,\ \text{KBasis}[\{1\}]\}]$

$\text{Table}[(BB\ //\ \Delta_{1\to2,3}\ //\ S_3\ //\ m_{2,3\to1}) - (BB\ //\ \epsilon_1),\ \{BB,\ \text{KBasis}[\{1\}]\}]$

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

In[∘]:= $\text{Table}[(BB\ //\ \Delta_{1\to0,3}\ //\ \Delta_{0\to1,2}) - (BB\ //\ \Delta_{1\to1,0}\ //\ \Delta_{0\to2,3}),\ \{BB,\ \text{KBasis}[\{1\}]\}]\ //\ \text{Expand}$

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

In[∘]:= $\text{Table}[(BB\ //\ m_{1,2\to0}\ //\ m_{0,3\to0}) - (BB\ //\ m_{2,3\to0}\ //\ m_{1,0\to0}),\ \{BB,\ \text{KBasis}[\{1, 2, 3\}]\}]\ //\ \text{Expand}$

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
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0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:= $\text{Table}[(BB\ //\ \Delta_{1\to a,b}\ //\ \Delta_{2\to c,d}\ //\ m_{a,c\to1}\ //\ m_{b,d\to2}) - (BB\ //\ m_{1,2\to0}\ //\ \Delta_{0\to1,2}),$
$\{BB,\ \text{KBasis}[\{1, 2\}]\}]\ //\ \text{Expand}$

Out[∘]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
(R_{1,2} // Δ_{1→r,s}) - (R_{r,2} R_{s,4} // m_{2,4→2}) // Expand
(R_{1,2} // Δ_{2→r,s}) - (R_{1,s} R_{3,r} // m_{1,3→1}) // Expand
Table[((BB // Δ_{1→2,3}) R_{5,4} // m_{2,4→a} // m_{3,5→b}) -
  ((BB // Δ_{1→3,2}) R_{5,4} // m_{4,2→a} // m_{5,3→b}), {BB, KBasis[{1}]}]
```

Out[∘]= 0

Out[∘]= 0

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
Cent = Table[c_i, {i, 1, 7}].KBasis[{j}];
Table[(Cent bas // m_{i,j→i}) - (Cent bas // m_{j,i→i}), {bas, KBasis[{i}]}]
```

Out[∘]= $\{2 c_2\, ef_i - 2 c_4\, ke_i,\ -2 c_1\, ef_i - 2 c_4\, kf_i,\ 0,\ 2 c_5\, e_i + 2 c_6\, f_i + 2 c_1\, ke_i + 2 c_2\, kf_i,\ -2 c_4\, e_i - 2 c_6\, ef_i,\ 2 c_5\, ef_i - 2 c_4\, f_i,\ 0\}$

In[∘]:=
```
Central = c_0 + c_1 ef_i + c_2 kef_i;
Table[(Central bas // m_{i,j→i}) - (Central bas // m_{j,i→i}), {bas, KBasis[{i}]}]
```

Out[∘]= {0, 0, 0, 0, 0, 0, 0}

In[∘]:=
```
u_{i_} := R_{1,2} // S_2 // m_{2,1→i}
```

In[∘]:=
```
u_3
```

Out[∘]= $k_3 - kef_3$

In[∘]:=
```
u_1 // S_1
```

Out[∘]= $k_1 - kef_1$

In[∘]:=
```
Z31 = R_{1,5} R_{6,2} R_{3,7} \bar{C}_4 \bar{Kink}_8 \bar{Kink}_9 \bar{Kink}_{10}; Do[Z31 = Z31 // m_{1,j→1}, {j, 2, 10}]; Z31
```

Out[∘]= 1

In[∘]:=
```
(u_1 (u_2 // S_2)) // m_{1,2→0}
```

Out[∘]= $1 - 2\, ef_0$

In[∘]:=
```
\bar{Kink}_1 \bar{Kink}_2 // m_{1,2→1}
```

Out[∘]= $1 - 2\, ef_1$

In[•]:=
```
Once[<< KnotTheory`];
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

e: Symbol e appears in multiple contexts {KnotTheory`FastKh`Tangles`, Global`}; definitions in context KnotTheory`FastKh`Tangles` may shadow or be shadowed by other definitions.

In[•]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm crossings
    xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots[[k]] is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

In[•]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :→ Piecewise[{{Xp[x[[4]], x[[1]]], PositiveQ@x}, {Xm[x[[2]], x[[1]]], True}}]];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :→ {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :→ (++rots[[l]]; {1 - l, k + 1, l}),
            _Xp | _Xm :→ {}
           }), {1}],
     Cases[front, k | -k] /. {k, -k} :→ --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[•]:=
```
rot[i_, 0] := 1;
rot[i_, n_] := rot[i, n, $k];
rot[i_, n_, k_] := Module[{j},
   rot[i, n, k] = If[n > 0, rot[i, n - 1] C_j, rot[i, n + 1] Overscript[C, _]_j] // m_{i,j→i}];
```

In[•]:=
```
Width[pd_PD] :=
 Max[Length /@ FoldList[Complement[#1 ⋃ #2, #1 ⋂ #2] &, {}, List @@ List @@@ pd]]
```

In[∘]:=
```
ThinPosition[K_] := Module[{todo, done, pd, c},
   todo = List @@ PD@K; done = {}; pd = PD[];
   While[todo =!= {},
    AppendTo[pd, c = RandomChoice@MaximalBy[todo, Length[done ⋂ List @@ #] &]];
    todo = DeleteCases[todo, c];
    done = done ⋃ List @@ c];
   pd ];
ThinPosition[K_, n_] := First@MinimalBy[Table[ThinPosition[K], n], Width];
```

In[∘]:=
```
Z[K_] := Z[RVK@Width@ThinPosition[K, 100]];
Z[rvk_RVK] := Module[{ξ, done, st, c, χ, i, j, k},
  ξ = 1; done = {}; st = Range[2 Length[rvk[[1]]]]; $M = {};
  Do[AppendTo[$M, c];
   {i, j} = List @@ c;
   χ = (c /. {_Xp :> R_{i,j} OverBar[Kink_0], _Xm :> OverBar[R]_{i,j} Kink_0}) // m_{j,0→j};
   Do[χ = (rot[0, rvk[[2, k]]] χ) // m_{0,k→k}, {k, {i, j}}];
   ξ *= χ;
   Do[
    If[MemberQ[done, k + 1], ξ = ξ // m_{k,k+1→k}; st = st /. k + 1 → k];
    If[MemberQ[done, k - 1], ξ = ξ // m_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]]],
    {k, {i, j}}];
   done = done ⋃ {i, j},
   {c, rvk[[1]]}
  ];
  ξ
 ]
```

In[∘]:=
```
Z /@ AllKnots[{3, 5}]
```

Out[∘]=
```
{1, 1, 1, 1}
```

In[∘]:=
```
Z /@ AllKnots[{3, 10}]
```

Out[∘]=
```
{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1,
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 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1}
```