Bulk-stitching the simplified 1co invariant (tangle version).

Data-structure of the tangle invariant: {ω,Q,Z0,Z2,Z4,n}, where ω is a Laurent polynomial in t, Q is a quadratic in $\xi_i \upsilon_j$ and Z0,Z2,Z4 are
the constant, quadratic and quartic parts of the perturbation. Finally, n is the number of strands in the tangle.
The tangle is assumed to have has components labeled 1,..,n, where n is the size of the matrix Q. P is a list of disjoint subsets P_i of {1,..n}.
After stitching all components in P_i are renamed *i*.

Input: {ω,Q,Z0,Z2,Z4,n} and an ordered partition P of the strand labels, assumed to be {1,..n}.
Output: The 1co-invariant of the tangle in the same format.

```
Pol2Mat[P_, n_] := Table[Coefficient[P, ξi υj], {i, 1, n}, {j, 1, n}]
(*Convert a 2n-multivariate quadratic to its coefficient matrix*)
Mat2Pol[M_] :=
 ({Table[ξi, {i, 1, Length[M]}]}.M.Transpose[{Table[υi, {i, 1, Length[M]}]}])[[1, 1]]
 (*Convert back*)
mP_[{ω_, Q_, Z0_, Z2_, Z4_, n_}] :=
 Block[{W, M, SwapPoly, A, B, H, G, newω, newQ, newZ0, newZ2, newZ4},
  W = Sum[If[Position[P, i][[1, 1]] == Position[P, j][[1, 1]] &&
      Position[P, i][[1, 2]] < Position[P, j][[1, 2]], ξi υj, 0], {i, 1, n}, {j, 1, n}];
  (*Instances of the Weyl commutation relation are collected in W, derived from P.*)
  A = Mat2Pol[Inverse[IdentityMatrix[n] - Pol2Mat[W, n].Pol2Mat[Q, n]]];
  B = Mat2Pol[Inverse[IdentityMatrix[n] - Pol2Mat[Q, n].Pol2Mat[W, n]]];
  G = Mat2Pol[Inverse[IdentityMatrix[n] - Pol2Mat[Q, n].Pol2Mat[W, n]].Pol2Mat[Q, n]];
  H = Mat2Pol[Pol2Mat[W, n].Inverse[IdentityMatrix[n] - Pol2Mat[Q, n].Pol2Mat[W, n]]];
  newω = ω Det[IdentityMatrix[n] - Pol2Mat[Q, n].Pol2Mat[W, n]];
  (*The Alexander tangle invt, const*)
  newQ = Mat2Pol[
     Inverse[IdentityMatrix[n] - Pol2Mat[Q, n].Pol2Mat[W, n]].Pol2Mat[Q, n]] // Together;
  (*The Alexander tangle invt, quadratic*)
  SwapPoly = (t + 1)/(t - 1) Expand[Sum[
      Sum[Sum[yi, {i, Take[P[[k]], Position[P[[k]], j][[1, 1]] - 1]}] xj
        (Sum[1/2 yi, {i, Take[P[[k]], Position[P[[k]], j][[1, 1]] - 1]}] + EEj) (1/2 xj +
           Sum[xi, {i, Take[P[[k]], {Position[P[[k]], j][[1, 1]] + 1, Length[P[[k]]]}] }] + FFj),
       {j, P[[k]]}]
      , {k, 1, Length[P]}]];
  (*What comes out of changing order of exponentials, ϵ-part.*)

  R40[i_, j_, k_, l_] :=
   Coefficient[H, ξl υi] Coefficient[H, ξj υk] + Coefficient[H, ξl υk] Coefficient[H, ξj υi];
  (*Contribution of Z4 to newZ0*)
```

```
R42[i_, j_, k_, l_] := Coefficient[H, ξl υi] Coefficient[B, υk] Coefficient[A, ξj] +
  Coefficient[H, ξj υi] Coefficient[B, υk] Coefficient[A, ξl] +
  Coefficient[H, ξl υk] Coefficient[B, υi] Coefficient[A, ξj] +
  Coefficient[H, ξj υk] Coefficient[B, υi] Coefficient[A, ξl];
(*Contribution of Z4 to newZ2*)
Swap2[i_, j_, k_, l_] := Coefficient[G, υi ξj] Coefficient[G, υk] Coefficient[G, ξl] +
  Coefficient[G, υk ξj] Coefficient[G, υi] Coefficient[G, ξl]
  + Coefficient[G, υi ξl] Coefficient[G, υk] Coefficient[G, ξj] +
  Coefficient[G, υk ξl] Coefficient[G, υi] Coefficient[G, ξj];
(*Contribution of SwapPoly to newZ2*)
Swap0[i_, j_, k_, l_] :=
 Coefficient[G, υi ξj] Coefficient[G, υk ξl] + Coefficient[G, υk ξj] Coefficient[G, υi ξl];
(*Contribution of SwapPoly to newZ2*)

newZ0 = Z0 + ((Z2 // Expand) /. {ξi_ υj_ :→ Coefficient[H, ξj υi]}) +
  (*Note these rules only work on polynomials, so need to apply Expand*)
  ((Z4 // Expand) /. {ξi_ ξk_ υj_ υl_ :→ R40[i, j, k, l], ξi_^2 υj_ υl_ :→ R40[i, j, i, l],
     ξi_ ξk_ υj_^2 :→ R40[i, j, k, j], ξi_^2 υj_^2 :→ R40[i, j, i, j]}) +
  + (SwapPoly /. {EEi_ → 0, FFi_ → 0, yi_ yk_ xj_ xl_ :→ Swap0[i, j, k, l], yi_^2 xj_ xl_ :→
       Swap0[i, j, i, l], yi_ yk_ xj_^2 :→ Swap0[i, j, k, j], yi_^2 xj_^2 :→ Swap0[i, j, i, j]});

newZ2 = ((Z2 // Expand) /. {ξi_ :→ Coefficient[B, υi], υi_ :→ Coefficient[A, ξi]}) +
  ((Z4 // Expand) /. {ξi_ ξk_ υj_ υl_ :→ R42[i, j, k, l], ξi_^2 υj_ υl_ :→ R42[i, j, i, l],
     ξi_ ξk_ υj_^2 :→ R42[i, j, k, j], ξi_^2 υj_^2 :→ R42[i, j, i, j]}) +
  (SwapPoly /. {yi_ yk_ xj_ xl_ :→ Swap2[i, j, k, l], yi_^2 xj_ xl_ :→ Swap2[i, j, i, l],
     yi_ yk_ xj_^2 :→ Swap2[i, j, k, j], yi_^2 xj_^2 :→ Swap2[i, j, i, j],
     yi_ yk_ xj_ FFl_ :→
      (Coefficient[G, υi ξj] Coefficient[G, υk] + Coefficient[G, υk ξj] Coefficient[G, υi])
       υl, yi_^2 xj_ FFl_ :→ 2 Coefficient[G, υi ξj] Coefficient[G, υi] υl,
     yi_ EEk_ xj_ xl_ :→ (Coefficient[G, υi ξj] Coefficient[G, ξl] +
         Coefficient[G, υi ξl] Coefficient[G, ξj]) ξk,
     yi_ xj_^2 EEl_ :→ 2 Coefficient[G, υi ξj] Coefficient[G, ξj] ξl,
     yi_ xj_ EEa_ FFb_ :→ Coefficient[G, υi ξj] ξa υb});

newZ4 = (Z4 /. {ξi_ :→ Coefficient[B, υi], υi_ :→ Coefficient[A, ξi], EEi_ → ξi, FFi_ → υi})
   + (SwapPoly /. {xi_ :→ Coefficient[G, ξi],
      yi_ :→ Coefficient[G, υi], EEi_ → ξi, FFi_ → υi}) // Together;

({newω, newQ, newZ0, newZ2, newZ4, Length[P]} /.
   {ξi_ :→ ξPosition[P,i][[1,1]], υi_ :→ υPosition[P,i][[1,1]]}) // Together
]
```

```
(*Bulk meta-associativity*)
({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 4}, {j, 1, 4}], 0, 0, 0, 4} // m_{{1},{2},{3,4}} // m_{{1,2},{3}}) -
  ({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 4}, {j, 1, 4}], 0, 0, 0, 4} // m_{{1,2},{3,4}}) // Together
({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 3}, {j, 1, 3}], 0, 0, 0, 3} // m_{{1,2},{3}} // m_{{1,2}}) -
  ({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 3}, {j, 1, 3}], 0, 0, 0, 3} // m_{{1,2,3}}) // Together
({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 3}, {j, 1, 3}], 0, 0, 0, 3} // m_{{1},{2,3}} // m_{{1,2}}) -
  ({1, Sum[a_{i,j} ξ_i υ_j, {i, 1, 3}, {j, 1, 3}], 0, 0, 0, 3} // m_{{1,2,3}}) // Together
```

{0, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0}

{0, 0, 0, 0, 0, 0}

```
(*Disjoint union of a list L of tangles. Each
 tangle is supposed to be labelled with 1,2,3,...
 DU renames the components 1,2,3.. in order of appearance*)
DU[L_] := Block[{OutTangle, NextTangle},
  OutTangle = L[[1]];
  Do[
   NextTangle = L[[k]] /. {ξ_i_ → ξ_{i+OutTangle[[6]]}, υ_i_ → υ_{i+OutTangle[[6]]}};
   OutTangle[[1]] *= NextTangle[[1]];
   OutTangle[[2]] += NextTangle[[2]];
   OutTangle[[3]] += NextTangle[[3]];
   OutTangle[[4]] += NextTangle[[4]];
   OutTangle[[5]] += NextTangle[[5]];
   OutTangle[[6]] += NextTangle[[6]];
   , {k, 2, Length[L]}];
  OutTangle
 ]
```

R-matrices and cuaps

(*$t^{-\frac{1}{2}}$*) $R^+ = \{1, (1-t)\,(\xi_1 - \xi_2)\,\upsilon_2, 0, 0, \left((1-t)\,\frac{\xi_1\upsilon_2}{2}\right)^2 - \left((1+t)\,\frac{\xi_2\upsilon_2}{2}\right)^2 + t\,\xi_1\,\xi_2\,\upsilon_2\,(\upsilon_1 + (1-t)\,\upsilon_2), 2\};$

(*$t^{\frac{1}{2}}$*) $R^- = \{1, (1-t^{-1})\,(\xi_1 - \xi_2)\,\upsilon_2, 0, 0, -\left((1-t^{-1})\,\frac{\xi_1\upsilon_2}{2}\right)^2 + \left((1+t^{-1})\,\frac{\xi_2\upsilon_2}{2}\right)^2 + t^{-1}\,\xi_1\,\upsilon_1\,\upsilon_2\,(-\xi_2 + (1-t)\,\xi_1), 2\};$

(*$t^{\frac{1}{2}}$*) $n = \{1, 0, 0, -\xi_1\,\upsilon_1, 0, 1\};$

(*$t^{-\frac{1}{2}}$*) $u = \{1, 0, 0, \xi_1\,\upsilon_1, 0, 1\};$

Reidemeister moves:

```
(*Rotated crossing*)
{(DU[{u, u, R⁺, n, n}] // m_{{1,3,5},{2,4,6}}) - R⁺, (DU[{u, u, R⁻, n, n}] // m_{{1,3,5},{2,4,6}}) - R⁻,
  (DU[{n, n, R⁺, u, u}] // m_{{1,3,5},{2,4,6}}) - R⁺,
  (DU[{n, n, R⁻, u, u}] // m_{{1,3,5},{2,4,6}}) - R⁻} // Expand
```

{{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

```
(*Reidemeister 1, note the t-correction is still necessary*)
{DU[{R⁺, n}] // m_{{1,3,2}}, DU[{R⁺, u}] // m_{{2,3,1}},
 DU[{R⁻, u}] // m_{{1,3,2}}, DU[{R⁻, n}] // m_{{2,3,1}}}
```

$\{\{1, 0, 0, 0, 0, 1\}, \{t, 0, 0, 0, 0, 1\}, \{1, 0, 0, 0, 0, 1\}, \{\frac{1}{t}, 0, 0, 0, 0, 1\}\}$

```
(*Reidemeister 2 (braid-like)*)
{DU[{R⁺, R⁻}] // m_{{1,3},{2,4}}, DU[{R⁻, R⁺}] // m_{{1,3},{2,4}}}
```

$\{\{1, 0, 0, 0, 0, 2\}, \{1, 0, 0, 0, 0, 2\}\}$

```
(*Reidemeister 2 (opposites)*)
{DU[{R⁻, R⁺, u, n}] // m_{{1,3},{4,5,2,6}}, DU[{R⁺, R⁻, u, n}] // m_{{5,1,6,3},{4,2}}}
```

$\{\{1, 0, 0, 0, 0, 2\}, \{1, 0, 0, 0, 0, 2\}\}$

```
(*Reidemeister 3*)
{(DU[{R⁺, R⁺, R⁺}] // m_{{1,3},{2,5},{4,6}}) - (DU[{R⁺, R⁺, R⁺}] // m_{{3,5},{1,6},{2,4}}),
  (DU[{R⁻, R⁻, R⁻}] // m_{{1,3},{2,5},{4,6}}) - (DU[{R⁻, R⁻, R⁻}] // m_{{3,5},{1,6},{2,4}})} // Expand
```

$\{\{0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0\}\}$

Some knots:

```
(*Trefoil*)
DU[{R⁺, R⁺, R⁺, n}] // m_{{1,4,5,7,2,3,6}}
```

$\left\{1 - t + t^2, 0, \frac{-t + 2t^2 - 3t^3 + 2t^4}{(1 - t + t^2)^2}, 0, 0, 1\right\}$

```
(*Figure-eight*)
DU[{R⁺, R⁺, R⁻, R⁻, n, u}] // m_{{2,3,8,9,5,4,10,1,6,7}}
```

$\left\{\frac{-1 + 3t - t^2}{t}, 0, \frac{-1 + t^2}{1 - 3t + t^2}, 0, 0, 1\right\}$