Notes on doubles

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1 Introduction

2 Classical double

For parameters β, γ consider the Lie bialgebra $\mathfrak{a}$ (the 2d affine algebra) generated by u, w subject to the relation $[u, w] = \beta w$ and with cobracket $\delta : \mathfrak{a} \to \Lambda^2\mathfrak{a}$ given by $\delta(u) = 0$ and $\delta(w) = \gamma w \wedge u$. It clearly satisfies the co-Jacobi identity and the cocycle identity is clear too: $\delta([u, w]) = \beta\gamma w \wedge u = (1 \otimes ad_u + ad_u \otimes 1 - 1 \otimes ad_w - ad_w \otimes 1)\delta(w)$.

The dual $\mathfrak{a}^*$ has dual basis n, m where m is dual to w and n is dual to u (notice the dual is the same symbol written upside down). The dual bracket and cobracket are $[\ ,\ ]_{\mathfrak{a}^*} = \delta^*$ and $\delta_{\mathfrak{a}^*} = [\ ,\ ]^*$ given by $[m, n] = \gamma m$ and $\delta(n) = 0,\ \delta(m) = \beta n \wedge m$

The classical double is the vector space $D\mathfrak{a} = \mathfrak{a} + \mathfrak{a}^{*op}$ together with the following Lie algebra structure. Here $\mathfrak{a}^{*op}$ is the dual with opposite cobracket $\delta_{\mathfrak{a}^{*op}} = -\delta_{\mathfrak{a}^*}$. The bracket on the double is defined by viewing $\mathfrak{a}$ and $\mathfrak{a}^*$ as Lie subalgebras of $D\mathfrak{a}$ and extending the pairing to a non-degenerate form given by $\langle \phi + a, \phi' + a' \rangle = \phi(a') + \phi'(a)$. Next we extend the Lie bracket to make sure that the pairing is invariant in the sense that: $\langle [\phi, a], a' \rangle = \langle \phi, [a, a'] \rangle$ and $\langle [a, \phi], \phi' \rangle = \langle a, [\phi, \phi'] \rangle$.

In our example this means

$$[n, u] = 0 \quad [m, u] = \beta m \quad [n, w] = \gamma w \quad [m, w] = -\beta n - \gamma u$$

since for example $\langle [m, u], n \rangle = -\langle u, [m, n] \rangle = 0$ and $\langle [m, u], m \rangle = -\langle u, [m, m] \rangle = 0$ and $\langle [m, u], u \rangle = \langle m, [u, u] \rangle = 0$ and $\langle [m, u], w \rangle = \langle m, [u, w] \rangle = \beta$.

The co-bracket on $D\mathfrak{a}$ is defined to be $\delta_{\mathfrak{a}} - \delta_{\mathfrak{a}^*}$

The canonical element $r = \sum_i e_i \otimes e^i \in D\mathfrak{a}$ is a quasitriangular structure: It satisfies CYBE, and $\partial r = \delta$ and $r + \sigma(r)$ is $D\mathfrak{a}$ invariant, where $\sigma(x \otimes y) = y \otimes x$. In our case we have $r = u \otimes n + w \otimes m$ and hence $\delta(u) = u.r = 0$ and $\delta(w) = w.r = -\beta w \otimes n - u \otimes \gamma w + w \otimes (\beta n + \gamma u) = \gamma w \wedge u$ and $\delta(m) = m.r = \beta m \otimes n - u \otimes \gamma m - \beta n \otimes m + \gamma u \otimes m = -\delta_{\mathfrak{a}^*}(m) = \delta_{\mathfrak{a}^{*op}}(m)$. And CYBE yields $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = [u, w] \otimes n \otimes m + [w, u] \otimes m \otimes n + u \otimes [n, w] \otimes m + w \otimes [m, u] \otimes n + w \otimes [m, w] \otimes m + u \otimes w \otimes [n, m] + w \otimes u \otimes [m, n] = 0$

Alternatively we could have derived the bialgebra structure on $D\mathfrak{a}$ from the assumptions that $D\mathfrak{a} = \mathfrak{a} \oplus \mathfrak{a}^*$ as vector space and both $\mathfrak{a}$ and $\mathfrak{a}^{*op}$ inject in the obvious way as sub Lie-bialgebras and the canonical element is a quasi-triangular structure.

3 The Hopf algebra structure on $\mathcal{U}(\mathfrak{a})$

We now consider the universal enveloping algebra $\mathcal{U}(\mathfrak{a})$ and attempt to deform it using power series in the variable h. This allows us to turn $\mathcal{U}(\mathfrak{a})[[h]]$ into a Hopf algebra that is a quantization of $\mathcal{U}(\mathfrak{a})$ in the following sense. Instead of the usual coproduct $\Delta_0(X) = X_1 + X_2$ we want a coproduct Δ that reflects the cobracket. We require $\Delta(a) - \Delta^{op}(a) = h\delta(a) \mod h^2$ and $\Delta = \Delta_0 \mod h$.

In our case we have $\delta(u) = 0$ so the simplest possible Δ sets $\Delta(u) = u_1 + u_2$. The simplest(?) choice for $\Delta(w)$ is to respect the w-grading on a that gives w degree 1 and u degree 0 so as to set $\Delta(w) = w_1 f_2 + g_1 w_2$ for $f, g \in \mathcal{U}(\mathfrak{a})[[h]]$ independent of w. Applying the counit (or $\Delta = \Delta_0 \mod h$) suggests $f, g = 1 \mod h$. Coassociativity of Δ forces $\Delta(f) = f \otimes f$ and the same for g. A short calculation then shows (Chari-Pressley p.193) that any grouplike f as above is of the form $e^{h\mu u}$ for some $\mu \in \mathbb{Q}[[h]]$. We may therefore set

$$\Delta(w) = e^{h\mu u_2} w_1 + e^{h\mu' u_1} w_2$$

To reflect the cobracket we must have $\mu' - \mu = \gamma \mod h$ and since there seems no point in adding higher order terms we settle for $\mu = \gamma$ and $\mu' = 0$. Other choices (of lowest order) are equivalent and may be obtained anyway by rescaling w by the group-like $e^{h\nu u}$. In conclusion we have

$$\Delta(u) = u_1 + u_2 \quad \Delta(w) = w_1 e^{h\gamma u_2} + w_2$$

The rest of the Hopf algebra structure easily follows, the counit is as usual $\epsilon(u) = \epsilon(w) = 0$ and the antipode is derived from the equation $M \circ (id \otimes S) \circ \Delta = 1 \circ \epsilon$, where M denotes multiplication. We must have $S(u) = -u$ and $we^{-h\gamma u} + S(w) = 0$ so $S(w) = -we^{-h\gamma u}$ and $S^{-1}(w) = -e^{-h\gamma u} w = -e^{-\beta\gamma h} we^{-h\gamma u} = -q^{-1} we^{-h\gamma u}$ where we set $q = e^{h\beta\gamma}$. Our preferred basis of monomials is $w^a u^b$.

4 Dual Hopf algebra

To get a suitable dual to $\mathcal{U}(\mathfrak{a})[[h]]$ we redefine n and m to be linear functionals on $\mathcal{U}(\mathfrak{a})[[h]]$ that are 1 on hu and hw respectively and zero on all other monomials. Then m, n (topologically) generate the quantum enveloping algebra dual that we denote $\mathcal{U}(\mathfrak{a})[[h]]^*$. To find the basis dual to the monomials $(hw)^a(hu)^b$ we argue as follows. First $\langle (hw)^a(hu)^b, n^{b'} \rangle = \langle \Delta^{(b')}((hw)^a(hu)^b), n^{\otimes b} \rangle$ is non-zero only if $a = 0$ and $b' = b$, in which case it is equal to the coefficient of $(hu)^{\otimes b}$ in $(\bigotimes_{j=1}^b hu_j)^b$. This coefficient is $b!$ so $\langle (hw)^a(hu)^b, n^{b'} \rangle = \delta_{b,b'}\delta_{0,a} b!$ Next,

$\langle (hw)^a(hu)^b, m^{a'}\rangle = \langle \Delta^{(t)}(hw)^a(hu)^b, m^{\otimes a}\rangle$ is again zero unless $a' = a, b = 0$. In that case it is equal to the coefficient of the term in $(\Delta^{(a)}(hw))^a$ that has precisely one hw in each tensor factor. If we set $q = e^{h\beta\gamma}$ then the coefficient is precisely $[a]!$ where $[k] = \frac{1-q^k}{1-q}$ (induction!).

The general case now follows, because $\langle (hw)^a(hu)^b, n^{b'}m^{a'}\rangle$ is the coefficient of $(\Delta^{(a'+b')}(hwhu))^{a+b}$ that has ends with a' factors hw and hence $a = a'$ and should begin with b' factors hu so also $b' = b$ and the coefficient is $[a]!b!$ as computed previously. In conclusion the basis dual to $\{(hw)^a(hu)^b\}$ is $\{\frac{n^b m^a}{[a]!b!}\}$

Note the commutation relation between m, n still is $[m, n] = \gamma m$. This is consistent with nm being dual to $hwhu$, while both $\langle hwhu, mn\rangle = 1$ and $\langle hw, mn\rangle = \gamma$ and zero for all other monomials, so $mn - nm - \gamma m$ pairs to 0 with any monomial in hw, hu.

The rest of the Hopf algebra structure transfers to the dual easily now. $\langle X \otimes Y, \Delta(n)\rangle = \langle XY, n\rangle$ so the monomials X, Y must be 1 and hu. In other words $\Delta(n) = n_1 + n_2$. Likewise $\langle X \otimes Y, \Delta(m)\rangle = \langle XY, m\rangle$ now to get something non-zero either X, Y are hw and 1 or $X = (hu)^k$ and $Y = hw$. In the latter case $XY = hw(hu + h\beta)^k$ and so $\langle XY, m\rangle = (h\beta)^k$. It follows that $\Delta(m) = m_1 + e^{\beta h n_1} m_2$. The antipode must be $S(n) = -n$ and $S(m) = -e^{-\beta h n} m$.

5 Quantum Double

Following Etingof-Schiffman the product in the quantum double $D\mathcal{U}(\mathfrak{a}) = \mathcal{U}(\mathfrak{a})[[h]] \otimes \mathcal{U}(\mathfrak{a})[[h]]^{*op}$ is given by:

$$\phi X = \sum X''\phi''\langle S^{-1}(X'), \phi'\rangle\langle X''', \phi'''\rangle$$

where $\phi \in \mathcal{U}(\mathfrak{a})[[h]]^*$ and $X \in \mathcal{U}(\mathfrak{a})[[h]]$ and $\Delta^{(2)}X = \sum X' \otimes X'' \otimes X'''$ and $\Delta^{op\ (2)}\phi = \sum \phi' \otimes \phi'' \otimes \phi'''$ Just as in the classical double, this product is derived/designed to make the canonical pairing be a solution to the Yang-Baxter equation. More precisely it should yield a quasi-triangular Hopf algebra.

In our case $\Delta^{op\ (2)}(n) = n_1 + n_2 + n_3$ and $S_1^{-1}\Delta^{(2)}(u) = -u_1 + u_2 + u_3$ and $S_1^{-1}\Delta^{(2)}(w) = -q^{-1}w_1 e^{h\gamma(-u_1+u_2+u_3)} + w_2 e^{h\gamma u_3} + w_3$ and $\Delta^{op\ (2)}(m) = e^{h\beta(n_2+n_3)}m_1 + e^{h\beta n_3}m_2 + m_3$. It follows that $nu = un$ and $nw = wn + \gamma w$ and $mu = um + \beta m$ just as in the classical double. Finally $mw = qwm + \frac{1}{h}(1 - e^{h\beta n}e^{h\gamma u})$. This does not agree with the classical double but it does to first order in h. By *definition* the universal R-matrix is $\sum_{a,b\geq 0} h^{a+b}w^a u^b \otimes \frac{n^b m^a}{[a]!b!}$.

Notice that the element $c = \beta n - \gamma u$ is central. Quotienting out by the two-sided ideal generated by it yields the usual quantum group $U_h\mathfrak{sl}_2$ For us it is convenient to keep u and express n in terms of it: $n = \frac{c+\gamma u}{\beta}$.

A more symmetric expression with a usual commutator is obtained by setting $M = e^{\frac{-h\gamma u}{2}}m$ and $W = we^{\frac{-h\gamma u}{2}}$ in these variables we find, with $t = e^{hc}$:

$$[M, W] = \frac{1}{h}(e^{-h\gamma u} - te^{h\gamma u})$$

$$[n,u]=0 \quad [M,n]=\gamma M \quad [M,u]=\beta M \quad [n,W]=\gamma W \quad [u,W]=\beta W$$

The capitals M,W are convenient since their antipode is very simple: $S(W)=S(e^{-\frac{h\gamma u}{2}})S(w)=e^{\frac{h\gamma u}{2}}(-we^{-h\gamma u})=-q^{\frac{1}{2}}W$. For M we have to take the opposite antipode corresponding to the opposite coproduct (written here in $D\mathcal{U}(\mathfrak{a})$ as S): $S(m)=-me^{-\beta hn}$. So $S(M)=S(m)e^{\frac{h\gamma u}{2}}=-me^{\frac{h\gamma u-2\beta hn}{2}}=-t^{-1}q^{-\frac{1}{2}}M$.

There is an interesting Lie algebra automorphism (involution) Θ on $D\mathcal{U}(\mathfrak{a})$ with $\Theta(M)=t^{-\frac{1}{2}}W$ and $\Theta(W)=t^{-\frac{1}{2}}M$ and $\Theta(n)=-n$, $\Theta(u)=-u$

In summary the Hopf algebra structure on the quantum double is given as

$$[M,W]=\frac{1}{h}(e^{-h\gamma u}-te^{h\gamma u})$$
$$[n,u]=0 \quad [M,n]=\gamma M \quad [M,u]=\beta M \quad [n,W]=\gamma W \quad [u,W]=\beta W$$
$$\Delta(W)=W_1e^{\frac{h\gamma u_2}{2}}+W_2e^{\frac{-h\gamma u_1}{2}} \quad \Delta(u)=u_1+u_2$$
$$\Delta(M)=e^{\frac{-h\gamma u_1}{2}}M_2+t_2e^{\frac{h\gamma u_2}{2}}M_1 \quad \Delta(c)=c_1+c_2$$
$$S(W)=-q^{\frac{1}{2}}W \quad S(u)=-u \quad S(c)=-c \quad S(M)=-t^{-1}q^{-\frac{1}{2}}M$$

The universal R-matrix takes the form

$$R=\sum_{a,b}\frac{h^{a+b}}{[a]_{q^{-1}}!b!\beta^b}W^ae^{\frac{ah\gamma u}{2}}u^b\otimes(c+\gamma u)^be^{\frac{ah\gamma u}{2}}M^a$$

6 The special case $\gamma^2=0$

To make use of the nilpotency of γ we set $h=1,\beta=1$ so $q=e^\gamma$ and set $t=e^c$. The commutation relation between M and W reads

$$[M,W]=e^{-\gamma u}-te^{\gamma u}=1-t-(1+t)\gamma u$$

Also $q=e^\gamma=1+\gamma$ so $q^k=1+k\gamma$ and $[k]=k(1+\gamma\frac{k-1}{2})$, $[k]!=k!(1+\frac{\gamma}{2}\binom{k}{2})$ and $[a]!^{-1}_{q^{-1}}=\frac{1}{a!}(1+\frac{\gamma}{2}\binom{a}{2})$. The R-matrix can be written

$$R=\sum_{a,b}\frac{1}{a!b!}(1+\frac{\gamma}{2}\binom{a}{2})W^a(1+\frac{a\gamma u}{2})u^b\otimes(c^b+bc^{b-1}\gamma u)(1+\frac{a\gamma u}{2})M^a$$

More concisely the γ independent part is $R=\sum_{a,b}\frac{1}{a!b!}W_1^au_1^bc_2^bM_2^a=\mathbb{O}(e^{W_1M_2+c_2u_1})$ Here the $\mathbb{O}$ denotes anti-alphabetic ordering of the variables in the power series. The whole R-matrix becomes

$$R=\mathbb{O}(e^{W_1M_2+c_2u_1}(1+\gamma(\frac{W_1^2M_2^2}{4}+\frac{W_1(u_1+u_2)M_2}{2}+u_1u_2)))$$

Applying $S\otimes id$ to R we get the inverse:

$$R^{-1}=\mathbb{O}(e^{-t_2W_1M_2-c_2u_1}(1+\gamma(-\frac{W_1^2M_2^2}{4}-\frac{W_1(-u_1+u_2)M_2}{2}-u_1u_2)))$$

6.1 Commutation relations

The commutator $K = [M, W] = 1-t-(1+t)\gamma u$ is an element that Q-commutes with both M and W as follows: $MK = QKM$ and $KW = QWK$. Here $Q = 1 + \frac{t+1}{t-1}\gamma$

lemma 1.

$$M^a W^b = \sum_j \frac{a!b!}{(a-j)!(b-j)!j!} W^{b-j}(1-t)^{j-1}(1-t-((\frac{a+b}{2}-1+u)j-\frac{3}{4}j(j-1))(t+1)\gamma)M^{a-j}$$

Proof. It follows by induction that

$$M^a W^b = \sum_j \frac{[a]_Q![b]_Q!}{[j]_Q![a-j]_Q![b-j]_Q!} W^{b-j} K^j M^{a-j}$$

More explicitly $\frac{[a]_Q![b]_Q!}{[j]_Q![a-j]_Q![b-j]_Q!} = \frac{a!b!}{(a-j)!(b-j)!j!}(1+((\frac{a+b}{2}-1)j-\frac{3}{4}j(j-1))\frac{t+1}{t-1}\gamma)$ and $K^j = (1-t)^j(1+j\frac{t+1}{t-1}\gamma u)$ so the result follows. □

Next define $\bar{\mathbb{O}}(f)$ to be the power series f written in alphabetical order. We will present a few lemmas that allow reordering of the exponentials one meets when stitching together R-matrices.

lemma 2. *If* $\nu = (1-\delta(1-t))^{-1}$ *then*

$$\bar{\mathbb{O}}(e^{\delta MW}) = \mathbb{O}\nu e^{\nu\delta WM}(1+\gamma P)$$

where

$$P = \nu^2\delta(t+1)(\nu^{-1}u + \frac{\delta}{2} + \delta W(\delta(u+2)+\nu)M + \delta^2\nu(1+\frac{\nu\delta}{4})W^2M^2))$$

Proof.

$$\sum_k \frac{\delta^k}{k!}M^kW^k = \sum_{j,k}\frac{k!}{(k-j)!^2 j!}\delta^{k-j}W^{k-j}\delta^j(1-t)^{j-1}(1-t-((\ell+u)j+\frac{1}{4}j(j-1))(t+1)\gamma)M^{k-j}$$

Set $\ell = k-j$, we get

$$\sum_\ell \frac{\delta^\ell}{\ell!}W^\ell \sum_j F(j,\ell)M^\ell$$

where

$$F(j,\ell) = \binom{\ell+j}{j}\delta^j(1-t)^{j-1}((\ell+u)j+\frac{1}{4}j(j-1))(t+1)\gamma)$$

using the binomial series and its derivatives we can carry out the sum over j explicitly: $\sum_j \binom{\ell+j}{j}x^j = \frac{1}{(1-x)^{\ell+1}}$ and $\sum_j \binom{\ell+j}{j}jx^{j-1} = \frac{\ell+1}{(1-x)^{\ell+2}}$ and $\sum_j \binom{\ell+j}{j}j(j-1)x^{j-2} = \frac{\ell(\ell-1)+4\ell+2}{(1-x)^{\ell+3}}$ Explicitly $\sum_j F(j,\ell) =$

$$\frac{1}{(1-\delta(1-t))^{\ell+1}} + \frac{\delta(t+1)\gamma}{(1-\delta(1-t))^{\ell+3}}((1-\delta(1-t))(\ell+1)(\ell+u)+\frac{\delta}{4}(\ell(\ell-1)+4\ell+2)) =$$

$$\nu^{\ell+1} + \delta(t+1)\gamma\nu^{\ell+3}(\nu^{-1}u + \frac{\delta}{2} + (\nu^{-1}(u+2)+\delta)\ell + (\nu^{-1}+\frac{\delta}{4})\ell(\ell-1))$$

setting $\nu = (1-\delta(1-t))^{-1}$

It follows that $\sum_k \frac{\delta^k}{k!} M^k W^k = \mathbb{O}\nu e^{\nu\delta WM}(1+\gamma P)$ where

$$P = \nu^2\delta(t+1)(\nu^{-1}u + \frac{\delta}{2} + \delta W(\delta(u+2)+\nu)M + \delta^2\nu(1+\frac{\nu\delta}{4})W^2M^2))$$

$\square$

lemma 3.
$$\bar{\mathbb{O}}(e^{\alpha M+\beta W}) = \mathbb{O}(e^{\alpha M+\beta W+\alpha\beta(1-t)}(1+\gamma P))$$

with

$$P = -(t+1)(\alpha\beta(\frac{\alpha+\beta}{2}+u) + \frac{\alpha^2\beta^2}{4}(1-t))))$$

More generally consider $\bar{\mathbb{O}}(e^{\alpha M+\beta W}Z(M,W))$ for some polynomial Z in W. We may rewrite it as $Z(\partial_\alpha,\partial_\beta)\bar{\mathbb{O}}(e^{\alpha M+\beta W})$ and hence as

$$Z(\partial_\alpha,\partial_\beta)\mathbb{O}(e^{\alpha M+\beta W+\alpha\beta(1-t)}(1+\gamma P))$$

Proof. The left hand side reads $\sum_{a,b}\frac{\alpha^a\beta^b}{a!b!}M^aW^b$. By the same procedure we may rewrite it as

$$\sum_{a,b,j}\frac{\alpha^{a-j}\beta^{b-j}}{(a-j)!(b-j)!j!}\alpha^j\beta^jW^{b-j}(1-t)^{j-1}(1-t-((\frac{a+b}{2}-1+u)j-\frac{3}{4}j(j-1))(t+1)\gamma)M^{a-j}$$

Setting $k=a-j$ and $\ell=b-j$ we get

$$\sum_{k,\ell,j}\frac{\alpha^k\beta^\ell}{k!\ell!}W^kF(k,\ell,j)M^\ell$$

where this time

$$F(k,\ell,j) = \frac{\alpha^j\beta^j}{j!}(1-t)^{j-1}(1-t-((\frac{k+\ell}{2}+u)j+\frac{1}{4}j(j-1))(t+1)\gamma)$$

Again we can carry out the summation over j explicitly to get

$$\sum_j F(k,\ell,j) = \mathbb{O}(e^{\alpha\beta(1-t)}(1-(t+1)\gamma(\alpha\beta(\frac{k+\ell}{2}+u)+\frac{\alpha^2\beta^2}{4}(1-t))))$$

Therefore the end result is

$$\mathbb{O}(e^{\alpha M+\beta W+\alpha\beta(1-t)}(1+\gamma P))$$

with

$$P = -(t+1)(\alpha\beta(\frac{\alpha+\beta}{2}+u)+\frac{\alpha^2\beta^2}{4}(1-t))))$$

$\square$

Finally we take care of u terms.

lemma 4.

$$\bar{\mathbb{O}}(e^{\phi u+\alpha M}Z(u,M)) = Z(\partial_\phi,\partial_\alpha)\mathbb{O}(e^{\phi u+e^\phi\alpha M})$$

$$\bar{\mathbb{O}}(e^{\phi u+\beta W}) = Z(\partial_\phi,\partial_\beta)\mathbb{O}(e^{\phi u+e^\phi\beta W})$$

Proof. We only prove the first formula, the proof of the second is the same. First it follows by induction that $M^a u^b = (u+a)^b M^a$. Therefore $\sum_{a,b}\frac{\alpha^a\phi^b}{a!b!}M^a u^b = \sum_{a,b}\frac{\alpha^a\phi^b}{a!b!}(u+a)^b M^a = \sum_a \frac{\alpha^a}{a!}e^{\phi(u+a)}M^a = \mathbb{O}(e^{\phi u+e^\phi\alpha M})$. Including a polynomial Z is straightforward as in the previous lemma. $\square$