Pensieve header: The “Speedy” engine.

In[∘]:=
```
Once[<< KnotTheory`];
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
PP_ = Identity; $k = 1; γ = 1; ℏ;
```

In[∘]:= tKink_1

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{T_1}}+\left(\frac{\hbar\, a_1}{\sqrt{T_1}}+\frac{\hbar\, a_1^2}{\sqrt{T_1}}-\frac{\hbar^3\, x_1^2\, y_1^2}{4\sqrt{T_1}}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\mathrm{QZip}_{\{x_1,\xi_1,y_1,\eta_1,x_2,\xi_2,y_2,\eta_2\}}[\mathbb{E}\ @@\ (\mathrm{kR}_{1,2}\ \mathrm{km}_{2,1\to5})]$

Out[∘]= $$\mathbb{E}\left[t\,\hbar\, a_2+a_5\,\alpha_1+a_5\,\alpha_2,\ 0,\ \frac{1}{T^2}+\frac{\hbar\, a_1\, a_2\,\epsilon}{T^2}+O[\epsilon]^2\right]$$

In[∘]:= $\mathrm{R}_{1,2}\ \mathrm{R}_{3,4}\ \mathrm{dm}_{1,3\to5}$

Out[∘]= $$\mathbb{E}_{\{1,3\}\to\{1,2,3,4,5\}}\Big[\hbar\, a_2\, b_1+\hbar\, a_4\, b_3+a_5\,\alpha_1+a_5\,\alpha_3+b_5\,\beta_1+b_5\,\beta_3,$$
$$\hbar\, x_2\, y_1+\hbar\, x_4\, y_3+y_5\,\eta_1+\frac{y_5\,\eta_3}{\mathcal{A}_1}+\frac{x_5\,\xi_1}{\mathcal{A}_3}+\frac{(1-B_5)\,\eta_3\,\xi_1}{\hbar}+x_5\,\xi_3,$$
$$1+\Big(-\frac{1}{4}\hbar^3\, x_2^2\, y_1^2-\frac{1}{4}\hbar^3\, x_4^2\, y_3^2-\frac{y_5\,\beta_1\,\eta_3}{\mathcal{A}_1}-\frac{x_5\,\beta_3\,\xi_1}{\mathcal{A}_3}+a_5\, B_5\,\eta_3\,\xi_1+\frac{\hbar\, x_5\, y_5\,\eta_3\,\xi_1}{\mathcal{A}_1\,\mathcal{A}_3}+$$
$$\frac{(1-3\,B_5)\, y_5\,\eta_3^2\,\xi_1}{2\,\mathcal{A}_1}+\frac{(1-3\,B_5)\, x_5\,\eta_3\,\xi_1^2}{2\,\mathcal{A}_3}+\frac{\left(1-4\,B_5+3\,B_5^2\right)\eta_3^2\,\xi_1^2}{4\,\hbar}\Big)\epsilon+O[\epsilon]^2\Big]$$

In[∘]:= $\mathrm{QZip}_{\{x_1,\xi_1,y_1,\eta_1,x_3,\xi_3,y_3,\eta_3\}}[\mathbb{E}\ @@\ (\mathrm{kR}_{1,2}\ \mathrm{kR}_{3,4}\ \mathrm{kR}_{5,6}\ \mathrm{km}_{1,3\to5})]$

Out[∘]= $$\mathbb{E}\left[t\,\hbar\, a_2+t\,\hbar\, a_4+t\,\hbar\, a_6+a_5\,\alpha_1+a_5\,\alpha_3,\ \hbar\, x_6\, y_5,\ 1+\left(\hbar\, a_1\, a_2+\hbar\, a_3\, a_4+\hbar\, a_5\, a_6-\frac{1}{4}\hbar^3\, x_6^2\, y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

The “Speedy” Engine

Internal Utilities

Canonical Form:

```
In[∘]:= CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
          Expand[ℰ] //. 𝕖^x_ 𝕖^y_ :→ 𝕖^(x+y) /. 𝕖^x_ :→ 𝕖^CCF[x]];
        CF[ℰ_List] := CF /@ ℰ;
        CF[sd_SeriesData] := MapAt[CF, sd, 3];
        CF[ℰ_] := Module[
          {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃ {y, b, t, a, x, η, β, τ, α, ξ}},
          Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :→ CCF[c] × (Times @@ vs^ps)]
         ];
        CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
In[∘]:= Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
In[∘]:= 𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
          CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
        𝔼 /: 𝔼[L1_, Q1_, P1_] × 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
        𝔼[L_, Q_, P_]_$k_ := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
In[∘]:= {t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
        {τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z}; (u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
In[∘]:= U2l = {B_i_^p_. :→ 𝕖^(-p ℏ γ b_i), B^p_. :→ 𝕖^(-p ℏ γ b), T_i_^p_. :→ 𝕖^(-p ℏ t_i), T^p_. :→ 𝕖^(-p ℏ t), 𝒜_i_^p_. :→ 𝕖^(p γ α_i), 𝒜^p_. :→ 𝕖^(p γ α)};
        l2U = {𝕖^(c_. b_i_ + d_.) :→ B_i^(-c/(ℏ γ)) 𝕖^d, 𝕖^(c_. b + d_.) :→ B^(-c/(ℏ γ)) 𝕖^d,
          𝕖^(c_. t_i_ + d_.) :→ T_i^(-c/ℏ) 𝕖^d, 𝕖^(c_. t + d_.) :→ T^(-c/ℏ) 𝕖^d,
          𝕖^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) 𝕖^d, 𝕖^(c_. α + d_.) :→ 𝒜^(c/γ) 𝕖^d,
          𝕖^ℰ_ :→ 𝕖^Expand@ℰ};
```

Derivatives in the presence of exponentiated variables:

```
In[∘]:= D_b[f_] := ∂_b f - ℏ γ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ℏ γ B_i ∂_B_i f;
        D_t[f_] := ∂_t f - ℏ T ∂_T f; D_t_i_[f_] := ∂_t_i f - ℏ T_i ∂_T_i f;
        D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + γ 𝒜_i ∂_𝒜_i f;
        D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
        D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

```
In[∘]:=
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip{}[P_] := P;
Zipζs_[Ps_List] := Zipζs /@ Ps;
Zip{ζ_,ζs___}[P_] :=
 (collect[P // Zip{ζs}, ζ] /. f_. ζ^d_. :> (D{ζ*,d}[f])) /. ζ* → 0 /.
  ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)\mathrm{e}^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}| \left\langle \left. P(z_i,\zeta^j)\mathrm{e}^{c+\eta^i z_i}\right|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$
$$= |\tilde{q}|\mathrm{e}^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

```
In[∘]:=
QZipζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδz,ζ* - ∂z,ζ Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zipζs[P /. (zrule ⋃ ζrule)]] ];
```

LZip implements the “L-level zips” on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single”P”. Here the z’s are b and α and the ζ’s are β and a.

In[ ]:=
```
LZip_{ξs_List}@𝔼[L_, Q_, P_] :=
  Module[{ξ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ξrule, Q1, EEQ, EQ},
   zs = Table[ξ*, {ξ, ξs}];
   Zs = zs /. {b → B, t → T, α → 𝒜};
   c = L /. Alternatives @@ (ξs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ξ (L /. Alternatives @@ zs → 0), {ξ, ξs}];
   ηs = Table[∂_z (L /. Alternatives @@ ξs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ξ*} - ∂_{z,ξ} L, {ξ, ξs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule = Join[zrule,
     zrule /. r_Rule :> ((U = r⟦1⟧ /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
   ξrule = Thread[ξs → ξs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ξrule);
   EEQ[ps___] := EEQ[ps] =
     (CF[e^{-Q1} D_{Thread[{zs,{ps}}]}[e^{Q1}]] /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
    Det[lt] (Zip_{ξs}[(EQ @@ zs) (P /. (Zrule ⋃ ξrule))] /.
        Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ]];
```

In[ ]:=
```
B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)_i → v_{n@i}, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)_i → v_{n@i}, {i, {is}}]
     ] // LZip_{Join@@Table[{β_{n@i},τ_{n@i},a_{n@i}},{i,{is}}]} // QZip_{Join@@Table[{ξ_{n@i},y_{n@i}},{i,{is}}]} ];
B_{is___}[L_, R_] := B_{{is}}[L, R];
```

𝔼 morphisms with domain and range.

In[ ]:=
```
B_{is_List}[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
  𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_{is}[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
  B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
  𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] × 𝔼[L2, Q2, P2]);
𝔼_{dr_}[L_, Q_, P_]_{$k_} := 𝔼_{dr} @@ 𝔼[L, Q, P]_{$k};
𝔼_[ℰ___][i_] := {ℰ}⟦i⟧;
```

𝔼[Λ]

In[∘]:=
```
𝔼dr_[Λ_] := CF@
  Module[{L, Λ0 = Limit[Λ, ϵ → 0]}, 𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, ⅇ^(Λ-Λ0)]$k /. l2U]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of \$k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

In[∘]:=
```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Objects

Symmetric Algebra Objects

In[∘]:=
```
sm_{i_,j_→k_} := 𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} := 𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
```

In[∘]:=
```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sϒ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Booting Up QU

In[∘]:=
```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

In[∘]:=
```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
  bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$. As a map $P:\mathbb{A}\otimes\mathbb{B}\to Q$.

$\overline{\text{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

In[∘]:=
```
Define[R_{i,j} = 𝔼_{{}→{i,j}}[ℏ a_j b_i + Σ_{k=1}^{$k+1} (1 - e^{γ ϵ ℏ})^k (ℏ y_i x_j)^k / (k (1 - e^{k γ ϵ ℏ}))],
  R̄_{i,j} = CF@𝔼_{{}→{i,j}}[-ℏ a_j b_i, -ℏ x_j y_i / B_i, 1 + If[$k == 0, 0, (R̄_{{i,j},$k-1})_{$k}[3] -
      (((R̄_{{i,j},0})_{$k} R_{1,2} (R̄_{{3,4},$k-1})_{$k}) // (bm_{i,1→i} am_{j,2→j}) // (bm_{i,3→i} am_{j,4→j}))[3]]],
  P_{i,j} = 𝔼_{{i,j}→{}}[β_j α_i / ℏ, η_j ξ_i / ℏ, 1 + If[$k == 0, 0, (P_{{i,j},$k-1})_{$k}[3] -
      (R_{1,2} // ((P_{{i,1},0})_{$k} (P_{{2,j},$k-1})_{$k}))[3]]]]
```

In[∘]:= `R_{1,2} // P_{2,3}`

Out[∘]= $\mathbb{E}_{\{3\}\to\{1\}}\left[b_1\,\beta_3,\ y_1\,\eta_3,\ 1+O[\epsilon]^3\right]$

In[∘]:= `(R_{1,2} // ((P_{{i,1},0})_2 (P_{{2,j},1})_2))[3]`

Out[∘]= $1+\left(-\frac{1}{8}\eta_j^2\,\xi_i^2-\frac{\eta_j^3\,\xi_i^3}{4\,\hbar}-\frac{\eta_j^4\,\xi_i^4}{16\,\hbar^2}\right)\epsilon^2+O[\epsilon]^3$

In[∘]:=
```
Define[aS_i = (aσ_{i→2} R̄_{1,i}) // P_{2,1},
  aS̄_i = 𝔼_{{i}→{i}}[-a_i α_i, -x_i 𝒜_i ξ_i, 1 + If[$k == 0, 0, (aS̄_{{i},$k-1})_{$k}[3] -
      ((aS̄_{{i},0})_{$k} // aS_i // (aS̄_{{i},$k-1})_{$k})[3]]]]
```

In[∘]:=
```
Define[bS_i = bσ_{i→1} R_{i,2} // aS_2 // P_{2,1},
  bS̄_i = bσ_{i→1} R_{i,2} // aS̄_2 // P_{2,1},
  aΔ_{i→j,k} = (R_{1,j} R_{2,k}) // bm_{1,2→3} // P_{i,3},
  bΔ_{i→j,k} = (R_{j,1} R_{k,2}) // am_{1,2→3} // P_{3,i}]
```

In[∘]:=
```
Define[
  dm_{i,j→k} = ((sΥ_{i→4,4,1,1} // aΔ_{1→1,2} // aΔ_{2→2,3} // aS̄_3) (sΥ_{j→-1,-1,-4,-4} // bΔ_{-1→-1,-2} // bΔ_{-2→-2,-3})) //
    (P_{1,-3} P_{3,-1} am_{2,-4→k} bm_{4,-2→k})]
```

NB. We use the co-algebra structure B tensor A^{cop}. This has the benefit of making our algebra quasi-triangular in the traditional sense of the word.

Watch out: $\Delta_{i\to j,k}$ means j is to the RIGHT of strand k and dS looks like an S.

In[∘]:=
```
Define[dσ_{i→j} = aσ_{i→j} bσ_{i→j},
  dϵ_i = sϵ_i, dη_i = sη_i,
  dS_i = sΥ_{i→1,1,2,2} // (bS_1 \overline{aS}_2) // dm_{2,1→i},
  \overline{dS}_i = sΥ_{i→1,1,2,2} // (\overline{bS}_1 aS_2) // dm_{2,1→i},
  dΔ_{i→j,k} = (bΔ_{i→1,3} aΔ_{i→4,2}) // (dm_{3,4→k} dm_{1,2→j})]
```

In[∘]:=
```
Define[C_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/2} e^{-ℏ ϵ a_i/2}]_{$k},
  \overline{C}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/2} e^{ℏ ϵ a_i/2}]_{$k},
  c_i = 𝔼_{{}→{i}}[0, 0, B_i^{1/4} e^{-ℏ ϵ a_i/4}]_{$k},
  \overline{c}_i = 𝔼_{{}→{i}}[0, 0, B_i^{-1/4} e^{ℏ ϵ a_i/4}]_{$k},
  Kink_i = (R_{1,3} \overline{C}_2) // dm_{1,2→1} // dm_{1,3→i},
  \overline{Kink}_i = (\overline{R}_{1,3} C_2) // dm_{1,2→1} // dm_{1,3→i},
  ρ_i = (c_1 \overline{c}_3 dS_i) // dm_{1,i→i} // dm_{i,3→i}] (*ρ reverses a strand*)
```

Note. $t = -\epsilon \mathrm{a} + \gamma \mathrm{b}$ and $b = t/\gamma + \epsilon \mathrm{a}/\gamma$

In[∘]:=
```
Define[b2t_i = 𝔼_{{i}→{i}}[α_i a_i + β_i (ϵ a_i + t_i) / γ + ξ_i x_i + η_i y_i],
  t2b_i = 𝔼_{{i}→{i}}[α_i a_i + τ_i (-ϵ a_i + γ b_i) + ξ_i x_i + η_i y_i]]
```

In[∘]:=
```
𝔼_{{}→{1}}[0, 0, x_1] // dΔ_{1→1,2}
𝔼_{{}→{1}}[0, 0, x_1] // dS_1
𝔼_{{}→{1}}[0, 0, y_1] // dS_1
𝔼_{{}→{1}}[0, 0, x_1] // \overline{dS}_1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\,0,\,(x_1+x_2)-\hbar\,a_2\,x_1\,\epsilon+\frac{1}{2}\,\hbar^2\,a_2^2\,x_1\,\epsilon^2+O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1+(\hbar\,x_1-\hbar\,a_1\,x_1)\,\epsilon+\left(-\frac{1}{2}\,\hbar^2\,x_1+\hbar^2\,a_1\,x_1-\frac{1}{2}\,\hbar^2\,a_1^2\,x_1\right)\epsilon^2+O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-\frac{y_1}{B_1}+O[\epsilon]^3\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1-\hbar\,a_1\,x_1\,\epsilon-\frac{1}{2}\left(\hbar^2\,a_1^2\,x_1\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:=
```
𝔼_{{}→{1}}[0, 0, (1 + ϵ a_1 ℏ) x_1] // dS_1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1+\left(\frac{\hbar^2\,x_1}{2}-\hbar^2\,a_1\,x_1+\frac{1}{2}\,\hbar^2\,a_1^2\,x_1\right)\epsilon^2+O[\epsilon]^3\right]$

In[∘]:=
```
((-1 + ℏ) x_1 + (1 - ℏ) a_1 x_1) // Expand
```

Out[∘]= $-x_1+\hbar\,x_1+a_1\,x_1-\hbar\,a_1\,x_1$

In[∘]:= $\mathbf{t2b_1\ t2b_2\ //\ P_{2,1}}$

Out[∘]= $\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2\,\tau_1}{\hbar},\ \frac{\eta_1\,\xi_2}{\hbar},\ 1+\left(\frac{\eta_1^2\,\xi_2^2}{4\,\hbar}-\frac{\tau_1\,\tau_2}{\hbar}\right)\in+O[\in]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{y_1}]\ //\ \mathbf{b\Delta_{1\to1,2}}$
$\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{y_1}]\ //\ \mathbf{d\Delta_{1\to1,2}}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (B_2\,y_1+y_2)+O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (B_2\,y_1+y_2)+O[\in]^2\right]$

In[∘]:= $(\mathbf{R_{1,2}\ //\ bS_1})\equiv\mathbf{\overline{R}_{1,2}}$
$(\mathbf{R_{1,2}\ //\ aS_2})\equiv\mathbf{\overline{R}_{1,2}}$

Out[∘]= True

Out[∘]= True

$\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{x_1}]\ //\ \mathbf{d\Delta_{1\to1,2}}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[0,\ 0,\ (x_1+x_2)-\hbar\,a_2\,x_1\in+O[\in]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{x_1}]\ //\ \mathbf{a\Delta_{1\to1,2}}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (x_1+x_2)-\hbar\,a_1\,x_2\in+O[\in]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{x_1}]\ //\ (\overline{\mathbf{aS}})_{\mathbf{1}}$
$\mathbb{E}_{\{\}\to\{1\}}[\mathbf{0},\mathbf{0},\mathbf{x_1}]\ //\ \mathbf{aS_1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1+(\hbar\,x_1-\hbar\,a_1\,x_1)\in+O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1-\hbar\,a_1\,x_1\in+O[\in]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{1,2\}}[\mathbf{0},\mathbf{0},b_1\,\mathbf{y_2}]\ //\ \mathrm{bm}_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ b_1\,y_1-y_1\in+O[\in]^2\right]$

In[∘]:= $\mathbf{a\Delta_{i\to1,2}\ //\ aS_1\ //\ am_{1,2\to1}}$
$\mathbf{a\Delta_{i\to1,2}\ //\ aS_2\ //\ am_{1,2\to1}}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1+O[\in]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1+O[\in]^2\right]$

In[∘]:= $\mathbf{a\Delta_{1\to1,2}}$

Out[∘]= $\mathbb{E}_{\{1\}\to\{1,2\}}\left[a_1\,\alpha_1+a_2\,\alpha_1,\ x_1\,\xi_1+x_2\,\xi_1,\ 1+\left(-\hbar\,a_1\,x_2\,\xi_1+\frac{1}{2}\,\hbar\,x_1\,x_2\,\xi_1^2\right)\in+O[\in]^2\right]$

Testing

co-associativity

In[∘]:= $(d\Delta_{1\to1,2}\ //\ d\Delta_{2\to2,3}) \equiv (d\Delta_{1\to2,3}\ //\ d\Delta_{2\to1,2})$

Out[∘]= True

algebra morphism

In[∘]:= $(d\Delta_{i\to1,2}\ d\Delta_{j\to3,4}\ //\ dm_{1,3\to i}\ //\ dm_{2,4\to j}) \equiv (dm_{i,j\to k}\ //\ d\Delta_{k\to i,j})$

Out[∘]= True

associativity

In[∘]:= $(dm_{1,2\to k}\ //\ dm_{k,3\to k}) \equiv (dm_{2,3\to k}\ //\ dm_{1,k\to k})$

Out[∘]= True

antipode

In[∘]:= $d\Delta_{i\to1,2}\ //\ dS_1\ //\ dm_{1,2\to1}$
$d\Delta_{i\to1,2}\ //\ dS_2\ //\ dm_{1,2\to1}$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{1\}}[0, 0, 1 + O[\epsilon]^2]$

quasi-triangular axioms

In[∘]:= $(R_{1,3}\ //\ d\Delta_{1\to1,2}) \equiv (R_{1,3}\ R_{2,4}\ //\ dm_{3,4\to3})$
$(R_{1,3}\ //\ d\Delta_{3\to2,3}) \equiv (R_{1,3}\ R_{0,2}\ //\ dm_{1,0\to1})$
$(d\Delta_{i\to k,j}\ R_{1,2}\ //\ dm_{j,1\to1}\ //\ dm_{k,2\to2}) \equiv (R_{1,2}\ d\Delta_{i\to j,k}\ //\ dm_{1,j\to1}\ //\ dm_{2,k\to2})$

Out[∘]= True

Out[∘]= True

Out[∘]= True

In[∘]:= $(R_{1,2}\ //\ aS_2) \equiv (\overline{R}_{1,2})$

Out[∘]= True

In[∘]:= $(R_{1,2}\ //\ dS_1) \equiv (\overline{R}_{1,2})$
$(R_{1,2}\ //\ \overline{dS}_2) \equiv (\overline{R}_{1,2})$

Out[∘]= True

Out[∘]= True

In[∘]:= $QQ_{s\_,r\_} := R_{11,22}\ R_{33,44}\ //\ dm_{11,44\to s}\ //\ dm_{22,33\to r}$
$\overline{QQ}_{s\_,r\_} := \overline{R}_{22,11}\ \overline{R}_{44,33}\ //\ dm_{11,44\to s}\ //\ dm_{22,33\to r}$

In[∘]:= $QQ_{1,2}\ \overline{QQ}_{3,4}\ //\ dm_{1,3\to1}\ //\ dm_{2,4\to2}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}[0, 0, 1 + O[\epsilon]^2]$

Drinfeld element u

In[∘]:= $u_{i\_} := R_{11,22} \,//\, dS_{22} \,//\, dm_{22,11\to i}$

$\overline{u}_{i\_} := \overline{R}_{11,22} \,//\, \overline{dS}_{22} \,//\, dm_{22,11\to i}$

$\overline{uu}_{i\_} := \overline{R}_{11,22} \,//\, \overline{dS}_{22} \,//\, dm_{11,22\to i}$

$\overline{u2}_{i\_} := \overline{R}_{11,22} \,//\, dS_{11} \,//\, dm_{11,22\to i}$

$\overline{u3}_{i\_} := R_{11,22} \,//\, dS_{11} \,//\, dS_{11} \,//\, dm_{22,11\to i}$

In[∘]:= $u_i\,\overline{u}_j \,//\, dm_{i,j\to i}$

$u_i\,\overline{uu}_j \,//\, dm_{i,j\to i}$

$u_i\,\overline{u2}_j \,//\, dm_{i,j\to i}$

$u_i\,\overline{u3}_j \,//\, dm_{i,j\to i}$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i-\hbar\,a_i\,B_i\,\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,1+O[\epsilon]^2\right]$

In[∘]:= $(u_1 \,//\, dS_1)$

$R_{11,22} \,//\, dS_{22} \,//\, dm_{11,22\to i}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\,a_1\,b_1,\,-\frac{\hbar\,x_1\,y_1}{B_1},\,1+\left(\frac{\hbar^2\,x_1\,y_1}{B_1}-\frac{\hbar^2\,a_1\,x_1\,y_1}{B_1}-\frac{3\,\hbar^3\,x_1^2\,y_1^2}{4\,B_1^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_1\,y_1}{2\,B_1}+\frac{\hbar^3\,a_1\,x_1\,y_1}{B_1}-\frac{\hbar^3\,a_1^2\,x_1\,y_1}{2\,B_1}+\frac{5\,\hbar^4\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{5\,\hbar^4\,a_1\,x_1^2\,y_1^2}{2\,B_1^2}+\frac{\hbar^4\,a_1^2\,x_1^2\,y_1^2}{2\,B_1^2}-\frac{67\,\hbar^5\,x_1^3\,y_1^3}{36\,B_1^3}+\frac{3\,\hbar^5\,a_1\,x_1^3\,y_1^3}{4\,B_1^3}+\frac{9\,\hbar^6\,x_1^4\,y_1^4}{32\,B_1^4}\right)\epsilon^2+O[\epsilon]^3\Big]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\Big[-\hbar\,a_i\,b_i,\,-\frac{\hbar\,x_i\,y_i}{B_i},\,1+\left(\frac{\hbar^2\,x_i\,y_i}{B_i}-\frac{\hbar^2\,a_i\,x_i\,y_i}{B_i}-\frac{3\,\hbar^3\,x_i^2\,y_i^2}{4\,B_i^2}\right)\epsilon+\left(-\frac{\hbar^3\,x_i\,y_i}{2\,B_i}+\frac{\hbar^3\,a_i\,x_i\,y_i}{B_i}-\frac{\hbar^3\,a_i^2\,x_i\,y_i}{2\,B_i}+\frac{5\,\hbar^4\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{5\,\hbar^4\,a_i\,x_i^2\,y_i^2}{2\,B_i^2}+\frac{\hbar^4\,a_i^2\,x_i^2\,y_i^2}{2\,B_i^2}-\frac{67\,\hbar^5\,x_i^3\,y_i^3}{36\,B_i^3}+\frac{3\,\hbar^5\,a_i\,x_i^3\,y_i^3}{4\,B_i^3}+\frac{9\,\hbar^6\,x_i^4\,y_i^4}{32\,B_i^4}\right)\epsilon^2+O[\epsilon]^3\Big]$$

In[∘]:= $$\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,B_1^{-1}\left(1+\epsilon\,a_1\,\hbar+\frac{\epsilon^2}{2}\,a_1^2\,\hbar^2\right)\right]u_2 \,//\, dm_{1,2\to1}\right)\equiv(u_1 \,//\, dS_1)$$

Out[∘]= True

In[∘]:= **u_1**

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-\hbar\, a_1 b_1,\ -\frac{\hbar\, x_1 y_1}{B_1},\ B_1+\left(-\hbar\, a_1 B_1-\hbar^2 x_1 y_1-\hbar^2 a_1 x_1 y_1-\frac{3\hbar^3 x_1^2 y_1^2}{4 B_1}\right)\in+\left(\frac{1}{2}\hbar^2 a_1^2 B_1-\frac{1}{2}\hbar^3 x_1 y_1+\frac{1}{2}\hbar^3 a_1^2 x_1 y_1-\frac{\hbar^4 x_1^2 y_1^2}{2 B_1}+\frac{\hbar^4 a_1 x_1^2 y_1^2}{4 B_1}+\frac{\hbar^4 a_1^2 x_1^2 y_1^2}{2 B_1}-\frac{13\hbar^5 x_1^3 y_1^3}{36 B_1^2}+\frac{3\hbar^5 a_1 x_1^3 y_1^3}{4 B_1^2}+\frac{9\hbar^6 x_1^4 y_1^4}{32 B_1^3}\right)\in^2+O[\in]^3\Big]$$

q

In[∘]:= **(u_1 // dS_1) $\overline{u3}_2$ // $dm_{1,2\to1}$**

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[0,\ 0,\ \frac{1}{B_1}+\frac{\hbar\, a_1 \in}{B_1}+O[\in]^2\Big]$$

In[∘]:= **(u_1 // $d\Delta_{1\to2,1}$) ≡ ($\overline{QQ}_{1,2}$ u_3 u_4 // $dm_{1,3\to1}$ // $dm_{2,4\to2}$)**

Out[∘]= True

In[∘]:= **$\mathbb{E}_{\{\}\to\{i\}}$[0, 0, x_i] // dS_i**

Out[∘]= $$\mathbb{E}_{\{\}\to\{i\}}\Big[0,\ 0,\ -x_i+(\hbar\, x_i-\hbar\, a_i x_i)\in+O[\in]^2\Big]$$

In[∘]:= **$Kink_1$**

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[\hbar\, a_1 b_1,\ \hbar\, x_1 y_1,\ \frac{1}{\sqrt{B_1}}+\left(\frac{\hbar\, a_1}{2\sqrt{B_1}}-\frac{\hbar^3 x_1^2 y_1^2}{4\sqrt{B_1}}\right)\in+O[\in]^2\Big]$$

In[∘]:= **(u_1 // dS_1) u_2 // $dm_{1,2\to1}$**
(u_1 // dS_1) u_2 // $dm_{2,1\to1}$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1\left(-2\hbar^2-\hbar^2 B_1\right) x_1 y_1}{B_1}+\frac{\left(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2\right) x_1^2 y_1^2}{4 B_1^3}\right)\in+O[\in]^2\Big]$$

Out[∘]= $$\mathbb{E}_{\{\}\to\{1\}}\Big[-2\hbar\, a_1 b_1,\ \frac{(-\hbar-\hbar B_1)\, x_1 y_1}{B_1^2},\ B_1+\left(-\hbar\, a_1 B_1+\frac{a_1\left(-2\hbar^2-\hbar^2 B_1\right) x_1 y_1}{B_1}+\frac{\left(-7\hbar^3-8\hbar^3 B_1-3\hbar^3 B_1^2\right) x_1^2 y_1^2}{4 B_1^3}\right)\in+O[\in]^2\Big]$$

In[∘]:=
```
(u1 // dS1)
u2
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ 1+\left(\frac{\hbar^2\, x_1\, y_1}{B_1}-\frac{\hbar^2\, a_1\, x_1\, y_1}{B_1}-\frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1^2}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[-\hbar\, a_2\, b_2,\ -\frac{\hbar\, x_2\, y_2}{B_2},\ B_2+\left(-\hbar\, a_2\, B_2-\hbar^2\, x_2\, y_2-\hbar^2\, a_2\, x_2\, y_2-\frac{3\,\hbar^3\, x_2^2\, y_2^2}{4\, B_2}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:=
```
R1,2
R̄1,2
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\, a_2\, b_1,\ \hbar\, x_2\, y_1,\ 1-\frac{1}{4}\left(\hbar^3\, x_2^2\, y_1^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2\}}\left[-\hbar\, a_2\, b_1,\ -\frac{\hbar\, x_2\, y_1}{B_1},\ 1+\left(-\frac{\hbar^2\, a_2\, x_2\, y_1}{B_1}-\frac{3\,\hbar^3\, x_2^2\, y_1^2}{4\, B_1^2}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:=
```
C1
```

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ \sqrt{B_1}-\frac{1}{2}\left(\hbar\, a_1\,\sqrt{B_1}\right)\epsilon+O[\epsilon]^2\right]$

The Knot Tensors

In[∘]:=
```
Define[kR_{i,j} = R_{i,j} // (b2t_i b2t_j) /. t_{i|j} → t,
 kR̄_{i,j} = R̄_{i,j} // (b2t_i b2t_j) /. {t_{i|j} → t, T_{i|j} → T},
 km_{i,j→k} = ((t2b_i t2b_j) // dm_{i,j→k} // b2t_k) /. {t_k → t, T_k → T, τ_{i|j} → 0},
 kC_i = (C_i // b2t_i) /. T_i → T,
 kC̄_i = (C̄_i // b2t_i) /. T_i → T,
 kKink_i = Kink_i // b2t_i /. {t_i → t, T_i → T},
 kKink̄_i = Kink̄_i // b2t_i /. {t_i → t, T_i → T}]
```

In[∘]:=
```
Define[BS_{i,j→k} =
   C_3 C_4 dΔ_{i→l1,r1} dΔ_{j→l2,r2} // dS̄_{r1} // dS_{r2} // dm_{l1,3→k} // dm_{k,r2→k} // dm_{k,r1→k} // dm_{k,4→k} // dm_{k,l2→k}]
Define[tBS_{i,j→k} = (t2b_i t2b_j) // C_3 C_4 dΔ_{i→l1,r1} dΔ_{j→l2,r2} // dS̄_{r1} // dS_{r2} // dm_{l1,3→k} // dm_{k,r2→k} //
      dm_{k,r1→k} // dm_{k,4→k} // dm_{k,l2→k} // b2t_k]
Define[tm_{i,j→k} = t2b_i // t2b_j // dm_{i,j→k} // b2t_k]
Define[tΔ_{i→j,k} = t2b_i // dΔ_{i→j,k} // b2t_j // b2t_k]
Define[tS_i = t2b_i // dS_i // b2t_i]
Define[tS̄_i = t2b_i // dS̄_i // b2t_i]
Define[tR_{i,j} = R_{i,j} // b2t_i // b2t_j, tR̄_{i,j} = R̄_{i,j} // b2t_i // b2t_j]
Define[tC_i = C_i // b2t_i , tC̄_i = C̄_i // b2t_i ]
Define[tKink_i = Kink_i // b2t_i , tKink̄_i = Kink̄_i // b2t_i]
```

In[∘]:= $R_{1,3}\, R_{2,6}$ // $dm_{3,6\to 3}$
$R_{1,3}$ // $d\Delta_{1\to 2,1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\hbar^3 B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\hbar^3 x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\ \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\ 1 + \left(-\frac{1}{4}\hbar^3 B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\hbar^3 x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:= $tR_{1,2}$ // $\overline{tS}_1$ // $\overline{tS}_1$ // $tm_{1,2\to 1}$
$(tR_{1,2}$ // $\overline{tS}_1$ // $\overline{tS}_1$ // $tm_{1,2\to 1}$ // $tS_1)$ $\mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2\,(1 - 2\,\epsilon\,\hbar\, a_1)]$ // $tm_{1,2\to 1}$
$(tR_{1,2}$ // $\overline{tS}_1$ // $\overline{tS}_1$ // $tm_{2,1\to 1})$ $\mathbb{E}_{\{\}\to\{2\}}[0, 0, T_2\,(1 - 2\,\epsilon\,\hbar\, a_1)]$ // $tm_{1,2\to 1}$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 + \hbar^2\, x_1\, y_1 - \frac{1}{4}\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 - \hbar^2\, x_1\, y_1 - \frac{1}{4}\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ 1 + \left(\hbar\, a_1^2 - \hbar^2\, x_1\, y_1 - \frac{1}{4}\hbar^3\, x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, x_2]$ // dS_2

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -x_2 - \hbar\, a_2\, x_2\, \epsilon + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2]$ // $\overline{dS}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, -\frac{y_2}{B_2} + O[\epsilon]^2\right]$

In[∘]:= $\mathbb{E}_{\{\}\to\{2\}}[0, 0, y_2]$ // $\overline{dS}_2$ // $\overline{dS}_2$

Out[∘]= $\mathbb{E}_{\{\}\to\{2\}}\left[0, 0, y_2 + \hbar\, y_2\, \epsilon + O[\epsilon]^2\right]$

In[∘]:= $tm_{i,j\to k}$
$tR_{i,j}$
$\overline{tR}_{i,j}$
tC_i
$\overline{tC}_i$
$tKink_i$
$\overline{tKink}_i$
$t\Delta_{i\to j,k}$
tS_i

Out[∘]= $\mathbb{E}_{\{i,j\}\to\{k\}}\left[a_k\,\alpha_i + a_k\,\alpha_j + t_k\,\tau_i + t_k\,\tau_j,\ y_k\,\eta_i + \frac{y_k\,\eta_j}{\mathcal{A}_i} + \frac{x_k\,\xi_i}{\mathcal{A}_j} + \frac{(1 - T_k)\,\eta_j\,\xi_i}{\hbar} + x_k\,\xi_j,\right.$
$\left.1 + \left(2\,a_k\,T_k\,\eta_j\,\xi_i + \frac{\hbar\, x_k\, y_k\,\eta_j\,\xi_i}{\mathcal{A}_i\,\mathcal{A}_j} + \frac{(1 - 3\,T_k)\,y_k\,\eta_j^2\,\xi_i}{2\,\mathcal{A}_i} + \frac{(1 - 3\,T_k)\,x_k\,\eta_j\,\xi_i^2}{2\,\mathcal{A}_j} + \frac{(1 - 4\,T_k + 3\,T_k^2)\,\eta_j^2\,\xi_i^2}{4\,\hbar}\right)\epsilon +\right.$
$\left.O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, t_i,\ \hbar\, x_j\, y_i,\ 1+\left(\hbar\, a_i\, a_j-\frac{1}{4}\hbar^3 x_j^2 y_i^2\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\, a_j\, t_i,\ -\frac{\hbar\, x_j\, y_i}{T_i},\ 1+\left(-\hbar\, a_i\, a_j-\frac{\hbar^2 a_i\, x_j\, y_i}{T_i}-\frac{\hbar^2 a_j\, x_j\, y_i}{T_i}-\frac{3\hbar^3 x_j^2 y_i^2}{4 T_i^2}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{T_i}-\hbar\, a_i\sqrt{T_i}\ \epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{T_i}}+\frac{\hbar\, a_i\,\epsilon}{\sqrt{T_i}}+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, t_i,\ \hbar\, x_i\, y_i,\ \frac{1}{\sqrt{T_i}}+\left(\frac{\hbar\, a_i}{\sqrt{T_i}}+\frac{\hbar\, a_i^2}{\sqrt{T_i}}-\frac{\hbar^3 x_i^2 y_i^2}{4\sqrt{T_i}}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\, a_i\, t_i,\ -\frac{\hbar\, x_i\, y_i}{T_i},\ \sqrt{T_i}+\left(-\hbar\, a_i\sqrt{T_i}-\hbar\, a_i^2\sqrt{T_i}-\frac{2\hbar^2 a_i\, x_i\, y_i}{\sqrt{T_i}}-\frac{3\hbar^3 x_i^2 y_i^2}{4 T_i^{3/2}}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{j,k\}}\Big[a_j\,\alpha_i+a_k\,\alpha_i+t_j\,\tau_i+t_k\,\tau_i,\ y_j\,\eta_i+T_j\, y_k\,\eta_i+x_j\,\xi_i+x_k\,\xi_i,$
$1+\left(-\hbar\, a_j\, T_j\, y_k\,\eta_i+\frac{1}{2}\hbar\, T_j\, y_j\, y_k\,\eta_i^2-\hbar\, a_j\, x_k\,\xi_i+\frac{1}{2}\hbar\, x_j\, x_k\,\xi_i^2\right)\epsilon+O[\epsilon]^2\Big]$

Out[∘]= $\mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i-t_i\,\tau_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{T_i}-x_i\,\mathcal{A}_i\,\xi_i+\frac{(\mathcal{A}_i-T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\, T_i},$
$1+\Big(\frac{\hbar\, y_i\,\mathcal{A}_i\,\eta_i}{T_i}-\frac{\hbar\, a_i\, y_i\,\mathcal{A}_i\,\eta_i}{T_i}-\frac{\hbar\, y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2T_i^2}-\hbar\, a_i\, x_i\,\mathcal{A}_i\,\xi_i+$
$\frac{2a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{T_i}-\frac{\hbar\, x_i\, y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{T_i}+\frac{(-\mathcal{A}_i+T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{T_i}+\frac{y_i\left(3\mathcal{A}_i^2-T_i\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i}{2T_i^2}-$
$\frac{1}{2}\hbar\, x_i^2\,\mathcal{A}_i^2\,\xi_i^2+\frac{x_i\left(3\mathcal{A}_i^2-T_i\,\mathcal{A}_i^2\right)\eta_i\,\xi_i^2}{2T_i}+\frac{\left(-3\mathcal{A}_i^2+4T_i\,\mathcal{A}_i^2-T_i^2\,\mathcal{A}_i^2\right)\eta_i^2\,\xi_i^2}{4\hbar\, T_i^2}\Big)\epsilon+O[\epsilon]^2\Big]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[a_i\, t_i,\ x_i\, y_i,\ \frac{1}{\sqrt{T_i}}+\left(\frac{a_i}{\sqrt{T_i}}+\frac{a_i^2}{\sqrt{T_i}}-\frac{x_i^2 y_i^2}{4\sqrt{T_i}}\right)\epsilon+O[\epsilon]^2\right]$

Out[∘]= $\mathbb{E}_{\{\}\to\{i\}}\left[-a_i\, t_i,\ -\frac{x_i\, y_i}{T_i},\ \sqrt{T_i}+\left(-a_i\sqrt{T_i}-a_i^2\sqrt{T_i}-\frac{2a_i\, x_i\, y_i}{\sqrt{T_i}}-\frac{3x_i^2 y_i^2}{4T_i^{3/2}}\right)\epsilon+O[\epsilon]^2\right]$

In[∘]:=

```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :→ { Xp[x[[4]], x[[1]]]  PositiveQ@x
                          Xm[x[[2]], x[[1]]]     True      ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten[front /. k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :→ {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :→ (++rots[[l]]; {1 - l, k + 1, l})
           })],
     Cases[front, k | -k] /. {k, -k} :→ --rots[[k + 1]];
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=

```
rot[i_, 0] := 𝔼_{{}→{i}}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC̄_j] // km_{i,j→i}];
```

In[∘]:=
```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼_{}→{0}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#〚1〛, #〚2〛, #〚1〛 - 1, #〚2〛 - 1}] &, 1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
      Xp, (kR_{i,j} OverBar[kKink]_k) // km_{j,k→j},
      Xm, (OverBar[kR]_{i,j} kKink_k) // km_{j,k→j}
     ];
   ξ1 = (rot[k, rots〚i〛] ξ1) // km_{k,i→i}; rots〚i〛 = 0;
   ξ1 = (ξ1 rot[k, rots〚i + 1〛]) // km_{i,k→i}; rots〚i + 1〛 = 0;
   ξ1 = (rot[k, rots〚j〛] ξ1) // km_{k,j→j}; rots〚j〛 = 0;
   ξ1 = (ξ1 rot[k, rots〚j + 1〛]) // km_{j,k→j}; rots〚j + 1〛 = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st〚i + 2〛 → st〚i + 1〛];
   If[MemberQ[done, i - 1], ξ = ξ // km_{st〚i〛,i→st〚i〛}; st = st /. st〚i + 1〛 → st〚i〛];
   If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st〚j + 2〛 → st〚j + 1〛];
   If[MemberQ[done, j - 1], ξ = ξ // km_{st〚j〛,j→st〚j〛}; st = st /. st〚j + 1〛 → st〚j〛];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
 ]
```

In[∘]:=
```
Z@Knot[3, 1]
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=
$$\mathbb{E}_{\{\}\to\{0\}}\left[0,0,\frac{T}{1-T+T^2}+\left(\frac{a\left(-2T\hbar+2T^3\hbar\right)}{1-2T+3T^2-2T^3+T^4}+\frac{-2T\hbar+3T^2\hbar-2T^3\hbar+T^4\hbar}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+\frac{x\,y\left(-2T\hbar^2-2T^2\hbar^2\right)}{1-2T+3T^2-2T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:=
```
R_{1,2} R_{3,4} // dm_{1,3→5}
```

Out[∘]=
$$\mathbb{E}_{\{\}\to\{2,4,5\}}\left[a_2 b_5+a_4 b_5,\ x_2 y_5+x_4 y_5,\ 1+\left(-a_2 x_4 y_5-\frac{1}{4}x_2^2 y_5^2-\frac{1}{4}x_4^2 y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

In[∘]:= $\overline{\text{kR}}_{1,2}\,\overline{\text{kR}}_{3,4}$ // $\text{tm}_{1,4\to 5}$

Out[∘]= $\mathbb{E}_{\{\}\to\{2,3,5\}}\Big[-t\,a_2 - t\,a_5,\ -\frac{x_5\,y_3}{T} - \frac{x_2\,y_5}{T},$
$1 + \left(-a_2\,a_5 - a_3\,a_5 - \frac{a_3\,x_5\,y_3}{T} - \frac{a_5\,x_5\,y_3}{T} - \frac{3\,x_5^2\,y_3^2}{4\,T^2} - \frac{a_2\,x_2\,y_5}{T} - \frac{a_5\,x_2\,y_5}{T} - \frac{3\,x_2^2\,y_5^2}{4\,T^2}\right)\epsilon + O[\epsilon]^2\Big]$

$\overline{\text{kR}}_{1,2}\,\overline{\text{kR}}_{3,4}$ // $\text{tm}_{1,4\to 5}$

In[∘]:=
```
(*Working Casimir, not unique!*)
Define[ω_i = 𝔼_{{}→{i}}[0, 0, Series[y e^(ϵ a) x + (e^(ϵ (a+1)) + e^(-ϵ a) T)/(e^ϵ - 1) - (T + 1) ϵ^-1, {ϵ, 0, 3}] /.
    {a → a_i, T → T_i, x → x_i, y → y_i}]
]
ωsq = ω_1 ω_2 // tm_{1,2→1};
ωcub = ωsq ω_2 // tm_{1,2→1};
ω4 = ωcub ω_2 // tm_{1,2→1};
(*Cleaned versions*)
ωc = ω_1[[3]] /. {T_1 → T, a_1 → a, x_1 → x, y_1 → y} // Normal;
ωsqc = ωsq[[3]] /. {T_1 → T, a_1 → a, x_1 → x, y_1 → y} // Normal;
ωcubc = ωcub[[3]] /. {T_1 → T, a_1 → a, x_1 → x, y_1 → y} // Normal;
ω4c = ω4[[3]] /. {T_1 → T, a_1 → a, x_1 → x, y_1 → y} // Normal;
```