Roland -

This is a demo of how to include Mathematica notebooks in a latex document. See also http://drorbn.net/AP/Projects/nb2tex/nb2tex.pdf (written in Groningen!).

The files used to construct this document are all at http://drorbn.net/AP/Projects/nb2tex/nb2tex.pdf. The "main" file is `Demo.tex`. It defines a few macros that detail how various types of Mathematica notebook cells should be formatted (especially see `\nbpdfInput` and `\nbpdfoutput`, it contains all the text down to the separator line below, and it inputs `TraditionalHopfStructure.tex`. That latter `.tex` file is produced automatically from the notebook `TraditionalHopfStructure@.nb` by running the code in the notebook `Make.nb`.

Here's how the notebook `TraditionalHopfStructure@.nb` works. I started from a copy of the notebook you sent me, the notebook `TraditionalHopfStructure.nb`. In the Mathematica *Cell→Cell Tags* I enabled *Show Cell Tags*, I selected the whole notebook, removed all the tags that were already there, and added a `pdf` tag to all cells. This tells `Make.nb` to create PDF files for all the cells and put the instructions to input them into the latex file `TraditionalHopfStructure.tex`.

But then I removed the `pdf` tag from the "Pensieve header" cell because there is no need to include it in the resulting document, and I've inserted a text cells with tag `tex` containing the paragraphs you are reading now. `Make.nb` simply copies cells with tag `tex` into `TraditionalHopfStructure.tex`, so it is easy to interlace latex with Mathematica. There's more information at http://drorbn.net/AP/Projects/nb2tex/nb2tex.pdf.

```
Once[<< KnotTheory`];
```

ParentDirectory: Argument File should be a positive machine-size integer, a nonempty string, or a File specification.

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ToFileName: String or list of strings expected at position 1 in ToFileName[{File, WikiLink, mathematica}].

ToFileName: String or list of strings expected at position 1 in ToFileName[{File, QuantumGroups}].

```
Loading KnotTheory` version of September 6, 2014, 13:37:37.2841.
Read more at http://katlas.org/wiki/KnotTheory.
```

$PP\_ = \text{Identity};\ \$k = 1;\ \gamma = 1;\ \hbar;$

tKink_1

$$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\ \hbar\, x_1\, y_1,\ \frac{1}{\sqrt{T_1}} + \left(\frac{\hbar\, a_1}{\sqrt{T_1}} + \frac{\hbar\, a_1^2}{\sqrt{T_1}} - \frac{\hbar^3\, x_1^2\, y_1^2}{4\sqrt{T_1}}\right)\epsilon + O[\epsilon]^2\right]$$

$\text{QZip}_{\{x_1,\xi_1,y_1,\eta_1,x_2,\xi_2,y_2,\eta_2\}}[\mathbb{E}\ @@\ (\text{kR}_{1,2}\ \text{km}_{2,1\to5})]$

$$\mathbb{E}\left[t\,\hbar\, a_2 + a_5\,\alpha_1 + a_5\,\alpha_2,\ 0,\ \frac{1}{T^2} + \frac{\hbar\, a_1\, a_2\,\epsilon}{T^2} + O[\epsilon]^2\right]$$

$R_{1,2}\ R_{3,4}\ \text{dm}_{1,3\to5}$

$$\mathbb{E}_{\{1,3\}\to\{1,2,3,4,5\}}\Big[\hbar\, a_2 b_1 + \hbar\, a_4 b_3 + a_5\alpha_1 + a_5\alpha_3 + b_5\beta_1 + b_5\beta_3,$$
$$\hbar\, x_2 y_1 + \hbar\, x_4 y_3 + y_5\eta_1 + \frac{y_5\eta_3}{\mathcal{A}_1} + \frac{x_5\xi_1}{\mathcal{A}_3} + \frac{(1-B_5)\,\eta_3\xi_1}{\hbar} + x_5\xi_3,$$
$$1 + \Big(-\frac{1}{4}\hbar^3 x_2^2 y_1^2 - \frac{1}{4}\hbar^3 x_4^2 y_3^2 - \frac{y_5\beta_1\eta_3}{\mathcal{A}_1} - \frac{x_5\beta_3\xi_1}{\mathcal{A}_3} + a_5 B_5\eta_3\xi_1 + \frac{\hbar\, x_5 y_5\eta_3\xi_1}{\mathcal{A}_1\mathcal{A}_3} +$$
$$\frac{(1-3B_5)\,y_5\eta_3^2\xi_1}{2\mathcal{A}_1} + \frac{(1-3B_5)\,x_5\eta_3\xi_1^2}{2\mathcal{A}_3} + \frac{\left(1-4B_5+3B_5^2\right)\eta_3^2\xi_1^2}{4\hbar}\Big)\epsilon + O[\epsilon]^2\Big]$$

```
QZip_{x1,ξ1,y1,η1,x3,ξ3,y3,η3}[𝔼 @@ (kR_{1,2} kR_{3,4} kR_{5,6} km_{1,3→5})]
```

$$\mathbb{E}\Big[t\,\hbar\, a_2 + t\,\hbar\, a_4 + t\,\hbar\, a_6 + a_5\alpha_1 + a_5\alpha_3,\ \hbar\, x_6 y_5,$$
$$1 + \Big(\hbar\, a_1 a_2 + \hbar\, a_3 a_4 + \hbar\, a_5 a_6 - \frac{1}{4}\hbar^3 x_6^2 y_5^2\Big)\epsilon + O[\epsilon]^2\Big]$$

The “Speedy” Engine

Internal Utilities

Canonical Form:

```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := Module[
   {vs = Cases[ℰ, (y | b | t | a | x | η | β | τ | α | ξ)_, ∞] ⋃
      {y, b, t, a, x, η, β, τ, α, ξ}},
   Total[CoefficientRules[Expand[ℰ], vs] /.
     (ps_ → c_) :> CCF[c] × (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

The Kronecker δ:

```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

Equality, multiplication, and degree-adjustment of perturbed Gaussians; $\mathbb{E}[L, Q, P]$ stands for $e^{L+Q} P$:

```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
   CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] × 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
𝔼[L_, Q_, P_]_{$k_} := 𝔼[L, Q, Series[Normal@P, {ϵ, 0, $k}]];
```

Zip and Bind

Variables and their duals:

```
{t*, b*, y*, a*, x*, z*} = {τ, β, η, α, ξ, ζ};
{τ*, β*, η*, α*, ξ*, ζ*} = {t, b, y, a, x, z};
(u_i_)* := (u*)_i;
```

Upper to lower and lower to Upper:

```
U2l = {B_i_^(p_.) :→ e^(-p ħ γ b_i), B^(p_.) :→ e^(-p ħ γ b), T_i_^(p_.) :→ e^(-p ħ t_i), T^(p_.) :→ e^(-p ħ t), 𝒜_i_^(p_.) :→ e^(p γ α_i),
    𝒜^(p_.) :→ e^(p γ α)};
l2U = {e^(c_. b_i_ + d_.) :→ B_i^(-c/(ħ γ)) e^d, e^(c_. b + d_.) :→ B^(-c/(ħ γ)) e^d,
    e^(c_. t_i_ + d_.) :→ T_i^(-c/ħ) e^d, e^(c_. t + d_.) :→ T^(-c/ħ) e^d,
    e^(c_. α_i_ + d_.) :→ 𝒜_i^(c/γ) e^d, e^(c_. α + d_.) :→ 𝒜^(c/γ) e^d,
    e^ℰ_ :→ e^(Expand@ℰ)};
```

Derivatives in the presence of exponentiated variables:

```
D_b[f_] := ∂_b f - ħ γ B ∂_B f; D_b_i_[f_] := ∂_b_i f - ħ γ B_i ∂_B_i f;
D_t[f_] := ∂_t f - ħ T ∂_T f; D_t_i_[f_] := ∂_t_i f - ħ T_i ∂_T_i f;
D_α[f_] := ∂_α f + γ 𝒜 ∂_𝒜 f; D_α_i_[f_] := ∂_α_i f + γ 𝒜_i ∂_𝒜_i f;
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f;
D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

Finite Zips:

```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] :=
  (collect[P // Zip_{ζs}, ζ] /. f_. ζ^(d_.) :→ (D_{ζ*,d}[f])) /. ζ* → 0 /.
   ((ζ* /. {b → B, t → T, α → 𝒜}) → 1)
```

QZip implements the “Q-level zips” on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i,\zeta^j)e^{c+\eta^i z_i+y_j\zeta^j+q^i_j z_i\zeta^j}\right\rangle = |\tilde{q}| \left\langle \left. P(z_i,\zeta^j)e^{c+\eta^i z_i}\right|_{z_i\to\tilde{q}^k_i(z_k+y_k)}\right\rangle$$

$$= |\tilde{q}|e^{c+\eta^i\tilde{q}^k_i y_k}\left\langle P\left(\tilde{q}^k_i(z_k+y_k),\zeta^j+\eta^i\tilde{q}^j_i\right)\right\rangle.$$

```
QZip_{ζs_List}@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
   zs = Table[ζ*, {ζ, ζs}];
   c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
   ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
   qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → CF[qt.(zs + ys)]];
   ζrule = Thread[ζs → ζs + ηs.qt];
   CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]]];
```

LZip implements the "L-level zips" on $\mathbb{E}(L, Q, P) = Pe^{L+Q}$. Such zips regard all of Pe^{Q} as a single"P". Here the z's are b and α and the ζ's are β and a.

```
LZip_{ζs_List}@𝔼[L_, Q_, P_] :=
  Module[{ζ, z, zs, Zs, c, ys, ηs, lt, zrule, Zrule, ζrule, Q1, EEQ, EQ},
   zs = Table[ζ*, {ζ, ζs}];
   Zs = zs /. {b → B, t → T, α → 𝒜};
   c = L /. Alternatives @@ (ζs ⋃ zs) → 0 /. Alternatives @@ Zs → 1;
   ys = Table[∂_ζ (L /. Alternatives @@ zs → 0), {ζ, ζs}];
   ηs = Table[∂_z (L /. Alternatives @@ ζs → 0), {z, zs}];
   lt = Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}L, {ζ, ζs}, {z, zs}];
   zrule = Thread[zs → lt.(zs + ys)];
   Zrule =
    Join[zrule,
     zrule /.
      r_Rule :> ((U = r〚1〛 /. {b → B, t → T, α → 𝒜}) → (U /. U2l /. r //. l2U))];
   ζrule = Thread[ζs → ζs + ηs.lt];
   Q1 = Q /. (Zrule ⋃ ζrule);
   EEQ[ps___] :=
    EEQ[ps] =
     (CF[e^-Q1 D_Thread[{zs,{ps}}][e^Q1]] /.
       {Alternatives @@ zs → 0, Alternatives @@ Zs → 1});
   CF@𝔼[c + ηs.lt.ys, Q1 /. {Alternatives @@ zs → 0, Alternatives @@ Zs → 1},
     Det[lt]
      (Zip_ζs[(EQ @@ zs) (P /. (Zrule ⋃ ζrule))] /.
         Derivative[ps___][EQ][___] :> EEQ[ps] /. _EQ → 1) ]];
```

```
B{}[L_, R_] := L R;
B{is__}[L_𝔼, R_𝔼] := Module[{n},
   Times[
      L /. Table[(v : b | B | t | T | a | x | y)i → vn@i, {i, {is}}],
      R /. Table[(v : β | τ | α | 𝒜 | ξ | η)i → vn@i, {i, {is}}]
    ] // LZipJoin@@Table[{βn@i,τn@i,an@i},{i,{is}}] //
   QZipJoin@@Table[{ξn@i,yn@i},{i,{is}}] ];
Bis___[L_, R_] := B{is}[L, R];
```

𝔼 morphisms with domain and range.

```
Bis_List[𝔼d1_→r1_[L1_, Q1_, P1_], 𝔼d2_→r2_[L2_, Q2_, P2_]] :=
  𝔼(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is]) @@ Bis[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] // 𝔼d2_→r2_[L2_, Q2_, P2_] :=
  Br1⋂d2[𝔼d1→r1[L1, Q1, P1], 𝔼d2→r2[L2, Q2, P2]];
𝔼d1_→r1_[L1_, Q1_, P1_] ≡ 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼d1_→r1_[L1_, Q1_, P1_] 𝔼d2_→r2_[L2_, Q2_, P2_] ^:=
  𝔼(d1⋃d2)→(r1⋃r2) @@ (𝔼[L1, Q1, P1] × 𝔼[L2, Q2, P2]);
𝔼dr_[L_, Q_, P_]$k_ := 𝔼dr @@ 𝔼[L, Q, P]$k;
𝔼_[ℰ___][i_] := {ℰ}[[i]];
```

𝔼[Λ]

```
𝔼dr_[Λ_] :=
 CF@Module[{L, Λ0 = Limit[Λ, ϵ → 0]},
   𝔼dr[L = Λ0 /. (η | y | ξ | x)_ → 0, Λ0 - L, 𝕖^(Λ-Λ0)]$k /. l2U]
```

“Define” Code

Define[lhs = rhs, ...] defines the lhs to be rhs, except that rhs is computed only once for each value of $k. Fancy Mathematica not for the faint of heart. Most readers should ignore.

```
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] :=
 Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
   ReleaseHold[Hold[
      SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
      SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{sis}];
     ] /. {SD → SetDelayed,
      isp → {is} /. {i → i_, j → j_, k → k_},
      nis → {is} /. {i → ii, j → jj, k → kk},
      nisp → {is} /. {i → ii_, j → jj_, k → kk_}
     }] ]]
```

The Objects

Symmetric Algebra Objects

```
sm_{i_,j_→k_} :=
  𝔼_{{i,j}→{k}}[b_k (β_i + β_j) + t_k (τ_i + τ_j) + a_k (α_i + α_j) + y_k (η_i + η_j) + x_k (ξ_i + ξ_j)];
sΔ_{i_→j_,k_} :=
  𝔼_{{i}→{j,k}}[β_i (b_j + b_k) + τ_i (t_j + t_k) + α_i (a_j + a_k) + η_i (y_j + y_k) + ξ_i (x_j + x_k)];
sS_{i_} := 𝔼_{{i}→{i}}[-β_i b_i - τ_i t_i - α_i a_i - η_i y_i - ξ_i x_i];
sϵ_{i_} := 𝔼_{{}→{i}}[0];
sη_{i_} := 𝔼_{{i}→{}}[0];
```

```
sσ_{i_→j_} := 𝔼_{{i}→{j}}[β_i b_j + τ_i t_j + α_i a_j + η_i y_j + ξ_i x_j];
sΥ_{i_→j_,k_,l_,m_} := 𝔼_{{i}→{j,k,l,m}}[β_i b_k + τ_i t_k + α_i a_l + η_i y_j + ξ_i x_m];
```

Booting Up QU

```
Define[aσ_{i→j} = 𝔼_{{i}→{j}}[a_j α_i + x_j ξ_i], bσ_{i→j} = 𝔼_{{i}→{j}}[b_j β_i + y_j η_i]]
```

```
Define[am_{i,j→k} = 𝔼_{{i,j}→{k}}[(α_i + α_j) a_k + (𝒜_j^{-1} ξ_i + ξ_j) x_k],
 bm_{i,j→k} = 𝔼_{{i,j}→{k}}[(β_i + β_j) b_k + (η_i + e^{-ϵ β_i} η_j) y_k]]
```

Three types of inverses appear below!

$\overline{R}$ is the inverse of R in the algebra $\mathbb{B}\otimes\mathbb{A}$.

P is the inverse of R as a quadratic form, like how an element of $V^*\otimes V^*$ can be the inverse of an element of $V\otimes V$. As a map $\mathrm{P}:\mathbb{A}\otimes\mathbb{B}\to Q$.

$\overline{\mathrm{aS}}$ is the inverse of aS as an operator form, like how an element of $V^*\otimes V$ can be the inverse of another element of $V^*\otimes V$.

$$\texttt{Define}\left[R_{i,j} = \mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, b_i + \sum_{k=1}^{\$k+1} \frac{\left(1 - e^{\gamma\,\epsilon\,\hbar}\right)^k \left(\hbar\, y_i\, x_j\right)^k}{k\left(1 - e^{k\,\gamma\,\epsilon\,\hbar}\right)}\right],\right.$$

$$\overline{R}_{i,j} = \texttt{CF@}\mathbb{E}_{\{\}\to\{i,j\}}\Big[-\hbar\, a_j\, b_i,\ -\hbar\, x_j\, y_i / B_i,\ 1 + \texttt{If}\big[\$k == 0,\ 0,\ (\overline{R}_{\{i,j\},\$k-1})_{\$k}[3] -$$
$$\Big(\Big((\overline{R}_{\{i,j\},0})_{\$k}\, R_{1,2}\, (\overline{R}_{\{3,4\},\$k-1})_{\$k}\Big) \,//\, (bm_{i,1\to i}\, am_{j,2\to j}) \,//\, (bm_{i,3\to i}\, am_{j,4\to j})\Big)[3]\big]\Big],$$

$$P_{i,j} = \mathbb{E}_{\{i,j\}\to\{\}}[\beta_j\, \alpha_i / \hbar,\ \eta_j\, \xi_i / \hbar,\ 1 + \texttt{If}[\$k == 0,\ 0,\ (P_{\{i,j\},\$k-1})_{\$k}[3] -$$
$$\left.(R_{1,2} \,//\, ((P_{\{i,1\},0})_{\$k}\ (P_{\{2,j\},\$k-1})_{\$k}))[3]]]\right]$$

$$R_{1,2} \,//\, P_{2,3}$$

$$\mathbb{E}_{\{3\}\to\{1\}}\left[b_1\, \beta_3,\ y_1\, \eta_3,\ 1 + O[\epsilon]^3\right]$$

$$(R_{1,2} \,//\, ((P_{\{i,1\},0})_2\ (P_{\{2,j\},1})_2))[3]$$

$$1 + \left(-\frac{1}{8}\, \eta_j^2\, \xi_i^2 - \frac{\eta_j^3\, \xi_i^3}{4\,\hbar} - \frac{\eta_j^4\, \xi_i^4}{16\,\hbar^2}\right)\epsilon^2 + O[\epsilon]^3$$

$$\texttt{Define}\Big[aS_i = \left(a\sigma_{i\to 2}\, \overline{R}_{1,i}\right) \,//\, P_{2,1},$$
$$\overline{aS}_i = \mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\, \alpha_i,\ -x_i\, \mathcal{A}_i\, \xi_i,\ 1 + \texttt{If}\big[\$k == 0,\ 0,\ (\overline{aS}_{\{i\},\$k-1})_{\$k}[3] -$$
$$\left((\overline{aS}_{\{i\},0})_{\$k} \,//\, aS_i \,//\, (\overline{aS}_{\{i\},\$k-1})_{\$k}\right)[3]\big]\Big]\Big]$$

$$\texttt{Define}\Big[bS_i = b\sigma_{i\to 1}\, R_{i,2} \,//\, aS_2 \,//\, P_{2,1},$$
$$\overline{bS}_i = b\sigma_{i\to 1}\, R_{i,2} \,//\, \overline{aS}_2 \,//\, P_{2,1},$$
$$a\Delta_{i\to j,k} = (R_{1,j}\, R_{2,k}) \,//\, bm_{1,2\to 3} \,//\, P_{i,3},$$
$$b\Delta_{i\to j,k} = (R_{j,1}\, R_{k,2}) \,//\, am_{1,2\to 3} \,//\, P_{3,i}\Big]$$

$$\texttt{Define}\Big[$$
$$dm_{i,j\to k} =$$
$$\Big(\big(s\Upsilon_{i\to 4,4,1,1} \,//\, a\Delta_{1\to 1,2} \,//\, a\Delta_{2\to 2,3} \,//\, \overline{aS}_3\big)$$
$$(s\Upsilon_{j\to -1,-1,-4,-4} \,//\, b\Delta_{-1\to -1,-2} \,//\, b\Delta_{-2\to -2,-3})\Big) \,//\, (P_{1,-3}\, P_{3,-1}\, am_{2,-4\to k}\, bm_{4,-2\to k})\Big]$$

NB. We use the co-algebra structure B tensor A^{cop}. This has the benefit of making our algebra quasi-triangular in the traditional sense of the word.

Watch out: $\Delta_{i\to j,k}$ means *j* is to the RIGHT of strand k and dS looks like an S.

$$\texttt{Define}\Big[d\sigma_{i\to j} = a\sigma_{i\to j}\, b\sigma_{i\to j},$$
$$d\epsilon_i = s\epsilon_i,\ \ d\eta_i = s\eta_i,$$
$$dS_i = s\Upsilon_{i\to 1,1,2,2} \,//\, \left(bS_1\, \overline{aS}_2\right) \,//\, dm_{2,1\to i},$$
$$\overline{dS}_i = s\Upsilon_{i\to 1,1,2,2} \,//\, \left(\overline{bS}_1\, aS_2\right) \,//\, dm_{2,1\to i},$$
$$d\Delta_{i\to j,k} = (b\Delta_{i\to 1,3}\, a\Delta_{i\to 4,2}) \,//\, (dm_{3,4\to k}\, dm_{1,2\to j})\Big]$$

$$\mathtt{Define}\Big[C_i = \mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i^{1/2}\,e^{-\hbar\,\epsilon\,a_i/2}\right]_{\$k},$$
$$\overline{C}_i = \mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i^{-1/2}\,e^{\hbar\,\epsilon\,a_i/2}\right]_{\$k},$$
$$c_i = \mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i^{1/4}\,e^{-\hbar\,\epsilon\,a_i/4}\right]_{\$k},$$
$$\overline{c}_i = \mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,B_i^{-1/4}\,e^{\hbar\,\epsilon\,a_i/4}\right]_{\$k},$$
$$\mathtt{Kink}_i = \left(R_{1,3}\,\overline{C}_2\right)\ //\ dm_{1,2\to1}\ //\ dm_{1,3\to i},$$
$$\overline{\mathtt{Kink}}_i = \left(\overline{R}_{1,3}\,C_2\right)\ //\ dm_{1,2\to1}\ //\ dm_{1,3\to i},$$
$$\rho_i = (c_1\,\overline{c}_3\,dS_i)\ //\ dm_{1,i\to i}\ //\ dm_{i,3\to i}\Big]$$

```
(*ρ reverses a strand*)
```

Note. $t = -\epsilon a + \gamma b$ and $b = t/\gamma + \epsilon a/\gamma$

$$\mathtt{Define}[b2t_i = \mathbb{E}_{\{i\}\to\{i\}}[\alpha_i\,a_i + \beta_i\,(\epsilon\,a_i + t_i)/\gamma + \xi_i\,x_i + \eta_i\,y_i],$$
$$t2b_i = \mathbb{E}_{\{i\}\to\{i\}}[\alpha_i\,a_i + \tau_i\,(-\epsilon\,a_i + \gamma\,b_i) + \xi_i\,x_i + \eta_i\,y_i]]$$

$$\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1]\ //\ d\Delta_{1\to1,2}$$
$$\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1]\ //\ dS_1$$
$$\mathbb{E}_{\{\}\to\{1\}}[0,0,y_1]\ //\ dS_1$$
$$\mathbb{E}_{\{\}\to\{1\}}[0,0,x_1]\ //\ \overline{dS}_1$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\,0,\,(x_1+x_2) - \hbar\,a_2\,x_1\,\epsilon + \frac{1}{2}\,\hbar^2\,a_2^2\,x_1\,\epsilon^2 + O[\epsilon]^3\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1 + (\hbar\,x_1 - \hbar\,a_1\,x_1)\,\epsilon + \left(-\frac{1}{2}\,\hbar^2\,x_1 + \hbar^2\,a_1\,x_1 - \frac{1}{2}\,\hbar^2\,a_1^2\,x_1\right)\epsilon^2 + O[\epsilon]^3\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-\frac{y_1}{B_1} + O[\epsilon]^3\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1 - \hbar\,a_1\,x_1\,\epsilon - \frac{1}{2}\,\left(\hbar^2\,a_1^2\,x_1\right)\epsilon^2 + O[\epsilon]^3\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}[0,0,(1+\epsilon\,a_1\,\hbar)\,x_1]\ //\ dS_1$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,-x_1 + \left(\frac{\hbar^2\,x_1}{2} - \hbar^2\,a_1\,x_1 + \frac{1}{2}\,\hbar^2\,a_1^2\,x_1\right)\epsilon^2 + O[\epsilon]^3\right]$$

$$((-1+\hbar)\,x_1 + (1-\hbar)\,a_1\,x_1)\ //\ \mathtt{Expand}$$

$$-x_1 + \hbar\,x_1 + a_1\,x_1 - \hbar\,a_1\,x_1$$

$$t2b_1\,t2b_2\ //\ P_{2,1}$$

$$\mathbb{E}_{\{1,2\}\to\{\}}\left[\frac{\alpha_2\,\tau_1}{\hbar},\,\frac{\eta_1\,\xi_2}{\hbar},\,1 + \left(\frac{\eta_1^2\,\xi_2^2}{4\,\hbar} - \frac{\tau_1\,\tau_2}{\hbar}\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}[0,0,y_1]\ //\ b\Delta_{1\to1,2}$$
$$\mathbb{E}_{\{\}\to\{1\}}[0,0,y_1]\ //\ d\Delta_{1\to1,2}$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\,0,\,(B_2\,y_1 + y_2) + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (B_2\, y_1 + y_2) + O[\epsilon]^2\right]$$

$$(R_{1,2}\ //\ bS_1) \equiv \overline{R}_{1,2}$$
$$(R_{1,2}\ //\ aS_2) \equiv \overline{R}_{1,2}$$

True

True

$$\mathbb{E}_{\{\}\to\{1\}}[0,\ 0,\ x_1]\ //\ d\Delta_{1\to 1,2}$$

$$\mathbb{E}_{\{1\}\to\{1,2\}}\left[0,\ 0,\ (x_1 + x_2) - \hbar\, a_2\, x_1\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}[0,\ 0,\ x_1]\ //\ a\Delta_{1\to 1,2}$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ (x_1 + x_2) - \hbar\, a_1\, x_2\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}[0,\ 0,\ x_1]\ //\ (\overline{aS})_1$$
$$\mathbb{E}_{\{\}\to\{1\}}[0,\ 0,\ x_1]\ //\ aS_1$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 + (\hbar\, x_1 - \hbar\, a_1\, x_1)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ -x_1 - \hbar\, a_1\, x_1\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1,2\}}[0,\ 0,\ b_1\, y_2]\ //\ bm_{1,2\to 1}$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ b_1\, y_1 - y_1\, \epsilon + O[\epsilon]^2\right]$$

$$a\Delta_{i\to 1,2}\ //\ aS_1\ //\ am_{1,2\to 1}$$
$$a\Delta_{i\to 1,2}\ //\ aS_2\ //\ am_{1,2\to 1}$$

$$\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$$

$$a\Delta_{1\to 1,2}$$

$$\mathbb{E}_{\{1\}\to\{1,2\}}\left[a_1\, \alpha_1 + a_2\, \alpha_1,\ x_1\, \xi_1 + x_2\, \xi_1,\ 1 + \left(-\hbar\, a_1\, x_2\, \xi_1 + \frac{1}{2}\, \hbar\, x_1\, x_2\, \xi_1^2\right) \epsilon + O[\epsilon]^2\right]$$

Testing

co-associativity

$$(d\Delta_{1\to 1,2}\ //\ d\Delta_{2\to 2,3}) \equiv (d\Delta_{1\to 2,3}\ //\ d\Delta_{2\to 1,2})$$

True

algebra morphism

$$(d\Delta_{i\to 1,2}\ d\Delta_{j\to 3,4}\ //\ dm_{1,3\to i}\ //\ dm_{2,4\to j}) \equiv (dm_{i,j\to k}\ //\ d\Delta_{k\to i,j})$$

True

associativity

$(dm_{1,2\to k}\ //\ dm_{k,3\to k}) \equiv (dm_{2,3\to k}\ //\ dm_{1,k\to k})$

True

antipode

$d\Delta_{i\to 1,2}\ //\ dS_1\ //\ dm_{1,2\to 1}$

$d\Delta_{i\to 1,2}\ //\ dS_2\ //\ dm_{1,2\to 1}$

$\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

$\mathbb{E}_{\{i\}\to\{1\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

quasi-triangular axioms

$(R_{1,3}\ //\ d\Delta_{1\to 1,2}) \equiv (R_{1,3}\, R_{2,4}\ //\ dm_{3,4\to 3})$

$(R_{1,3}\ //\ d\Delta_{3\to 2,3}) \equiv (R_{1,3}\, R_{0,2}\ //\ dm_{1,0\to 1})$

$(d\Delta_{i\to k,j}\, R_{1,2}\ //\ dm_{j,1\to 1}\ //\ dm_{k,2\to 2}) \equiv (R_{1,2}\, d\Delta_{i\to j,k}\ //\ dm_{1,j\to 1}\ //\ dm_{2,k\to 2})$

True

True

True

$(R_{1,2}\ //\ aS_2) \equiv (\overline{R}_{1,2})$

True

$(R_{1,2}\ //\ dS_1) \equiv (\overline{R}_{1,2})$

$\left(R_{1,2}\ //\ \overline{dS}_2\right) \equiv (\overline{R}_{1,2})$

True

True

$QQ_{s\_,r\_}\ := R_{11,22}\, R_{33,44}\ //\ dm_{11,44\to s}\ //\ dm_{22,33\to r}$

$\overline{QQ}_{s\_,r\_}\ := \overline{R}_{22,11}\, \overline{R}_{44,33}\ //\ dm_{11,44\to s}\ //\ dm_{22,33\to r}$

$QQ_{1,2}\, \overline{QQ}_{3,4}\ //\ dm_{1,3\to 1}\ //\ dm_{2,4\to 2}$

$\mathbb{E}_{\{\}\to\{1,2\}}\left[0,\ 0,\ 1 + O[\epsilon]^2\right]$

Drinfeld element u

$u_{i\_}\ := R_{11,22}\ //\ dS_{22}\ //\ dm_{22,11\to i}$

$\overline{u}_{i\_}\ := \overline{R}_{11,22}\ //\ \overline{dS}_{22}\ //\ dm_{22,11\to i}$

$\overline{uu}_{i\_}\ := \overline{R}_{11,22}\ //\ \overline{dS}_{22}\ //\ dm_{11,22\to i}$

$\overline{u2}_{i\_}\ := \overline{R}_{11,22}\ //\ dS_{11}\ //\ dm_{11,22\to i}$

$\overline{u3}_{i\_}\ := R_{11,22}\ //\ dS_{11}\ //\ dS_{11}\ //\ dm_{22,11\to i}$

$u_i\ \overline{u}_j$ // $dm_{i,j\to i}$
$u_i\ \overline{uu}_j$ // $dm_{i,j\to i}$
$u_i\ \overline{u2}_j$ // $dm_{i,j\to i}$
$u_i\ \overline{u3}_j$ // $dm_{i,j\to i}$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ B_i-\hbar\, a_i\, B_i\, \epsilon+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ B_i-\hbar\, a_i\, B_i\, \epsilon+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ 1+O[\epsilon]^2\right]$$

(u_1 // dS_1)
$R_{11,22}$ // dS_{22} // $dm_{11,22\to i}$

$$\mathbb{E}_{\{\}\to\{1\}}\Bigg[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ 1+\left(\frac{\hbar^2\, x_1\, y_1}{B_1}-\frac{\hbar^2\, a_1\, x_1\, y_1}{B_1}-\frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1^2}\right)\epsilon+\left(-\frac{\hbar^3\, x_1\, y_1}{2\, B_1}+\frac{\hbar^3\, a_1\, x_1\, y_1}{B_1}-\frac{\hbar^3\, a_1^2\, x_1\, y_1}{2\, B_1}+\frac{5\,\hbar^4\, x_1^2\, y_1^2}{2\, B_1^2}-\frac{5\,\hbar^4\, a_1\, x_1^2\, y_1^2}{2\, B_1^2}+\frac{\hbar^4\, a_1^2\, x_1^2\, y_1^2}{2\, B_1^2}-\frac{67\,\hbar^5\, x_1^3\, y_1^3}{36\, B_1^3}+\frac{3\,\hbar^5\, a_1\, x_1^3\, y_1^3}{4\, B_1^3}+\frac{9\,\hbar^6\, x_1^4\, y_1^4}{32\, B_1^4}\right)\epsilon^2+O[\epsilon]^3\Bigg]$$

$$\mathbb{E}_{\{\}\to\{i\}}\Bigg[-\hbar\, a_i\, b_i,\ -\frac{\hbar\, x_i\, y_i}{B_i},\ 1+\left(\frac{\hbar^2\, x_i\, y_i}{B_i}-\frac{\hbar^2\, a_i\, x_i\, y_i}{B_i}-\frac{3\,\hbar^3\, x_i^2\, y_i^2}{4\, B_i^2}\right)\epsilon+\left(-\frac{\hbar^3\, x_i\, y_i}{2\, B_i}+\frac{\hbar^3\, a_i\, x_i\, y_i}{B_i}-\frac{\hbar^3\, a_i^2\, x_i\, y_i}{2\, B_i}+\frac{5\,\hbar^4\, x_i^2\, y_i^2}{2\, B_i^2}-\frac{5\,\hbar^4\, a_i\, x_i^2\, y_i^2}{2\, B_i^2}+\frac{\hbar^4\, a_i^2\, x_i^2\, y_i^2}{2\, B_i^2}-\frac{67\,\hbar^5\, x_i^3\, y_i^3}{36\, B_i^3}+\frac{3\,\hbar^5\, a_i\, x_i^3\, y_i^3}{4\, B_i^3}+\frac{9\,\hbar^6\, x_i^4\, y_i^4}{32\, B_i^4}\right)\epsilon^2+O[\epsilon]^3\Bigg]$$

$$\left(\mathbb{E}_{\{\}\to\{1\}}\left[0,\ 0,\ B_1^{-1}\left(1+\epsilon\, a_1\,\hbar+\frac{\epsilon^2}{2}\, a_1{}^2\,\hbar^2\right)\right]\ u_2\ //\ dm_{1,2\to 1}\right)\equiv(u_1\ //\ dS_1)$$

True

u_1

$$\mathbb{E}_{\{\}\to\{1\}}\Bigg[-\hbar\, a_1\, b_1,\ -\frac{\hbar\, x_1\, y_1}{B_1},\ B_1+\left(-\hbar\, a_1\, B_1-\hbar^2\, x_1\, y_1-\hbar^2\, a_1\, x_1\, y_1-\frac{3\,\hbar^3\, x_1^2\, y_1^2}{4\, B_1}\right)\epsilon+\left(\frac{1}{2}\,\hbar^2\, a_1^2\, B_1-\frac{1}{2}\,\hbar^3\, x_1\, y_1+\frac{1}{2}\,\hbar^3\, a_1^2\, x_1\, y_1-\frac{\hbar^4\, x_1^2\, y_1^2}{2\, B_1}+\frac{\hbar^4\, a_1\, x_1^2\, y_1^2}{4\, B_1}+\frac{\hbar^4\, a_1^2\, x_1^2\, y_1^2}{2\, B_1}-\frac{13\,\hbar^5\, x_1^3\, y_1^3}{36\, B_1^2}+\frac{3\,\hbar^5\, a_1\, x_1^3\, y_1^3}{4\, B_1^2}+\frac{9\,\hbar^6\, x_1^4\, y_1^4}{32\, B_1^3}\right)\epsilon^2+O[\epsilon]^3\Bigg]$$

q

(u_1 // dS_1) $\overline{u3}_2$ // $dm_{1,2\to 1}$

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\frac{1}{B_1}+\frac{\hbar\,a_1\,\epsilon}{B_1}+O[\epsilon]^2\right]$$

$(u_1\ //\ d\Delta_{1\to 2,1}) \equiv (\overline{QQ}_{1,2}\ u_3\ u_4\ //\ dm_{1,3\to 1}\ //\ dm_{2,4\to 2})$

True

$\mathbb{E}_{\{\}\to\{i\}}[0,\,0,\,x_i]\ //\ dS_i$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\,0,\,-x_i+(\hbar\,x_i-\hbar\,a_i\,x_i)\,\epsilon+O[\epsilon]^2\right]$$

$Kink_1$

$$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\,a_1\,b_1,\,\hbar\,x_1\,y_1,\,\frac{1}{\sqrt{B_1}}+\left(\frac{\hbar\,a_1}{2\sqrt{B_1}}-\frac{\hbar^3\,x_1^2\,y_1^2}{4\sqrt{B_1}}\right)\epsilon+O[\epsilon]^2\right]$$

$(u_1\ //\ dS_1)\ u_2\ //\ dm_{1,2\to 1}$

$(u_1\ //\ dS_1)\ u_2\ //\ dm_{2,1\to 1}$

$$\mathbb{E}_{\{\}\to\{1\}}\left[-2\,\hbar\,a_1\,b_1,\,\frac{(-\hbar-\hbar\,B_1)\,x_1\,y_1}{B_1^2},\,B_1+\left(-\hbar\,a_1\,B_1+\frac{a_1\left(-2\,\hbar^2-\hbar^2\,B_1\right)x_1\,y_1}{B_1}+\frac{\left(-7\,\hbar^3-8\,\hbar^3\,B_1-3\,\hbar^3\,B_1^2\right)x_1^2\,y_1^2}{4\,B_1^3}\right)\epsilon+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[-2\,\hbar\,a_1\,b_1,\,\frac{(-\hbar-\hbar\,B_1)\,x_1\,y_1}{B_1^2},\,B_1+\left(-\hbar\,a_1\,B_1+\frac{a_1\left(-2\,\hbar^2-\hbar^2\,B_1\right)x_1\,y_1}{B_1}+\frac{\left(-7\,\hbar^3-8\,\hbar^3\,B_1-3\,\hbar^3\,B_1^2\right)x_1^2\,y_1^2}{4\,B_1^3}\right)\epsilon+O[\epsilon]^2\right]$$

$(u_1\ //\ dS_1)$

u_2

$$\mathbb{E}_{\{\}\to\{1\}}\left[-\hbar\,a_1\,b_1,\,-\frac{\hbar\,x_1\,y_1}{B_1},\,1+\left(\frac{\hbar^2\,x_1\,y_1}{B_1}-\frac{\hbar^2\,a_1\,x_1\,y_1}{B_1}-\frac{3\,\hbar^3\,x_1^2\,y_1^2}{4\,B_1^2}\right)\epsilon+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{2\}}\left[-\hbar\,a_2\,b_2,\,-\frac{\hbar\,x_2\,y_2}{B_2},\,B_2+\left(-\hbar\,a_2\,B_2-\hbar^2\,x_2\,y_2-\hbar^2\,a_2\,x_2\,y_2-\frac{3\,\hbar^3\,x_2^2\,y_2^2}{4\,B_2}\right)\epsilon+O[\epsilon]^2\right]$$

$R_{1,2}$

$\overline{R}_{1,2}$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[\hbar\,a_2\,b_1,\,\hbar\,x_2\,y_1,\,1-\frac{1}{4}\left(\hbar^3\,x_2^2\,y_1^2\right)\epsilon+O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1,2\}}\left[-\hbar\,a_2\,b_1,\,-\frac{\hbar\,x_2\,y_1}{B_1},\,1+\left(-\frac{\hbar^2\,a_2\,x_2\,y_1}{B_1}-\frac{3\,\hbar^3\,x_2^2\,y_1^2}{4\,B_1^2}\right)\epsilon+O[\epsilon]^2\right]$$

C_1

$$\mathbb{E}_{\{\}\to\{1\}}\left[0,\,0,\,\sqrt{B_1}-\frac{1}{2}\left(\hbar\,a_1\sqrt{B_1}\right)\epsilon+O[\epsilon]^2\right]$$

The Knot Tensors

$$\text{Define}\Big[kR_{i,j} = R_{i,j} \,//\, (b2t_i\, b2t_j) \;/.\; t_{i|j} \to t,$$
$$\overline{kR}_{i,j} = \overline{R}_{i,j} \,//\, (b2t_i\, b2t_j) \;/.\; \{t_{i|j} \to t,\, T_{i|j} \to T\},$$
$$km_{i,j\to k} = ((t2b_i\, t2b_j) \,//\, dm_{i,j\to k} \,//\, b2t_k) \;/.\; \{t_k \to t,\; T_k \to T,\; \tau_{i|j} \to 0\},$$
$$kC_i = (C_i \,//\, b2t_i) \;/.\; T_i \to T,$$
$$\overline{kC}_i = (\overline{C}_i \,//\, b2t_i) \;/.\; T_i \to T,$$
$$kKink_i = Kink_i \,//\, b2t_i \;/.\; \{t_i \to t,\; T_i \to T\},$$
$$\overline{kKink}_i = \overline{Kink}_i \,//\, b2t_i \;/.\; \{t_i \to t,\; T_i \to T\}\Big]$$

$$\text{Define}\Big[$$
$$BS_{i,j\to k} = C_3\, C_4\, d\Delta_{i\to 11,r1}\, d\Delta_{j\to 12,r2} \,//\, \overline{dS}_{r1} \,//\, dS_{r2} \,//\, dm_{11,3\to k} \,//\, dm_{k,r2\to k} \,//\, dm_{k,r1\to k} \,//$$
$$dm_{k,4\to k} \,//\, dm_{k,12\to k}\Big]$$
$$\text{Define}\Big[$$
$$tBS_{i,j\to k} =$$
$$(t2b_i\, t2b_j) \,//\, C_3\, C_4\, d\Delta_{i\to 11,r1}\, d\Delta_{j\to 12,r2} \,//\, \overline{dS}_{r1} \,//\, dS_{r2} \,//\, dm_{11,3\to k} \,//\, dm_{k,r2\to k} \,//$$
$$dm_{k,r1\to k} \,//\, dm_{k,4\to k} \,//\, dm_{k,12\to k} \,//\, b2t_k\Big]$$
$$\text{Define}[tm_{i,j\to k} = t2b_i \,//\, t2b_j \,//\, dm_{i,j\to k} \,//\, b2t_k]$$
$$\text{Define}[t\Delta_{i\to j,k} = t2b_i \,//\, d\Delta_{i\to j,k} \,//\, b2t_j \,//\, b2t_k]$$
$$\text{Define}[tS_i = t2b_i \,//\, dS_i \,//\, b2t_i]$$
$$\text{Define}\big[\overline{tS}_i = t2b_i \,//\, \overline{dS}_i \,//\, b2t_i\big]$$
$$\text{Define}\big[tR_{i,j} = R_{i,j} \,//\, b2t_i \,//\, b2t_j,\; \overline{tR}_{i,j} = \overline{R}_{i,j} \,//\, b2t_i \,//\, b2t_j\big]$$
$$\text{Define}\big[tC_i = C_i \,//\, b2t_i\,,\; \overline{tC}_i = \overline{C}_i \,//\, b2t_i\big]$$
$$\text{Define}\big[tKink_i = Kink_i \,//\, b2t_i\,,\; \overline{tKink}_i = \overline{Kink}_i \,//\, b2t_i\big]$$

$$R_{1,3}\, R_{2,6} \,//\, dm_{3,6\to 3}$$
$$R_{1,3} \,//\, d\Delta_{1\to 2,1}$$

$$\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\; \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\; 1 + \left(-\frac{1}{4}\hbar^3 B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\hbar^3 x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1,2,3\}}\left[\hbar\, a_3\, b_1 + \hbar\, a_3\, b_2,\; \hbar\, B_2\, x_3\, y_1 + \hbar\, x_3\, y_2,\; 1 + \left(-\frac{1}{4}\hbar^3 B_2^2\, x_3^2\, y_1^2 - \frac{1}{4}\hbar^3 x_3^2\, y_2^2\right)\epsilon + O[\epsilon]^2\right]$$

$$tR_{1,2} \,//\, \overline{tS}_1 \,//\, \overline{tS}_1 \,//\, tm_{1,2\to 1}$$
$$(tR_{1,2} \,//\, \overline{tS}_1 \,//\, \overline{tS}_1 \,//\, tm_{1,2\to 1} \,//\, tS_1)\; \mathbb{E}_{\{\}\to\{2\}}[0,\, 0,\, T_2\,(1 - 2\,\epsilon\,\hbar\, a_1)] \,//\, tm_{1,2\to 1}$$
$$(tR_{1,2} \,//\, \overline{tS}_1 \,//\, \overline{tS}_1 \,//\, tm_{2,1\to 1})\; \mathbb{E}_{\{\}\to\{2\}}[0,\, 0,\, T_2\,(1 - 2\,\epsilon\,\hbar\, a_1)] \,//\, tm_{1,2\to 1}$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\; \hbar\, x_1\, y_1,\; 1 + \left(\hbar\, a_1^2 + \hbar^2 x_1\, y_1 - \frac{1}{4}\hbar^3 x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\; \hbar\, x_1\, y_1,\; 1 + \left(\hbar\, a_1^2 - \hbar^2 x_1\, y_1 - \frac{1}{4}\hbar^3 x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{1\}}\left[\hbar\, a_1\, t_1,\; \hbar\, x_1\, y_1,\; 1 + \left(\hbar\, a_1^2 - \hbar^2 x_1\, y_1 - \frac{1}{4}\hbar^3 x_1^2\, y_1^2\right)\epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{2\}}[0,\, 0,\, x_2] \,//\, dS_2$$

$$\mathbb{E}_{\{\}\to\{2\}}\left[0,\ 0,\ -x_2 - \hbar\, a_2\, x_2\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{2\}}[0,\ 0,\ y_2]\ //\ \overline{dS}_2$$

$$\mathbb{E}_{\{\}\to\{2\}}\left[0,\ 0,\ -\frac{y_2}{B_2} + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{2\}}[0,\ 0,\ y_2]\ //\ \overline{dS}_2\ //\ \overline{dS}_2$$

$$\mathbb{E}_{\{\}\to\{2\}}\left[0,\ 0,\ y_2 + \hbar\, y_2\, \epsilon + O[\epsilon]^2\right]$$

$tm_{i,j\to k}$
$tR_{i,j}$
$\overline{tR}_{i,j}$
tC_i
$\overline{tC}_i$
$tKink_i$
$\overline{tKink}_i$
$t\Delta_{i\to j,k}$
tS_i

$$\mathbb{E}_{\{i,j\}\to\{k\}}\Bigg[a_k\, \alpha_i + a_k\, \alpha_j + t_k\, \tau_i + t_k\, \tau_j,\ y_k\, \eta_i + \frac{y_k\, \eta_j}{\mathcal{A}_i} + \frac{x_k\, \xi_i}{\mathcal{A}_j} + \frac{(1 - T_k)\, \eta_j\, \xi_i}{\hbar} + x_k\, \xi_j,\ 1 + \Bigg(2\, a_k\, T_k\, \eta_j\, \xi_i + \frac{\hbar\, x_k\, y_k\, \eta_j\, \xi_i}{\mathcal{A}_i\, \mathcal{A}_j} + \frac{(1 - 3\, T_k)\, y_k\, \eta_j^2\, \xi_i}{2\, \mathcal{A}_i} + \frac{(1 - 3\, T_k)\, x_k\, \eta_j\, \xi_i^2}{2\, \mathcal{A}_j} + \frac{\left(1 - 4\, T_k + 3\, T_k^2\right)\, \eta_j^2\, \xi_i^2}{4\, \hbar}\Bigg)\, \epsilon + O[\epsilon]^2\Bigg]$$

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[\hbar\, a_j\, t_i,\ \hbar\, x_j\, y_i,\ 1 + \left(\hbar\, a_i\, a_j - \frac{1}{4}\, \hbar^3\, x_j^2\, y_i^2\right)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i,j\}}\left[-\hbar\, a_j\, t_i,\ -\frac{\hbar\, x_j\, y_i}{T_i},\ 1 + \left(-\hbar\, a_i\, a_j - \frac{\hbar^2\, a_i\, x_j\, y_i}{T_i} - \frac{\hbar^2\, a_j\, x_j\, y_i}{T_i} - \frac{3\, \hbar^3\, x_j^2\, y_i^2}{4\, T_i^2}\right)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \sqrt{T_i} - \hbar\, a_i\, \sqrt{T_i}\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[0,\ 0,\ \frac{1}{\sqrt{T_i}} + \frac{\hbar\, a_i\, \epsilon}{\sqrt{T_i}} + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[\hbar\, a_i\, t_i,\ \hbar\, x_i\, y_i,\ \frac{1}{\sqrt{T_i}} + \left(\frac{\hbar\, a_i}{\sqrt{T_i}} + \frac{\hbar\, a_i^2}{\sqrt{T_i}} - \frac{\hbar^3\, x_i^2\, y_i^2}{4\, \sqrt{T_i}}\right)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{\}\to\{i\}}\left[-\hbar\, a_i\, t_i,\ -\frac{\hbar\, x_i\, y_i}{T_i},\ \sqrt{T_i} + \left(-\hbar\, a_i\, \sqrt{T_i} - \hbar\, a_i^2\, \sqrt{T_i} - \frac{2\, \hbar^2\, a_i\, x_i\, y_i}{\sqrt{T_i}} - \frac{3\, \hbar^3\, x_i^2\, y_i^2}{4\, T_i^{3/2}}\right)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{i\}\to\{j,k\}}\left[a_j\, \alpha_i + a_k\, \alpha_i + t_j\, \tau_i + t_k\, \tau_i,\ y_j\, \eta_i + T_j\, y_k\, \eta_i + x_j\, \xi_i + x_k\, \xi_i,\ 1 + \left(-\hbar\, a_j\, T_j\, y_k\, \eta_i + \frac{1}{2}\, \hbar\, T_j\, y_j\, y_k\, \eta_i^2 - \hbar\, a_j\, x_k\, \xi_i + \frac{1}{2}\, \hbar\, x_j\, x_k\, \xi_i^2\right)\, \epsilon + O[\epsilon]^2\right]$$

$$\mathbb{E}_{\{i\}\to\{i\}}\Big[-a_i\,\alpha_i - t_i\,\tau_i,\ -\frac{y_i\,\mathcal{A}_i\,\eta_i}{T_i} - x_i\,\mathcal{A}_i\,\xi_i + \frac{(\mathcal{A}_i - T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{\hbar\,T_i},$$
$$1 + \Big(\frac{\hbar\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,a_i\,y_i\,\mathcal{A}_i\,\eta_i}{T_i} - \frac{\hbar\,y_i^2\,\mathcal{A}_i^2\,\eta_i^2}{2\,T_i^2} - \hbar\,a_i\,x_i\,\mathcal{A}_i\,\xi_i + \frac{2\,a_i\,\mathcal{A}_i\,\eta_i\,\xi_i}{T_i} -$$
$$\frac{\hbar\,x_i\,y_i\,\mathcal{A}_i^2\,\eta_i\,\xi_i}{T_i} + \frac{(-\mathcal{A}_i + T_i\,\mathcal{A}_i)\,\eta_i\,\xi_i}{T_i} + \frac{y_i\,(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i}{2\,T_i^2} - \frac{1}{2}\,\hbar\,x_i^2\,\mathcal{A}_i^2\,\xi_i^2 +$$
$$\frac{x_i\,(3\,\mathcal{A}_i^2 - T_i\,\mathcal{A}_i^2)\,\eta_i\,\xi_i^2}{2\,T_i} + \frac{(-3\,\mathcal{A}_i^2 + 4\,T_i\,\mathcal{A}_i^2 - T_i^2\,\mathcal{A}_i^2)\,\eta_i^2\,\xi_i^2}{4\,\hbar\,T_i^2}\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$$\mathbb{E}_{\{\}\to\{i\}}\Big[a_i\,t_i,\ x_i\,y_i,\ \frac{1}{\sqrt{T_i}} + \Big(\frac{a_i}{\sqrt{T_i}} + \frac{a_i^2}{\sqrt{T_i}} - \frac{x_i^2\,y_i^2}{4\,\sqrt{T_i}}\Big)\,\epsilon + O[\epsilon]^2\Big]$$

$$\mathbb{E}_{\{\}\to\{i\}}\Big[-a_i\,t_i,\ -\frac{x_i\,y_i}{T_i},\ \sqrt{T_i} + \Big(-a_i\,\sqrt{T_i} - a_i^2\,\sqrt{T_i} - \frac{2\,a_i\,x_i\,y_i}{\sqrt{T_i}} - \frac{3\,x_i^2\,y_i^2}{4\,T_i^{3/2}}\Big)\,\epsilon + O[\epsilon]^2\Big]$$

```
RVK[pd_PD] := PP_RVK@Module[{n, xs, x, rots, front = {0}, k},
    n = Length@pd; rots = Table[0, {2 n}];
    xs = Cases[pd, x_X :> { Xp[x[[4]], x[[1]]]   PositiveQ@x
                            Xm[x[[2]], x[[1]]]   True        ];
    For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
      front = Flatten[front /. k → (xs /. {
              Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
              Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots[[l]];
                {1 - l, k + 1, l})
            })],
      Cases[front, k | -k] /. {k, -k} :> --rots[[k + 1]];
    ]];
    RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

```
rot[i_, 0] := 𝔼_{}→{i}[0, 0, 1];
rot[i_, n_] := Module[{j},
   rot[i, n] = If[n > 0, rot[i, n - 1] kC_j, rot[i, n + 1] kC̄_j] // km_{i,j→i}];
```

```
Z[K_] := Z[RVK@K];
Z[rvk_RVK] := (*Z[rvk] =*)
 Module[{todo, n, rots, ξ, done, st, cx, ξ1, i, j, k, k1, k2, k3},
  {todo, rots} = List @@ rvk;
  AppendTo[rots, 0];
  n = Length[todo];
  ξ = 𝔼{}→{0}[0, 0, 1];
  done = {0};
  st = Range[0, 2 n + 1];
  While[{} =!= ($M = todo),
   {cx} = MaximalBy[todo, Length[done ⋂ {#⟦1⟧, #⟦2⟧, #⟦1⟧ - 1, #⟦2⟧ - 1}] &,
     1];
   {i, j} = List @@ cx;
   ξ1 = Switch[Head[cx],
      Xp, (kR_{i,j} OverBar[kKink_k]) // km_{j,k→j},
      Xm, (OverBar[kR_{i,j}] kKink_k) // km_{j,k→j}
     ];
   ξ1 = (rot[k, rots⟦i⟧] ξ1) // km_{k,i→i}; rots⟦i⟧ = 0;
   ξ1 = (ξ1 rot[k, rots⟦i + 1⟧]) // km_{i,k→i}; rots⟦i + 1⟧ = 0;
   ξ1 = (rot[k, rots⟦j⟧] ξ1) // km_{k,j→j}; rots⟦j⟧ = 0;
   ξ1 = (ξ1 rot[k, rots⟦j + 1⟧]) // km_{j,k→j}; rots⟦j + 1⟧ = 0;
   ξ *= ξ1;
   If[MemberQ[done, i], ξ = ξ // km_{i,i+1→i}; st = st /. st⟦i + 2⟧ → st⟦i + 1⟧];
   If[MemberQ[done, i - 1], ξ = ξ // km_{st⟦i⟧,i→st⟦i⟧}; st = st /. st⟦i + 1⟧ → st⟦i⟧]];
   If[MemberQ[done, j], ξ = ξ // km_{j,j+1→j}; st = st /. st⟦j + 2⟧ → st⟦j + 1⟧];
   If[MemberQ[done, j - 1], ξ = ξ // km_{st⟦j⟧,j→st⟦j⟧}; st = st /. st⟦j + 1⟧ → st⟦j⟧]];
   done = done ⋃ {i - 1, i, j - 1, j};
   todo = DeleteCases[todo, cx]
  ];
  CF /@ (ξ /. {x_0 → x, y_0 → y, a_0 → a})
 ]
```

```
Z@Knot[3, 1]
```

KnotTheory: Loading precomputed data in PD4Knots`.

$$\mathbb{E}_{\{\}\to\{0\}}\left[0,\,0,\,\frac{T}{1-T+T^2}+\left(\frac{a\left(-2\,T\,\hbar+2\,T^3\,\hbar\right)}{1-2\,T+3\,T^2-2\,T^3+T^4}+\frac{-2\,T\,\hbar+3\,T^2\,\hbar-2\,T^3\,\hbar+T^4\,\hbar}{1-3\,T+6\,T^2-7\,T^3+6\,T^4-3\,T^5+T^6}+\frac{x\,y\left(-2\,T\,\hbar^2-2\,T^2\,\hbar^2\right)}{1-2\,T+3\,T^2-2\,T^3+T^4}\right)\epsilon+O[\epsilon]^2\right]$$

```
R_{1,2} R_{3,4} // dm_{1,3→5}
```

$$\mathbb{E}_{\{\}\to\{2,4,5\}}\left[a_2\,b_5+a_4\,b_5,\;x_2\,y_5+x_4\,y_5,\;1+\left(-a_2\,x_4\,y_5-\frac{1}{4}\,x_2^2\,y_5^2-\frac{1}{4}\,x_4^2\,y_5^2\right)\epsilon+O[\epsilon]^2\right]$$

```
kR̄1,2 kR̄3,4 // tm1,4→5
```

$$\mathbb{E}_{\{\}\to\{2,3,5\}}\left[-t\,a_2-t\,a_5,\;-\frac{x_5\,y_3}{T}-\frac{x_2\,y_5}{T},\right.$$
$$\left.1+\left(-a_2\,a_5-a_3\,a_5-\frac{a_3\,x_5\,y_3}{T}-\frac{a_5\,x_5\,y_3}{T}-\frac{3\,x_5^2\,y_3^2}{4\,T^2}-\frac{a_2\,x_2\,y_5}{T}-\frac{a_5\,x_2\,y_5}{T}-\frac{3\,x_2^2\,y_5^2}{4\,T^2}\right)\epsilon+O[\epsilon]^2\right]$$

```
kR̄1,2 kR̄3,4 // tm1,4→5
```

```
(*Working Casimir, not unique!*)
Define[
  ωi = E{}→{i}[0, 0, Series[y e^(ϵ a) x + (e^(ϵ (a+1)) + e^(-ϵ a) T)/(e^ϵ - 1) - (T + 1) ϵ^-1, {ϵ, 0, 3}] /.
     {a → ai, T → Ti, x → xi, y → yi}]
]
ωsq = ω1 ω2 // tm1,2→1;
ωcub = ωsq ω2 // tm1,2→1;
ω4 = ωcub ω2 // tm1,2→1;
(*Cleaned versions*)
ωc = ω1[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωsqc = ωsq[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ωcubc = ωcub[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
ω4c = ω4[[3]] /. {T1 → T, a1 → a, x1 → x, y1 → y} // Normal;
```