Roland: I wrote a small note on the central element w = YX+F(a) for some explicit F(a,T,epsilon) computed in the attached mathematica file.
Effective use of w may very well simplify stitching dramatically.
For example Y^nX^n = (w-F(a))(w-F(a-1))...(w-F(a-n+1))
This is a way to hide epsilon from the commutation relations since now ONLY Xa = (a-1)X and Ya = (a+1)Y are used.

Notes on the Mixed order alternative

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1 The central element ω

$$[X,a] = -\gamma X \quad [a,Y] = -\gamma Y \quad [X,Y] = \frac{A^{-1}T^{-1} - AT}{h}$$

Is it always possible to recover γ, h??

we set $\gamma = h = 1$. Here $A = e^{-\epsilon a} = 1 - \epsilon a + \frac{\epsilon^2 a^2}{2}$ and $XA = qAX$ and $AY = qYA$ with $q = e^{\epsilon}$. We are looking for a central element $\omega = YX + F(a,T)$. It suffices to check $X\omega = \omega X$. Since $XF(a) = F(a-1)X$ we must have

$$X\omega = XYX + XF(a) = (YX + A^{-1}T^{-1} - AT + F(a-1))X = (\omega + A^{-1}T^{-1} - AT + F(a-1) - F(a))X = \omega X$$

Provided that F satisfies the difference equation

$$F(a) - F(a-1) = A^{-1}T^{-1} - AT = e^{\epsilon a}T^{-1} - e^{-\epsilon a}T$$

For $\epsilon = 0$ we see that we must have $F(a) = (T^{-1} - T)a$. To deal with the general case we first solve the easier problem $g(a) - g(a-1) = A^{-1}T^{-1} = e^{\epsilon a}T^{-1}$. The easiest solution is $g(a) = \lambda e^{\epsilon a}$ where plugging in reveals $\lambda = (1 - e^{-\epsilon})^{-1}$ however this is not suitable for nilpotent ϵ. Looking at the $\epsilon = 0$ case we might do better with a solution that starts like that. Let us now assume $\epsilon^N = 0$ for some fixed $N \in \mathbb{N}$.

Our ansatz is $F(a) = \sum_{k=0}^{N-1} \frac{f_k(a)}{k!}\epsilon^k$. So for all $0 \leq k < N$ we should solve

$$f_k(a) - f_k(a-1) = (T^{-1} - (-1)^k T)a^k$$

Write $f_k(a) = \sum_{j=1}^{k+1} c_{k,j} a^j$ so that

$$f_k(a) - f_k(a-1) = \sum_{j=1}^{k+1} c_{k,j}(a^j - (a-1)^j) = \sum_{j=1}^{k+1} c_{k,j} \sum_{i=0}^{j-1} \binom{j}{i} (-1)^{j-i+1} a^i =$$

$$\sum_{i=0}^{k} \sum_{j=i+1}^{k+1} c_{k,j} \binom{j}{i} (-1)^{j-i+1} a^i = (T^{-1} - (-1)^k T)a^k$$

This means that for all $i < k$ we have

$$\sum_{j=i+1}^{k+1} c_{k,j} \binom{j}{i} (-1)^{j-i+1} = 0 \quad c_{k,k+1} = \frac{T^{-1} - (-1)^k T}{k+1}$$

For fixed k, finding the coefficient vector $c_k = (c_{k,1}, \dots, c_{k,k+1})$ can be written in terms of the $(k+1) \times (k+1)$ matrix $M(k)$ with $M_{i,j} = \binom{j}{i}(-1)^{j-i+1}\delta_{j>i}$. We need to solve $Mc_k = (0,..,0,T^{-1}-(-1)^k T)^t$, where the t means transpose. For example the value of $\omega = YX + F(a)$ to order 4 in ϵ is, so $\epsilon^5 = 0$, $(N = 5)$:

$$\omega = YX+F(a) = YX+a(T^{-1}-T)+\frac{1}{2}a(a+1)(T^{-1}+T)\epsilon+\frac{1}{12}a(a+1)(2a+1)(T^{-1}-T)\epsilon^2$$

$$+\frac{1}{24}a^2(a+1)^2(T^{-1}+T)\epsilon^3+\frac{1}{720}a(a+1)(2a+1)\left(3a^2+3a-1\right)(T^{-1}-T)\epsilon^4$$

2 A new normal form

In our YAX algebra we may use the central elements T, ω to write every power series as $\sum_n P_n(a)X^n + Y^n Q_n(a)$. Here P_n, Q_n are power series (or just polynomials?) in a with coefficients polynomials in T, ω.

Using $YX = \omega - F(a)$ we may rewrite

$$Y^m X^n = \begin{cases} Y^{m-n}\prod_{j=0}^{|m-n|-1}(\omega - F(a-j)) & \text{if } m > n \\ \prod_{j=0}^{|m-n|-1}(\omega - F(a-j))X^{n-m} & \text{if } m \le n \end{cases}$$