Finding a dequantizator

November 6, 2017

1 Asymmetric dequantizator

We seek an isomorphism between the algebras $U_{h,\epsilon}$ and $U(\mathfrak{g}_\epsilon)[[h]]$. The algebra $U_{h,\epsilon}$ is generated by $x, \tilde{y}, a, t$ subject to the relations

$$xa = (a-1)x \quad a\tilde{y} = \tilde{y}(a-1) \quad x\tilde{y} = q\tilde{y}x + \frac{1-TA^2}{h}$$

Here $q = e^{\epsilon h}$ and $T = e^{th}$ and $A = e^{-\epsilon a h}$ as in Dror's monoblog (with $\gamma = 1$).

The algebra $U(\mathfrak{g}_\epsilon)[[h]]$ has generators x, y, a, t subject to the relations

$$xa = (a-1)x \quad ay = y(a-1) \quad xy = yx - t + 2\epsilon a$$

Notice that $\mathfrak{g}_\epsilon$ has a central 'Casimir' element $\omega = yx + j(a) = xy + j(a-1)$ where $j(a) = \epsilon a^2 - (t-\epsilon)a$ (we may add a constant to j of course). (Do ω, t generate the center of $U(\mathfrak{g}_\epsilon)[[h]]$?). To check this just write $x\omega - \omega x = xyx + xj(a) - yx^2 - j(a)x = (-t + 2\epsilon a + j(a-1) - j(a))x = 0$ etc.

As a first step we make the presentation of $U_{h,\epsilon}$ more symmetric by the substitution $Y = \frac{1}{2}T^{-\frac{1}{2}}A^{-1}\tilde{y}$. This simplifies the presentation of $U_{h,\epsilon}$ to

$$xa = (a-1)x \quad aY = Y(a-1) \quad xY = Yx + \frac{e^{-\epsilon h}}{h}\sinh(h(-\frac{t}{2} + \epsilon a))$$

Indeed, this follows from $xA^{-1} = q^{-1}A^{-1}x$ and $xY = x\frac{1}{2}T^{-\frac{1}{2}}A^{-1}\tilde{y} = \frac{q^{-1}}{2}T^{-\frac{1}{2}}A^{-1}x\tilde{y} = \frac{q^{-1}}{2}T^{-\frac{1}{2}}A^{-1}(q\tilde{y}x + \frac{1-TA^2}{h}) = Yx + \frac{q^{-1}}{2h}T^{-\frac{1}{2}}A^{-1}(1-TA^2) = Yx + (hq)^{-1}\sinh(h(-\frac{t}{2} + \epsilon a))$.

As the first two relations are identical it is reasonable to look for a homomorphism $D : U_{h,\epsilon} \to U(\mathfrak{g}_\epsilon)[[h]]$ given by $D(x) = x, D(a) = a$ and $D(Y) = f(a,\omega,t)y$ for some f. All we need to check is that $[D(x), D(Y)] = e^{-\epsilon h}\sinh(h(-\frac{t}{2} + \epsilon a))$. We now find an f that makes this happen.

Using the Casimir element we rewrite the commutator as a difference:

$$[D(x), D(Y)] = xf(a)y - f(a)yx = f(a-1)xy - f(a)yx = f(a-1)(\omega - j(a-1)) - f(a)(\omega - j(a)) = F(a-1) - F(a)$$

where $F(a) = f(a)(\omega - j(a))$. To find D we just have to solve the equation $F(a-1) - F(a) = \frac{e^{-\epsilon h}}{h}\sinh(h(-\frac{t}{2} + \epsilon a))$ which is not hard given the identity $\cosh(r) - \cosh(s) = 2\sinh(\frac{r+s}{2})\sinh(\frac{r-s}{2})$. We find a solution $F(a) =$

$\frac{e^{-\epsilon h}}{h}\frac{\cosh(h(\frac{\epsilon-t}{2}+\epsilon a))+c(\omega,t)}{2\sinh\frac{-\epsilon h}{2}}$ for any central c. And since (formally) $f = F/(\omega - j)$ also complete the definition of D if the inverse makes sense:

$$D(Y) = fy = \frac{e^{-\epsilon h}}{h}\frac{\cosh(h(\frac{\epsilon-t}{2}+\epsilon a))+c(\omega,t)}{2\sinh(\frac{-\epsilon h}{2})(\omega - j(a))}y$$

This formula makes sense as a formal power series if we choose $c = -\cosh(h\sqrt{(\frac{t-\epsilon}{2})^2+\epsilon\omega})$. Indeed, $\frac{\cosh hu - \cosh hv}{u^2-v^2}$ makes sense as a power series in u^2, v^2 and in this case we set $u = \frac{\epsilon-t}{2}+\epsilon a$ and $v = \sqrt{(\frac{t-\epsilon}{2})^2+\epsilon\omega}$. It's not hard to see that $u^2 - v^2 = -\epsilon(\omega - j(a))$. The additional $-\epsilon$ that appears here can be taken from the $\sinh(-\epsilon/2)$ term in the denominator. In conclusion, an explicit formula for a dequantizator is:

Theorem 1. *The homomorphism $D : U_{h,\epsilon} \to U(\mathfrak{g}_\epsilon)[[h]]$ is defined by $x \mapsto x, a \mapsto a$ and*

$$D(\tilde{y}) = e^{h(\frac{t}{2}-\epsilon-\epsilon a)}\frac{\cosh(h(\frac{\epsilon-t}{2}+\epsilon a)) - \cosh(h\sqrt{(\frac{t-\epsilon}{2})^2+\epsilon\omega})}{h\sinh(\frac{-\epsilon h}{2})(\omega - \epsilon a^2 + (t-\epsilon)a)}y$$

and $D(\tilde{y}) = y \mod h$

Also, Chari-Pressley p.197 suggests to construct a Casimir for U_ϵ to find the inverse of D. It appears this can be done by solving more easy difference equations.

2 Cartan-symmetric dequantizator

In this section we attempt to find a special dequantizator homomorphism $\mathbb{D} : U_{h,\epsilon} \to U(\mathfrak{g}_\epsilon)[[h]]$ with the additional property that $\Theta /\!/ \mathbb{D} = \mathbb{D} /\!/ \theta$. Here the Cartan involutions are $\theta : U_{h,\epsilon} \to U_{h,\epsilon}$ given by $\Theta(\tilde{y}, a, \tilde{x}, t) = (-B^{-1}T^{\frac{1}{2}}\tilde{x}, -a, -A^{-1}T^{-\frac{1}{2}}\tilde{y}, -t)$ (note the sign of the exponent of T). with $B = TA$ and $\theta : U(\mathfrak{g}_\epsilon)[[h]] \to U(\mathfrak{g}_\epsilon)[[h]]$ given by $\theta(y, a, x, t) = (-x, -a, -y, -t)$.

As we would like to use the same technique we need to know how ω is moved by θ. $\theta(\omega) = xy + \epsilon a^2 - (t+\epsilon)a = \omega - t$. For this reason we may want to switch to $\bar{\omega} = \omega - \frac{t}{2} = yx + \epsilon a^2 - (t-\epsilon)a - \frac{t}{2} = yx + \bar{j}$ as it is Cartan invariant, where $\bar{j} = \epsilon a^2 - (t-\epsilon)a - \frac{t}{2}$

To avoid confusion we now use $\tilde{x}, \tilde{y}$ for the generators of $U_{h,\epsilon}$ with relations including $\tilde{x}\tilde{y} - q\tilde{y}\tilde{x} = (1 - TA^2)/h = \frac{1-e^{h(t-2\epsilon a)}}{h}$. To find $\mathbb{D}$ we assume $\mathbb{D}(a) = a$ and $\mathbb{D}(t) = t$. Now suppose $\mathbb{D}(\tilde{x}) = f(a,t,\bar{\omega})x$ and $D(\tilde{y}) = g(a,t,\bar{\omega})y$. From $\Theta /\!/ \mathbb{D} /\!/ \theta = \mathbb{D}$ it follows that $\theta(\mathbb{D}(\Theta(\tilde{x})) = \theta(\mathbb{D}(-A^{-1}T^{-\frac{1}{2}}\tilde{y})) = AT^{\frac{1}{2}}g(-a,-t)x = \mathbb{D}(\tilde{x}) = f(a,t)x$ so $f(a,t) = e^{-\epsilon a h}T^{\frac{1}{2}}g(-a,-t)$. It remains to find an expression for g.

$$\mathbb{D}(\tilde{x}\tilde{y} - q\tilde{y}\tilde{x}) = fxgy - qgyfx = f(a)g(a-1)xy - qg(a)f(a+1)yx$$

$$= f(a)g(a-1)(\bar{\omega} - \bar{j}(a-1)) - qf(a+1)g(a)(\bar{\omega} - \bar{j}(a)) = G(a-1) - qG(a)$$

if we set

$$G(a) = f(a+1)g(a)(\bar{\omega} - \bar{j}(a)) = e^{-\epsilon(a+1)h}T^{\frac{1}{2}}g(-a-1,-t)g(a,t)(\bar{\omega} - \bar{j}(a))$$

Or getting rid of the q:

$$\mathbb{D}(\tilde{x}\tilde{y} - q\tilde{y}\tilde{x}) = G(a-1) - qG(a) = e^{h(\frac{t}{2}-\epsilon ah)}(H(a-1) - H(a))$$

with $H(a) = g(-a-1,-t)g(a,t)(\bar{\omega} - \bar{j}(a))$ Our objective is to solve, first for H and then for g:

$$e^{h(\frac{t}{2}-\epsilon ah)}(H(a-1) - H(a)) = \frac{1 - e^{h(t-2\epsilon a)}}{h}$$

or equivalently:

$$H(a-1) - H(a) = e^{h(-\frac{t}{2}+\epsilon ah)}\frac{1 - e^{h(t-2\epsilon a)}}{h} = \frac{2}{h}\sinh(h(-\frac{t}{2} + \epsilon a))$$

As in the previous section the solution is $H(a) = \frac{1}{h}\frac{\cosh(h(\frac{\epsilon-t}{2}+\epsilon a))+c(\bar{\omega},t)}{\sinh\frac{-\epsilon h}{2}}$ for any central c. The c is chosen so that $H(a)/(\bar{\omega} - \bar{j}(a))$ makes sense as a series. As before we try to write it as $\frac{\cosh(hu)-\cosh(hv)}{u^2-v^2}$. More precisely set $u = \frac{\epsilon-t}{2} + \epsilon a$ and find v such that $u^2 - v^2 = -\epsilon(\bar{\omega} - \bar{j}(a))$, we get $v = \sqrt{\frac{t^2+\epsilon^2}{4} + \epsilon\omega}$. Therefore the following series makes sense:

$$\frac{H(a)}{\bar{\omega} - \bar{j}(a)} = \frac{1}{h}\frac{\cosh(h(\frac{\epsilon-t}{2} + \epsilon a)) - \cosh h(\sqrt{\frac{t^2+\epsilon^2}{4} + \epsilon\omega})}{\sinh(\frac{-\epsilon h}{2})(\bar{\omega} - \bar{j}(a))}$$

If we can find g such that $P(a,t) = \frac{H(a)}{\bar{\omega}-\bar{j}(a)} = g(-a-1,-t)g(a,t)$ then we are done. Making the substitution $c = a + \frac{1}{2}$ and $\gamma(c,t) = g(c - \frac{1}{2}, t)$ and $Q(c,t) = P(a - \frac{1}{2}, t)$ we get $Q(c,t) = \gamma(c,t)\gamma(-c,-t)$. Since $Q(c,t) = Q(-c,-t)$ (why?) we find a solution $\gamma(c,t) = \sqrt{Q(c,t)}$ and hence $g(a,t) = \sqrt{Q(a + \frac{1}{2}), t}$. Other solutions may be obtained by multiplying γ with multiplicatively odd functions.