A twisted Yetter-Drinfeld version of ρ_1

Mixing Garoufalidis-Kashaev section 6 with Gaussians and q = 1+h. Scroll to the very end of this file to see the knot invariants. They (experimentally) match exactly the 2-loop polynomials previously computed by other means by Ohtsuki, Rozansky and many others.

In[∘]:= **Internal Utilities and familiar things from our Gaussian calculus**

Out[∘]=

```
and calculus familiar from Gaussian Internal our things Utilities
```

In[∘]:=
```
Quiet@Once[<< KnotTheory`];
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
CCF[ℰ_] := ExpandDenominator@ExpandNumerator@Together[
    Expand[ℰ] //. e^x_ e^y_ :> e^(x+y) /. e^x_ :> e^CCF[x]];
CF[ℰ_List] := CF /@ ℰ;
CF[sd_SeriesData] := MapAt[CF, sd, 3];
CF[ℰ_] := Module[
   {vs = Cases[ℰ, (x | ξ)_, ∞] ⋃ {x, ξ}},
   Total[CoefficientRules[Expand[ℰ], vs] /. (ps_ → c_) :> CCF[c] (Times @@ vs^ps)]
  ];
CF[ℰ_𝔼] := CF /@ ℰ; CF[𝔼_sp___[ℰs___]] := CF /@ 𝔼_sp[ℰs];
```

In[∘]:=
```
Kδ /: Kδ_{i_,j_} := If[i === j, 1, 0];
```

In[∘]:=
```
𝔼 /: 𝔼[L1_, Q1_, P1_] ≡ 𝔼[L2_, Q2_, P2_] :=
  CF[L1 == L2] ∧ CF[Q1 == Q2] ∧ CF[Normal[P1 - P2] == 0];
𝔼 /: 𝔼[L1_, Q1_, P1_] 𝔼[L2_, Q2_, P2_] := 𝔼[L1 + L2, Q1 + Q2, P1 * P2];
```

In[∘]:=
```
x* = ξ; ξ* = x; (u_i_)* := (u*)_i;
```

In[∘]:=
```
D_v_[f_] := ∂_v f; D_{v_,0}[f_] := f; D_{}[f_] := f; D_{v_,n_Integer}[f_] := D_v[D_{v,n-1}[f]];
D_{l_List,ls___}[f_] := D_{ls}[D_l[f]];
```

In[∘]:=
```
collect[sd_SeriesData, ζ_] := MapAt[collect[#, ζ] &, sd, 3];
collect[ℰ_, ζ_] := Collect[ℰ, ζ];
Zip_{}[P_] := P;
Zip_ζs_[Ps_List] := Zip_ζs /@ Ps;
Zip_{ζ_,ζs___}[P_] :=
 (collect[P // Zip_{ζs}, ζ] /. f_. ζ^d_. :> (D_{ζ*,d}[f])) /. ζ* → 0 /. (ζ* → 1)
```

QZip implements the "Q-level zips" on $\mathbb{E}(L, Q, P) = e^{L+Q} P(\epsilon)$. Such zips regard the L variables as scalars.

$$\left\langle P(z_i, \zeta^j) e^{c+\eta^i z_i + y_j \zeta^j + q^i_j z_i \zeta^j} \right\rangle = |\tilde{q}| \left\langle P(z_i, \zeta^j) e^{c+\eta^i z_i} \Big|_{z_i \to \tilde{q}^k_i (z_k + y_k)} \right\rangle$$
$$= |\tilde{q}| e^{c+\eta^i \tilde{q}^k_i y_k} \left\langle P\left(\tilde{q}^k_i (z_k + y_k), \zeta^j + \eta^i \tilde{q}^j_i\right) \right\rangle.$$

```
In[]:= QZip_ζs_List@𝔼[L_, Q_, P_] := Module[{ζ, z, zs, c, ys, ηs, qt, zrule, ζrule, out},
    zs = Table[ζ*, {ζ, ζs}];
    c = CF[Q /. Alternatives @@ (ζs ⋃ zs) → 0];
    ys = CF@Table[∂_ζ (Q /. Alternatives @@ zs → 0), {ζ, ζs}];
    ηs = CF@Table[∂_z (Q /. Alternatives @@ ζs → 0), {z, zs}];
    (*Echo@MatrixForm@Table[∂_{z,ζ}Q,{ζ,ζs},{z,zs}];*)
    qt = CF@Inverse@Table[Kδ_{z,ζ*} - ∂_{z,ζ}Q, {ζ, ζs}, {z, zs}];
    zrule = Thread[zs → CF[qt.(zs + ys)]];
    ζrule = Thread[ζs → ζs + ηs.qt];
    CF /@ 𝔼[L, c + ηs.qt.ys, Det[qt] Zip_ζs[P /. (zrule ⋃ ζrule)]] ];
```

```
In[]:= B_{}[L_, R_] := L R;
B_{is__}[L_𝔼, R_𝔼] := Module[{n},
    Times[
      L /. Table[x_i → x_{n@i}, {i, {is}}],
      R /. Table[ξ_i → ξ_{n@i}, {i, {is}}]
     ] // QZip_{Join@@Table[{ξ_{n@i}},{i,{is}}]} ];
B_is___[L_, R_] := B_{is}[L, R];
```

```
In[]:= B_is_List[𝔼_{d1_→r1_}[L1_, Q1_, P1_], 𝔼_{d2_→r2_}[L2_, Q2_, P2_]] :=
   𝔼_{(d1⋃Complement[d2,is])→(r2⋃Complement[r1,is])} @@ B_is[𝔼[L1, Q1, P1], 𝔼[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] // 𝔼_{d2_→r2_}[L2_, Q2_, P2_] :=
   B_{r1⋂d2}[𝔼_{d1→r1}[L1, Q1, P1], 𝔼_{d2→r2}[L2, Q2, P2]];
𝔼_{d1_→r1_}[L1_, Q1_, P1_] ≡ 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
   (d1 == d2) ∧ (r1 == r2) ∧ (𝔼[L1, Q1, P1] ≡ 𝔼[L2, Q2, P2]);
𝔼_{d1_→r1_}[L1_, Q1_, P1_] 𝔼_{d2_→r2_}[L2_, Q2_, P2_] ^:=
   𝔼_{(d1⋃d2)→(r1⋃r2)} @@ (𝔼[L1, Q1, P1] 𝔼[L2, Q2, P2]);
𝔼_[ℰ___][i_] := {ℰ}〚i〛;
```

In[∘]:=
```
$k = 0;
SetAttributes[Define, HoldAll];
Define[def_, defs__] := (Define[def]; Define[defs];);
Define[op_is__ = ℰ_] := Module[{SD, ii, jj, kk, isp, nis, nisp, sis}, Block[{i, j, k},
  ReleaseHold[Hold[
    SD[op_{nisp,$k_Integer}, Block[{i, j, k}, op_{isp,$k} = ℰ; op_{nis,$k}]];
    SD[op_isp, op_{{is},$k}]; SD[op_sis__, op_{{sis}}];
   ] /. {SD → SetDelayed,
    isp → {is} /. {i → i_, j → j_, k → k_},
    nis → {is} /. {i → ii, j → jj, k → kk},
    nisp → {is} /. {i → ii_, j → jj_, k → kk_}
   }] ]]
```

The new objects:

In[∘]:=
```
$n = 1; Define[
 nI_i = 𝔼_{{i}→{i}}[0, ξ_i x_i, 1],
 nm_{i,j→k} = 𝔼_{{i,j}→{k}}[0, (ξ_i + ξ_j) x_k, 1],
 nτ_{i,j} = 𝔼_{{i,j}→{i,j}}[0, ξ_i x_i + ξ_j x_j, 1 + ℏ ξ_i x_i ξ_j x_j + O[ℏ]^2],
 \overline{nτ}_{i,j} = 𝔼_{{i,j}→{i,j}}[0, ξ_i x_i + ξ_j x_j, 1 - ℏ ξ_i x_i ξ_j x_j + O[ℏ]^2] (**),
 nS_i = 𝔼_{{i}→{i}}[0, -ξ_i x_i, 1 + ℏ (ξ_i^2 x_i^2)/2 + O[ℏ]^2],
 \overline{nS}_i = 𝔼_{{i}→{i}}[0, -ξ_i x_i, 1 - ℏ (ξ_i^2 x_i^2)/2 + O[ℏ]^2],
 nΔ_{i→j,k} = 𝔼_{{i}→{j,k}}[0, (x_j + x_k) ξ_i, 1 + ℏ (ξ_i^2 x_j x_k)/2 + O[ℏ]^2],
 nϕ_i = 𝔼_{{i}→{i}}[0, T ξ_i x_i, 1],
 \overline{nϕ}_i = 𝔼_{{i}→{i}}[0, T^-1 ξ_i x_i, 1],
 nδ_{i→j,k} = nΔ_{i→1,2} // nΔ_{2→k,3} // nS_3 // nϕ_3 // nτ_{k,3} // nm_{1,3→j},
 nR_{i,j} = nϕ_j // nδ_{i→1,i} // nτ_{i,j} // nm_{1,j→j},
 \overline{nR}_{i,j} = nδ_{i→z1,i} // \overline{nτ}_{z1,j} // \overline{nS}_{z1} // \overline{nτ}_{i,j} // \overline{nτ}_{i,z1} // nm_{z1,j→j} // \overline{nϕ}_j,
 nC_i = 𝔼_{{i}→{i}}[0, ξ_i x_i, 1 + ξ_i x_i ℏ + O[ℏ]^2], \overline{nC}_i = 𝔼_{{i}→{i}}[0, ξ_i x_i, 1 - ξ_i x_i ℏ + O[ℏ]^2],
 nKinkr_i = (nR_{1,3} \overline{nC}_2 // nμ_{1,2→1} // nμ_{1,3→i}),
 nKinkl_i = (nR_{3,1} nC_2 // nμ_{1,2→1} // nμ_{1,3→i}),
 \overline{nKinkr}_i = (\overline{nR}_{3,1} \overline{nC}_2 // nμ_{1,2→1} // nμ_{1,3→i}),
 \overline{nKinkl}_i = (\overline{nR}_{1,3} nC_2 // nμ_{1,2→1} // nμ_{1,3→i}) ]
```

In[∘]:=
```
(*For composing tensors, leg by leg*)
nμ_{i_,j_→k_}[𝔼_{d_→r_}[0, Q_, P_]] := Module[{v},
  QZip_{{ξv}}[
   𝔼[0, Q /. {ξj → ξv, xi → xv, ξi → ξk, xj → xk}, P /. {ξj → ξv, xi → xv, ξi → ξk, xj → xk}]]
  ] /. 𝔼 → 𝔼_{(d/.{i→Nothing,j→k})→(r/.{i→Nothing,j→k})}
```

In[∘]:= $n\delta_{i\to j,k}$

Out[∘]=

$$\mathbb{E}_{\{i\}\to\{j,k\}}\left[0,\ (1-T)\,x_j\,\xi_i + x_k\,\xi_i,\ 1+\left(\frac{1}{2}\left(-T+T^2\right)x_j^2\,\xi_i^2+\frac{1}{2}\,(1-3\,T)\,x_j\,x_k\,\xi_i^2\right)\hbar+O[\hbar]^2\right]$$

In[∘]:= $nR_{i,j}$

Out[∘]=

$$\mathbb{E}_{\{i,j\}\to\{i,j\}}\left[0,\ x_i\,\xi_i+(1-T)\,x_j\,\xi_i+T\,x_j\,\xi_j,\ 1+\left(\frac{1}{2}\,(1-3\,T)\,x_i\,x_j\,\xi_i^2+\frac{1}{2}\left(-T+T^2\right)x_j^2\,\xi_i^2+T\,x_i\,x_j\,\xi_i\,\xi_j\right)\hbar+O[\hbar]^2\right]$$

In[∘]:= $n\phi_j$ // $n\delta_{i\to 1,i}$ // $n\tau_{i,j}$ // $nm_{1,j\to j}$

Out[∘]=

$$\mathbb{E}_{\{i,j\}\to\{i,j\}}\left[0,\ x_i\,\xi_i+(1-t)\,x_j\,\xi_i+t\,x_j\,\xi_j,\ 1+\left(\frac{1}{2}\,(1-3\,t)\,x_i\,x_j\,\xi_i^2+\frac{1}{2}\left(-t+t^2\right)x_j^2\,\xi_i^2+t\,x_i\,x_j\,\xi_i\,\xi_j\right)\hbar+O[\hbar]^2\right]$$

In[∘]:= nC_1 // $\overline{nC}_1$

Out[∘]=

$$\mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ 1+O[\hbar]^2\right]$$

In[∘]:=

$$\text{Table}\Big[\partial_{\xi_v}\partial_{x_u}\Big(x_{il}\,\xi_{il}+(1-T)\,x_{jl}\,\xi_{il}+(-1+T)\,x_{jr}\,\xi_{il}+\frac{x_{ir}\,\xi_{ir}}{T}+(1-T)\,x_{jr}\,\xi_{ir}+\frac{(-1+T)\,x_{ir}\,\xi_{jl}}{T}+T\,x_{jl}\,\xi_{jl}+x_{jr}\,\xi_{jr}\Big),$$
$$\{u,\{il,ir,jl,jr\}\},\{v,\{il,ir,jl,jr\}\}\Big]\ //\ \text{Expand}\ //\ \text{MatrixForm}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{T} & 1-\frac{1}{T} & 0 \\ 1-T & 0 & T & 0 \\ -1+T & 1-T & 0 & 1 \end{pmatrix}$$

In[∘]:=

$$\text{Det}\left[\text{IdentityMatrix}[6]-\begin{pmatrix} 0 & T & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & T & 0 \\ 0 & 1-T & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right]$$

Out[∘]=

$$1-T+T^2$$

In[∘]:= $\text{QZip}_{\text{Table}[\xi_v,\{v,1,6\}]}\left[\mathbb{E}\ @@\ \left(\left(nR_{1,5}\, nR_{6,2}\, nR_{3,7}\, \overline{nC}_4\right)\ /.\ \{\xi_{u\_} \mapsto \xi_{u-1}\}\right)\right]$

Out[∘]=

$$\mathbb{E}\left[0,\ x_7\,\xi_0,\ \frac{1}{1-T+T^2}+\frac{\left(-T+2T^2-3T^3+2T^4\right)\hbar}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+O[\hbar]^2\right]$$

In[∘]:= $nR_{1,5}\, nR_{6,2}\, nR_{3,7}\, \overline{nC}_4\ //\ n\mu_{1,2\to1}\ //\ n\mu_{1,3\to1}\ //\ n\mu_{1,4\to1}\ //\ n\mu_{1,5\to1}\ //\ n\mu_{1,6\to1}\ //\ n\mu_{1,7\to1}$

Out[∘]=

$$\mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ \frac{1}{1-T+T^2}+\frac{\left(-T+2T^2-3T^3+2T^4\right)\hbar}{1-3T+6T^2-7T^3+6T^4-3T^5+T^6}+O[\hbar]^2\right]$$

In[∘]:= $Mx_{i\_,j\_} := x_i\,\xi_i + (1-T)\,x_j\,\xi_i + T\,x_j\,\xi_j$

$\overline{Mx}_{i\_,j\_} := x_i\,\xi_i + (1-T)\,x_j\,\xi_i + T\,x_j\,\xi_j$

In[∘]:= $\text{Mat} = \text{Table}\left[\partial_{\xi_j}\partial_{x_i}\left(Mx_{1,4}+Mx_{5,2}+Mx_{3,6}\right),\ \{i,1,6\},\ \{j,1,6\}\right];$

% // MatrixForm

$\text{Mat} = \text{Table}\left[\partial_{\xi_j}\partial_{x_{i-1}}\left(Mx_{1,4}+Mx_{5,2}+Mx_{3,6}\right),\ \{i,2,7\},\ \{j,2,7\}\right];$

% // MatrixForm

Det[IdentityMatrix[6] − Mat]

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 1-T & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1-T & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1-T & 0 & 0 & T \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ T & 0 & 0 & 1-T & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1-T & 0 & 0 & T & 0 \end{pmatrix}$$

Out[∘]=

$1-T+T^2$

In[∘]:= $nR_{a,3}\, n\Delta_{b\to1,2}\ //\ n\mu_{a,b\to c}$

In[∘]:= $\mathbb{E}_{\{3,c\}\to\{1,2,3\}}\Big[0,\ T\,x_3\,\xi_3 + (1-T)\,\xi_c + x_1\,\xi_c + x_2\,\xi_c,\ 1 + \Big(t\,x_1\,x_3\,\xi_3\,\xi_c + t\,x_2\,x_3\,\xi_3\,\xi_c + \frac{1}{2}\left(-t+t^2\right)\xi_c^2 + \frac{1}{2}(1-3t)\,x_1\,\xi_c^2 + \frac{1}{2}(1-3t)\,x_2\,\xi_c^2 + \frac{1}{2}x_1\,x_2\,\xi_c^2\Big)\hbar + O[\hbar]^2 \;/.\; t\to T\Big]$

Out[∘]=

$$\mathbb{E}_{\{3,c\}\to\{1,2,3\}}\Big[0,\ T\,x_3\,\xi_3 + (1-T)\,\xi_c + x_1\,\xi_c + x_2\,\xi_c,\ 1 + \Big(T\,x_1\,x_3\,\xi_3\,\xi_c + T\,x_2\,x_3\,\xi_3\,\xi_c + \frac{1}{2}\left(-T+T^2\right)\xi_c^2 + \frac{1}{2}(1-3T)\,x_1\,\xi_c^2 + \frac{1}{2}(1-3T)\,x_2\,\xi_c^2 + \frac{1}{2}x_1\,x_2\,\xi_c^2\Big)\hbar + O[\hbar]^2\Big]$$

In[∘]:= $nR_{a,3}$ // $n\Delta_{a\to1,2}$

Out[∘]=

$$\mathbb{E}_{\{3,a\}\to\{1,2,3\}}\Big[0,\ T\,x_3\,\xi_3 + x_1\,\xi_a + x_2\,\xi_a + (1-T)\,x_3\,\xi_a,\ 1 + \Big(T\,x_1\,x_3\,\xi_3\,\xi_a + T\,x_2\,x_3\,\xi_3\,\xi_a + \frac{1}{2}x_1\,x_2\,\xi_a^2 + \frac{1}{2}(1-3T)\,x_1\,x_3\,\xi_a^2 + \frac{1}{2}(1-3T)\,x_2\,x_3\,\xi_a^2 + \frac{1}{2}\left(-T+T^2\right)x_3^2\,\xi_a^2\Big)\hbar + O[\hbar]^2\Big]$$

In[∘]:= $nR_{1,3}\,nR_{2,4}$ // $n\mu_{4,3\to3}$
$nR_{2,3}$ // $nR_{1,3}$

Out[∘]=

$$\mathbb{E}_{\{1,2,3\}\to\{1,2,3\}}\Big[0,\ x_1\,\xi_1 + (1-T)\,x_3\,\xi_1 + x_2\,\xi_2 + \left(T-T^2\right)x_3\,\xi_2 + T^2\,x_3\,\xi_3,\ 1 + \Big(\frac{1}{2}(1-3T)\,x_1\,x_3\,\xi_1^2 + \frac{1}{2}\left(-T+T^2\right)x_3^2\,\xi_1^2 + \left(T-T^2\right)x_1\,x_3\,\xi_1\,\xi_2 + \frac{1}{2}\left(T-3T^2\right)x_2\,x_3\,\xi_2^2 + \frac{1}{2}\left(-T^3+T^4\right)x_3^2\,\xi_2^2 + T^2\,x_1\,x_3\,\xi_1\,\xi_3 + T^2\,x_2\,x_3\,\xi_2\,\xi_3\Big)\hbar + O[\hbar]^2\Big]$$

Out[∘]=

$$\mathbb{E}_{\{1,2,3\}\to\{1,2,3\}}\Big[0,\ x_1\,\xi_1 + (1-T)\,x_3\,\xi_1 + x_2\,\xi_2 + \left(T-T^2\right)x_3\,\xi_2 + T^2\,x_3\,\xi_3,\ 1 + \Big(\frac{1}{2}(1-3T)\,x_1\,x_3\,\xi_1^2 + \frac{1}{2}\left(-T+T^2\right)x_3^2\,\xi_1^2 + \left(T-T^2\right)x_1\,x_3\,\xi_1\,\xi_2 + \frac{1}{2}\left(T-3T^2\right)x_2\,x_3\,\xi_2^2 + \frac{1}{2}\left(-T^3+T^4\right)x_3^2\,\xi_2^2 + T^2\,x_1\,x_3\,\xi_1\,\xi_3 + T^2\,x_2\,x_3\,\xi_2\,\xi_3\Big)\hbar + O[\hbar]^2\Big]$$

Testing:

In[∘]:=
$\mathbf{nR_{1,2}}$
$\overline{\mathbf{nR}}_{1,2}$

Out[∘]=
$$\mathbb{E}_{\{1,2\}\to\{1,2\}}\Big[0,\ x_1\,\xi_1+(1-T)\,x_2\,\xi_1+T\,x_2\,\xi_2,\ 1+\left(\frac{1}{2}\,(1-3\,T)\,x_1\,x_2\,\xi_1^2+\frac{1}{2}\,\left(-T+T^2\right)\,x_2^2\,\xi_1^2+T\,x_1\,x_2\,\xi_1\,\xi_2\right)\hbar+O[\hbar]^2\Big]$$

Out[∘]=
$$\mathbb{E}_{\{1,2\}\to\{1,2\}}\Big[0,\ x_1\,\xi_1+\frac{(-1+T)\,x_2\,\xi_1}{T}+\frac{x_2\,\xi_2}{T},\ 1+\left(\frac{(1+T)\,x_1\,x_2\,\xi_1^2}{2\,T}+\frac{(-1+T)\,x_2^2\,\xi_1^2}{2\,T^2}-\frac{x_1\,x_2\,\xi_1\,\xi_2}{T}+\frac{(1-T)\,x_2^2\,\xi_1\,\xi_2}{T^2}\right)\hbar+O[\hbar]^2\Big]$$

In[∘]:= (*Reidemeister 3 in two ways*)
$(\mathbf{nR_{1,2}\ //\ nR_{1,3}\ //\ nR_{2,3}}) \equiv (\mathbf{nR_{2,3}\ //\ nR_{1,3}\ //\ nR_{1,2}})$

$(\mathbf{nR_{1,2}\,nR_{4,3}\,nR_{5,6}\ //\ n\mu_{1,4\to1}\ //\ n\mu_{2,5\to2}\ //\ n\mu_{3,6\to3}}) \equiv (\mathbf{nR_{1,6}\,nR_{2,3}\,nR_{4,5}\ //\ n\mu_{1,4\to1}\ //\ n\mu_{2,5\to2}\ //\ n\mu_{3,6\to3}})$

Out[∘]=
True

Out[∘]=
True

In[∘]:= (*Reidemeister 2b in two ways*)
$\left(\mathbf{nR_{1,2}\,\overline{nR}_{3,4}\ //\ n\mu_{1,3\to1}\ //\ n\mu_{2,4\to2}}\right) \equiv \mathbf{nI_1\,nI_2}$
$\left(\mathbf{nR_{1,2}\ //\ \overline{nR}_{1,2}}\right) \equiv \mathbf{nI_1\,nI_2}$

Out[∘]=
True

Out[∘]=
True

(*Reidemeister 2c works with an almost trivial C...*)

In[∘]:=
$\left(\mathbf{\overline{nR}_{1,4}\,nR_{3,2}\,nC_6\ //\ n\mu_{1,3\to1}\ //\ n\mu_{2,6\to2}\ //\ n\mu_{2,4\to2}}\right) \equiv \mathbf{nI_1\,nC_2}$

Out[∘]=
True

In[∘]:=
```
(*Explicit Figure eight knot*)
Z41 = nR̄8,1 nR10,3 nR5,9 nR̄2,6 nC4 nC̄7;
Do[Z41 = Z41 // nμ1,j→1, {j, 2, 10}];
Z41 // Factor
```

Out[∘]=

$$\mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ -\frac{T}{1-3T+T^2}+\frac{\left(T-T^3\right)\hbar}{1-6T+11T^2-6T^3+T^4}+O[\hbar]^2\right]$$

In[∘]:=
```
(*Kinks don't match well, so all four need to be considered*)
{nKinkl1, nKinkr1, nKinkl̄1, nKinkr̄1}
```

Out[∘]=

$$\left\{\mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ \frac{1}{T}+O[\hbar]^2\right],\ \mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ 1+O[\hbar]^2\right],\ \mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ 1+O[\hbar]^2\right],\ \mathbb{E}_{\{1\}\to\{1\}}\left[0,\ x_1\,\xi_1,\ T+O[\hbar]^2\right]\right\}$$

In[∘]:=
```
RVK::usage =
  "RVK[xs, rots] represents a Rotational Virtual Knot with a list of n Xp/Xm
    crossings xs and a length 2n list of rotation numbers rots. Crossing
    sites are indexed 1 through 2n, and rots⟦k⟧ is the rotation
    between site k-1 and site k. RVK is also a casting operator
    converting to the RVK presentation from other knot presentations.";
```

In[∘]:=
```
RVK[pd_PD] := Module[{n, xs, x, rots, front = {0}, k},
   n = Length@pd; rots = Table[0, {2 n}];
   xs = Cases[pd, x_X :> { Xp[x⟦4⟧, x⟦1⟧]  PositiveQ@x
                         { Xm[x⟦2⟧, x⟦1⟧]  True        ];
   For[k = 0, k < 2 n, ++k, If[k == 0 ∨ FreeQ[front, -k],
     front = Flatten@Replace[front, k → (xs /. {
            Xp[k + 1, l_] | Xm[l_, k + 1] :> {l, k + 1, 1 - l},
            Xp[l_, k + 1] | Xm[k + 1, l_] :> (++rots⟦l⟧;
              {1 - l, k + 1, l}),
            _Xp | _Xm :> {}
           }), {1}],
     Cases[front, k | -k] /. {k, -k} :> --rots⟦k + 1⟧;
    ]];
   RVK[xs, rots] ];
RVK[K_] := RVK[PD[K]];
```

In[∘]:=
```
rot[i_, 0] := nIi;
rot[i_, n_] := rot[i, n, $k];
rot[i_, n_, k_] := Module[{j},
   rot[i, n, k] = If[n > 0, rot[i, n - 1] nCj, rot[i, n + 1] nC̄j] // nμi,j→i];
```

```
In[◦]:= Width[pd_PD] :=
         Max[Length /@ FoldList[Complement[#1 ⋃ #2, #1 ⋂ #2] &, {}, List @@ List @@@ pd]]
```

```
In[◦]:= ThinPosition[K_] := Module[{todo, done, pd, c},
           todo = List @@ PD@K; done = {}; pd = PD[];
           While[todo =!= {},
            AppendTo[pd, c = RandomChoice@MaximalBy[todo, Length[done ⋂ List @@ #] &]];
            todo = DeleteCases[todo, c];
            done = done ⋃ List @@ c];
           pd];
        ThinPosition[K_, n_] := First@MinimalBy[Table[ThinPosition[K], n], Width];
```

```
In[◦]:= Z[K_] := Z[RVK@ThinPosition[K, 100]];
        Z[rvk_RVK] := Monitor[Module[{ζ, done, st, c, χ, i, j, k, totrot = Total[rvk⟦2⟧]},
           ζ = 1; done = {}; st = Range[2 Length[rvk⟦1⟧]]; $M = {};
           Do[AppendTo[$M, c];
            {i, j} = List @@ c;
            If[Head[c] === Xp, If[totrot ≤ 0, totrot++;
               χ = OverBar[nKinkl_0] nR_{i,j} // nμ_{j,0→j}, totrot--;
               χ = OverBar[nKinkr_0] nR_{i,j} // nμ_{j,0→j}] ,
             If[totrot ≤ 0, totrot++;
               χ = nKinkl_0 OverBar[nR]_{i,j} // nμ_{j,0→j}, totrot--;
               χ = nKinkr_0 OverBar[nR]_{i,j} // nμ_{j,0→j}]
              ];
            Do[χ = (rot[0, rvk⟦2, k⟧] χ) // nμ_{0,k→k}, {k, {i, j}}];
            ζ *= χ;
            Do[
             If[MemberQ[done, k + 1], ζ = ζ // nμ_{k,k+1→k}; st = st /. k + 1 → k];
             If[MemberQ[done, k - 1], ζ = ζ // nμ_{st⟦k-1⟧,k→st⟦k-1⟧}; st = st /. k → st⟦k - 1⟧],
             {k, {i, j}}];
            done = done ⋃ {i, j},
            {c, rvk⟦1⟧}
           ];
           CF /@ ζ
          ], {Length@$M, $M}]
        aTerm[K_] := T (D[Alexander[K][T], T])/((Alexander[K][T])^2)
        ρ_1[K_] := Module[{z = Z[K]⟦3⟧, Al}, Al = 1 / (z /. ℏ → 0);
           (-T Al^3)/((T - 1)^2) (Coefficient[z, ℏ] - T (D[Al, T])/(Al^2))] // Factor
```

Knot table. Note how ρ_1 matches precisely with the original ρ_1 from the Drinfeld

double.

In[∘]:=
```
Quiet@ρ1 /@ AllKnots[{3, 8}] // Column
```

KnotTheory: Loading precomputed data in PD4Knots`.

Out[∘]=

$$\frac{1+T^2}{T}$$
$$0$$
$$\frac{\left(1+T^2\right)\left(2+T^2+2T^4\right)}{T^3}$$
$$\frac{5-4T+5T^2}{T}$$
$$\frac{1-4T+T^2}{T}$$
$$\frac{1-4T+4T^2-4T^3+4T^4-4T^5+T^6}{T^3}$$
$$0$$
$$\frac{\left(1+T^2\right)\left(3+2T^2+4T^4+2T^6+3T^8\right)}{T^5}$$
$$\frac{2\left(7-8T+7T^2\right)}{T}$$
$$-\frac{9-8T+16T^2-12T^3+16T^4-8T^5+9T^6}{T^3}$$
$$-\frac{8\left(3-4T+3T^2\right)}{T}$$
$$\frac{9-16T+29T^2-28T^3+29T^4-16T^5+9T^6}{T^3}$$
$$\frac{1-8T+19T^2-20T^3+19T^4-8T^5+T^6}{T^3}$$
$$-\frac{3-8T+3T^2}{T}$$
$$\frac{5-16T+5T^2}{T}$$
$$\frac{2-8T+10T^2-12T^3+13T^4-12T^5+13T^6-12T^7+10T^8-8T^9+2T^{10}}{T^5}$$
$$0$$
$$\frac{3-8T+6T^2-4T^3+6T^4-8T^5+3T^6}{T^3}$$
$$-\frac{\left(1+T^2\right)\left(2-8T+11T^2-12T^3+11T^4-12T^5+11T^6-8T^7+2T^8\right)}{T^5}$$
$$\frac{5-20T+28T^2-32T^3+28T^4-20T^5+5T^6}{T^3}$$
$$-\frac{1-4T+10T^2-12T^3+13T^4-12T^5+13T^6-12T^7+10T^8-4T^9+T^{10}}{T^5}$$
$$-\frac{1-4T+12T^2-16T^3+12T^4-4T^5+T^6}{T^3}$$
$$0$$
$$-\frac{\left(1-T+T^2\right)^2\left(1+T+T^2\right)\left(1-3T+6T^2-3T^3+T^4\right)}{T^5}$$
$$\frac{5-24T+39T^2-44T^3+39T^4-24T^5+5T^6}{T^3}$$
$$0$$
$$-\frac{1-4T+14T^2-20T^3+14T^4-4T^5+T^6}{T^3}$$
$$\frac{(-1+T)^2\left(5-18T+16T^2-18T^3+5T^4\right)}{T^3}$$
$$\frac{\left(3-4T+3T^2\right)\left(7-12T+17T^2-12T^3+7T^4\right)}{T^3}$$
$$\frac{\left(1-T+T^2\right)\left(1-5T+11T^2-12T^3+12T^4-12T^5+11T^6-5T^7+T^8\right)}{T^5}$$
$$0$$

0

$$-\frac{\left(1+T^4\right)\left(3+4T^3+3T^6\right)}{T^5}$$

$$\frac{4\left(1-T+T^2\right)}{T}$$

$$\frac{1-8T+16T^2-20T^3+16T^4-8T^5+T^6}{T^3}$$