umm

```
γ /: γ^x_ /; x > 1 := 0;
PBWBasis = {W, u, M};

B[U@u, U@M] = - (B[U@M, U@u] = U@M);
B[U@W, U@u] = - (B[U@u, U@W] = U@W);

B[U@W, U@M] = - (B[U@M, U@W] = (1 - t) U[] - γ (1 + t) U[u]);
```

```
UU[l___, x_^n_, r___] := UU[l, Sequence @@ Table[x, {n}], r];
UU[l___, 1, r___] := UU[l, r];
UU[] = U[]; UU[l_, r___] := U[l] ** UU[r];
U_i_[ℰ_] := ℰ /. {t → t_i, u_U :→ UU @@ Replace[u, x_ :→ x_i, 1]};
```

```
B[x_, x_] = 0;
B[U[(x_)_i_], U[(y_)_i_]] := U_i[B[U@x, U@y]];
B[U[(x_)_i_], U[(y_)_j_]] /; i =!= j := 0;
B[x_, y_] := x ** y - y ** x;
```

```
OrderedQ[{u, W}]
```

```
True
```

```
UU[M, W]
```

```
(1 - t) U[] - (1 + t) γ U[u] + U[W, M]
```

```
B[U[u], U[W]]
```

```
U[W]
```

```
x_ ≤ y_ := OrderedQ[{y, x}]; x_ ≺ y_ := ! OrderedQ[{x, y}]; (*Anti-alphabetic ordering*)
Simp[ℰ_] := Collect[ℰ, _U, Together];
```

```
Unprotect[NonCommutativeMultiply];
NonCommutativeMultiply[x_] := x;
0 ** _ = _ ** 0 = 0;
x_ ** U[] := x; U[] ** x_ := x;
(a_ * x_U) ** (b_ * y_U) := If[a b === 0, 0, Simp[a b (x ** y)]];
(a_ * x_U) ** y_ := Simp[a (x ** y)]; x_ ** (a_ * y_U) := Simp[a (x ** y)];
(x_Plus) ** y_ := (# ** y) & /@ x; x_ ** (y_Plus) := (x ** #) & /@ y;
U[x_] ** U[y_] := (*U[x]**U[y] =*) If[x ≺ y, U[x, y], U[y, x] + B[U@x, U@y]];
U[x_] ** U[y1_, yy__] := (*U[x] ** U[y1,yy] =*)
  If[x ≤ y1, U[x, y1, yy], (U@x ** U@y1) ** U@yy];
U[xx__, xn_] ** U[yy__] := (*U[xx,xn]**U[yy] =*) U@xx ** (U@xn ** U@yy);
```

```
σ[i_, j_][ℰ_] := ℰ /. {W_i → W_j, t_i → t_j, M_i → M_j, c_i → c_j, u_i → u_j};
```

```
mul[i_, j_][ℰ_] :=
  Simp[ℰ /. x_U ⧴ DeleteCases[x, _j] ** U @@ Cases[x, y_j ⧴ y_i] /. {c_j → c_i, t_j → t_i}];
mul[i_, j_, k_][ℰ_] := ℰ // mul[i, j] // σ[i, k];
```

```
(*Jacobi*)
B[B[U@W, U@u], U@M] + B[B[U@M, U@W], U@u] + B[B[U@u, U@M], U@W]
```

0

```
Δ[i_, j_, k_][ℰ_] := Simp[ℰ /. {
    z_. U[] ⧴ (z /. {c_i → c_j + c_k, t_i → t_j t_k}) U[],
    z_. x_U ⧴ (z /. {c_i → c_j + c_k, t_i → t_j t_k}) NonCommutativeMultiply @@ (x /. {
          u_i → U@u_j + U@u_k,
          W_i → U[W_j] + ϒ/2 U[W_j, u_k] + U[W_k] - ϒ/2 U[W_k, u_j],
          M_i → t_k U[M_j] + t_k ϒ/2 U[u_k, M_j] + U[M_k] - ϒ/2 U[u_j, M_k],
          y_l_ ⧴ U@y_l
        })
  }]
```

```
S[i_][ℰ_] := Simp[ℰ /. {z_. x_U ⧴ (z /. {c_i → -c_i, t_i → t_i^-1}) S[i][x]}];
S[i_][U[]] = U[];
S[i_][U[y_j_, more___]] /; i ≠ j := U[y_j] ** S[i][U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
S[i_][U[u_i_, more___]] := S[i][U[more]] ** (-U@u_i);
S[i_][U[M_i_, more___]] := S[i][U[more]] ** (-t_i^-1 (1 - ϒ/2) U[M_i]);
S[i_][U[W_i_, more___]] := S[i][U[more]] ** (-(1 + ϒ/2) U[W_i]);
```

```
Sinv[i_][ℰ_] := Simp[ℰ /. {z_. x_U ⧴ (z /. {c_i → -c_i, t_i → t_i^-1}) Sinv[i][x]}];
Sinv[i_][U[]] = U[];
Sinv[i_][U[y_j_, more___]] /; i ≠ j := U[y_j] ** Sinv[i][U[more]];
(*Careful! if mathematica cannot decide i==j or not then we get an error*)
Sinv[i_][U[u_i_, more___]] := Sinv[i][U[more]] ** (-U@u_i);
Sinv[i_][U[M_i_, more___]] := Sinv[i][U[more]] ** (-t_i^-1 (1 + ϒ/2) U[M_i]);
Sinv[i_][U[W_i_, more___]] := Sinv[i][U[more]] ** (-(1 - ϒ/2) U[W_i]);
```

```
CoUnit[i_][ℰ_] :=
  Simp[ℰ /. {z_. x_U :> (z /. {t → 1, c → 0, c_i → 0, t_i → 1}) CoUnit[i][x]}] ;
CoUnit[i_][U[]] = U[];
CoUnit[i_][U[y_j_, more___]] /; i ≠ j := U[y_j] ** CoUnit[i][U[more]];
CoUnit[i_][U[y_i_, more___]] /; y ≠ a := 0;
CoUnit[i_][U[a_i_, more___]] := CoUnit[i][U[more]];
```

```
UExp[ℰ_, n_] := Module[{t = U[], k}, U[] + Sum[(t = t ** ℰ)/k!, {k, n}]] // Simp
```

```
ToDegree[n_][ℰ_] := (Simp[ℰ] /.
     {γ → ℏ γ, c_i_ :> ℏ c_i, t_i_ :> e^(ℏ c_i), c → ℏ c, t :> e^(ℏ c), x_U :> ℏ^(Count[x,W|W_]+Count[x,u|u_]) x} /.
   a_. x_U :> Normal[Series[a, {ℏ, 0, n}]] * x) /. ℏ → 1
```

## Hopf algebra axioms

```
(*Sinv is the inverse of S *)
Sinv[1][S[1][U@M1]] // Simp
Sinv[1][S[1][U@u1]] // Simp
Sinv[1][S[1][U@W1]] // Simp
```

U[$M_1$]

U[$u_1$]

U[$W_1$]

```
(*Coassociativity only works properly with t_i instead of t!*)
(U@W1 // Δ[1, x, yy] // Δ[yy, y, z]) -
  (U@W1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@u1 // Δ[1, x, yy] // Δ[yy, y, z]) -
  (U@u1 // Δ[1, xx, z] // Δ[xx, x, y])
(U@M1 // Δ[1, x, yy] // Δ[yy, y, z]) -
  (U@M1 // Δ[1, xx, z] // Δ[xx, x, y])
```

0

0

0

```
(*Convolution inverse*)
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[W1]]]]
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[u1]]]]
mul[2, 3, 1][S[2][Δ[1, 2, 3][U[M1]]]]

mul[2, 3, 1][S[3][Δ[1, 2, 3][U[W1]]]]
mul[2, 3, 1][S[3][Δ[1, 2, 3][U[u1]]]]
mul[2, 3, 1][S[3][Δ[1, 2, 3][U[M1]]]]
```

0

0

0

0

0

0

## Testing Yang-Baxter

$$R[i\_, j\_, d\_] := \mathrm{Sum}\left[\frac{1}{a!\,b!} UU[W_i{}^a] ** \left(U[] + \frac{\gamma}{2} a\, U[u_i]\right) ** UU[u_i{}^b] ** \left(c_j{}^b U[] + \gamma\, b\, c_j{}^{b-1} U[u_j]\right) ** \left(U[] + \frac{\gamma}{2} a\, U[u_j]\right) ** UU[M_j{}^a] \left(1 + \gamma \frac{1}{4} (a-1)\, a\right), \{b, 0, d\}, \{a, 0, d-b\}\right] \;//\; \mathrm{ToDegree}[d]$$

```
R3[d_] := (ToDegree[d][R[1, 2, d] ** R[1, 3, d]] ** R[2, 3, d]) -
  (ToDegree[d][R[2, 3, d] ** R[1, 3, d]] ** R[1, 2, d])
```

```
R3[2] // ToDegree[2]
```

0

```
Timing[R3[3] // ToDegree[2]]
```

{0.404, 0}

```
Timing[R3[4] // ToDegree[3]]
```

{2.732, 0}

```
Timing[R3[5] // ToDegree[4]]
```

$\left\{20.952, -\frac{1}{6} c_2^3\, U[W_1, M_3]\right\}$

```
Timing[R3[6] // ToDegree[4]]
```

{160.32, 0}

```
Timing[R3[7] // ToDegree[5]]
```

$Aborted

```
(*Verifying the inverse is given by (S tensor id) (R) *)
(S[1][R[1, 2, 7]] ** R[1, 2, 7] - U[]) // ToDegree[4]
(*Verifying the inverse is given by (id tensor S^{-1}) (R)  *)
```

0

```
(Sinv[2][R[1, 2, 7]] ** R[1, 2, 7] - U[]) // ToDegree[4]
```

0

```
Rinv[i_, j_, d_] := S[i][R[i, j, d]] // Expand
```

Rinv2[i_, j_, d_] := Sum[

$$\frac{1}{a!\,b!}(-1)^{a+b}\left(1+\frac{a}{2}\gamma\right)\left(U[] - \frac{\gamma}{2}a\,U[u_i]\right) ** UU\left[u_i{}^b\right] ** UU\left[W_i{}^a\right] ** \left(c_j{}^b\,U[] + \gamma\,b\,c_j{}^{b-1}\,U[u_j]\right) ** \left(U[] + \frac{\gamma}{2}a\,U[u_j]\right) ** UU\left[M_j{}^a\right]\left(1+\gamma\frac{1}{4}(a-1)\,a\right)$$

, {b, 0, d}, {a, 0, d - b}] // ToDegree[d]

```
(Rinv[1, 2, 9] - Rinv2[1, 2, 9]) // Simp // ToDegree[5]
```

0

Rinv3[i_, j_, d_] :=
Sum[

$$\frac{1}{a!\,b!}(-1)^{a+b}\,UU\left[u_i{}^b\right] ** UU\left[W_i{}^a\right] ** \left(U[] - \frac{\gamma}{2}a\,U[u_i]\right) ** \left(c_j{}^b\,U[] + \gamma\,b\,c_j{}^{b-1}\,U[u_j]\right) ** \left(U[] + \frac{\gamma}{2}a\,U[u_j]\right) ** UU\left[M_j{}^a\right]\left(1-\gamma\frac{1}{4}(a-1)\,a\right)$$

, {b, 0, d}, {a, 0, d - b}] // ToDegree[d]

```
(Rinv[1, 2, 9] - Rinv3[1, 2, 9]) // Simp // ToDegree[5]
```

0

Rinv4[i_, j_, d_] := Sum[

$$\frac{c_j{}^b}{a!\,b!}(-1)^{a+b}\,t_j{}^{-a}\,UU\left[W_i{}^a\right] ** \left(U[] - \frac{\gamma}{2}a\,U[u_i]\right) ** UU\left[u_i{}^b\right] ** (U[] - \gamma\,U[u_j, u_i] - a\,\gamma\,U[u_j]) ** \left(U[] + \frac{\gamma}{2}a\,U[u_j]\right) ** UU\left[M_j{}^a\right]\left(1-\gamma\frac{1}{4}(a-1)\,a\right)$$

, {b, 0, d}, {a, 0, d - b}] // ToDegree[d]

```
(Rinv[1, 2, 11] - Rinv4[1, 2, 11]) // Simp // ToDegree[6]
```

0

$UU[u^2, W^2]$ // Simp

4 U[W, W] + 4 U[W, W, u] + U[W, W, u, u]

```
(*Alternative formulas for Rinv, Rinv4 is the definitive one*)

TestS[m_, n_] := -S[i][UU[a_i^n] ** (U[] + ϵ n U[c_i]) ** (b_i^m U[] + ϵ m b_i^(m-1) U[c_i])] +
   (-1)^m (b_i^m U[] + ϵ m b_i^(m-1) U[c_i]) ** (U[] - ϵ n U[c_i]) ** San[n, i]
TestS[4, 3] // Together
0
San[n_, i_] := (-1)^n t_i^-n UU[a_i^n] ** (U[] + n (n + 1)/2 ϵ U[] + n ϵ U[c_i]) // Expand
TestSan[n_] := -S[i][UU[a_i^n]] + San[n, i]
(TestSan[6] // Together)
0
Rinv2[i_, j_, d_] :=
 Sum[1/(m! n!) (-1)^m (b_i^m U[] + ϵ m b_i^(m-1) U[c_i]) ** (U[] - ϵ n U[c_i]) ** San[n, i] ** UU[c_j^m, w_j^n]
    (1 + ϵ 1/4 (n - 1) n), {m, 0, d}, {n, 0, d - m}] // ToDegree[d]
Rinv3[i_, j_, d_] :=
 Sum[1/(m! n!) (-1)^m San[n, i] ** (b_i^m U[] + ϵ m b_i^(m-1) U[c_i] + ϵ m n b_i^(m-1) U[]) ** (U[] - ϵ n U[c_i] -
        ϵ n^2 U[]) ** UU[c_j^m, w_j^n] (1 + ϵ 1/4 (n - 1) n), {m, 0, d}, {n, 0, d - m}] // ToDegree[d]
Rinv4[i_, j_, d_] :=
 Sum[1/(m! n!) (-1)^(m+n) t_i^-n UU[a_i^n] ** (b_i^m U[] + ϵ m b_i^(m-1) U[c_i] + ϵ m n b_i^(m-1) U[]) ** UU[c_j^m, w_j^n]
    (1 - ϵ 1/4 (n - 1) n), {m, 0, d}, {n, 0, d - m}] // ToDegree[d]
(Rinv[1, 2, 6] - Rinv2[1, 2, 6]) // Simp // ToDegree[6]
Rinv[1, 2, 6] - Rinv3[1, 2, 6] // ToDegree[6]
Rinv[1, 2, 6] - Rinv4[1, 2, 6] // ToDegree[6]
0

0

0
```

## Quasi triangularity axioms

```
(*Check the three quasi-triangularity axioms*)

(Δ[i, k, l][R[i, j, 5]] - R[k, j, 5] ** R[l, j, 5]) // ToDegree[5]
0

(Δ[j, k, l][R[i, j, 5]] - R[i, l, 5] ** R[i, k, 5]) // ToDegree[5]
0
```

```
CheckRDR[x_, d_] :=
  (R[2, 3, d] ** Δ[1, 2, 3][x] ** Rinv[2, 3, d] - σ[2, 3][Δ[1, 2, 3][x]]) // ToDegree[d]

CheckRDR[U@c_i, 4]
CheckRDR[U@w_i, 4]
CheckRDR[U@a_i, 4]
```

0

0

0

## Drinfeld element

```
(*Drinfeld element*)
Dr[d_] := R[1, 2, d] // S[2] // mul[2, 1, 1]

(*Check that S(Dr) and Dr commute*)
S[1][Dr[4]] ** Dr[4] - S[1][Dr[4]] ** Dr[4]
```

0

```
S[1][Dr[2]] // ToDegree[2]
```

$U[\,] - b_1 U[c_1] + (-1 + \epsilon + b_1) U[a_1, w_1] + \left(-\epsilon + \frac{b_1^2}{2}\right) U[c_1, c_1] + (\epsilon + b_1) U[a_1, c_1, w_1] + \epsilon b_1 U[c_1, c_1, c_1] + \frac{1}{2} U[a_1, a_1, w_1, w_1] + \epsilon U[a_1, c_1, c_1, w_1]$

```
(*S(Dr) = t^-1 e^-2εc Dr *)
S[1][Dr[5]] - (U[] - 2 ε U[c_1]) ** (Dr[5] t_1^-1 // ToDegree[5]) // ToDegree[5]
```

0

```
(U[] - ε U[c_1]) ** Dr[4] - Dr[4] ** (U[] - ε U[c_1]) // ToDegree[4]
```

0

Therefore the Ribbon element v is implied by $v^2$ = S(Dr)Dr = $t^{-1} e^{-2\epsilon c}$ Dr$^2$ so choose $v = t^{-1/2} e^{-\epsilon c}$ Dr , note Dr commutes with $e^{-\epsilon c}$.
According to Ohtsuki p.72 read upside down, we should set the left-moving cup and cap to 1 and the right-moving cap nr should be $v\,\mathrm{Dr}^{-1} = t^{-1/2} e^{-\epsilon c}$ and
the right-moving cup ur should be Dr $v^{-1} = t^{1/2} e^{\epsilon c}$.

```
(*Square of antipode*)
S[1]@S[1]@U@c_1
S[1]@S[1]@U@w_1
S[1]@S[1]@U@a_1
```

$U[c_1]$

$(1 - \epsilon)\, U[w_1]$

$(1 + \epsilon)\, U[a_1]$

```
(U[] + ∈ U[c]) ** U[c] ** (U[] - ∈ U[c]) // Simp
(U[] + ∈ U[c]) ** U[w] ** (U[] - ∈ U[c]) // Simp
(U[] + ∈ U[c]) ** U[a] ** (U[] - ∈ U[c]) // Simp
```

U[c]

$(1-\epsilon)$ U[w]

$(1+\epsilon)$ U[a]

## Logos

```
B[U@w, U@a]
```

$(1-t)$ U[] $-$ $(1+t)$ $\epsilon$ U[c]

```
(*η /: η^n_/;n>1 :=0; *)
η = (t + 1)/(t - 1) ∈;
q = 1 + η;
qI[k_] := k (1 + η (k - 1) / 2) // Expand
qFac[n_] := n! (1 + η (n - 1) n / 4) // Expand
InvqFac[n_] := (1 - η (n - 1) n / 4) / n! // Expand
qBin[n_, k_] := Binomial[n, k] (1 + η k (n - k) / 2) // Expand
```

```
(*Checking the commutation relation for powers of a,w*)
WmAn[m_, n_] := Sum[(1 - t)^j qBin[m, j] qBin[n, j]
   qFac[j] UU[a^(n-j)] ** (U[] + j η U[c]) ** UU[w^(m-j)], {j, 0, Min[m, n]}]
TestWmAn[m_, n_] := -UU[w^m, a^n] + WmAn[m, n]
TestWmAn[5, 3] // Together
```

0

```
(*Guess qLogos first at q=1*)
ToDegh[F_, x_] := Series[F /. {α → α h, δ → δ h, β → β h}, {h, 0, x}]
d = 8;
LHS = Sum[α^m δ^k β^n InvqFac[m] InvqFac[k] InvqFac[n] WmAn[m + k, n + k],
    {m, 0, d}, {k, 0, d - m}, {n, 0, d - m - k}] // Simp;
(*powers of nu as power series*)
nuA[z_] := Sum[Binomial[z - 1 + x, x] (1 - t)^x δ^x, {x, 0, d}];
RHS =
  Sum[(1 - t)^j nuA[m + k + n + j + 1] α^(m+j) δ^k β^(n+j) InvqFac[m] InvqFac[k] InvqFac[n] InvqFac[j]
     UU[a^(n+k), w^(m+k)], {m, 0, d}, {k, 0, d - m}, {n, 0, d - m - k}, {j, 0, d - m - n - k}] // Simp;
ToDegh[(LHS - RHS) /. {∈ → 0} // Simp, 8]
```

$O[h]^9$

```
(*Now set up the LHS and RHS to find the Logos relating them.*)
d = 11;
```

$$\text{LHS} = \text{Sum}\left[\alpha^m\,\delta^k\,\beta^n\,\frac{\text{WmAn}[m+k,\,n+k]}{m!\,k!\,n!},\ \{m,0,d\},\ \{k,0,d-m\},\ \{n,0,d-m-k\}\right]\ //\ \text{Simp};$$

$$\text{RHS} = \text{Sum}\left[(1-t)^j\,\mu^{-(m+k+n+j+1)}\,\frac{\alpha^{m+j}\,\delta^k\,\beta^{n+j}}{m!\,k!\,n!\,j!}\,\text{UU}\left[a^{n+k},\,w^{m+k}\right],\ \{m,0,d\},\ \{k,0,d-m\},\ \{n,0,d-m-k\},\ \{j,0,d-m-n-k\}\right];$$

$$\text{CRHS} = \text{Sum}\left[(1-t)^j\,\mu^{-(m+k+n+j+1)}\,\frac{\alpha^{m+j}\,\delta^k\,\beta^{n+j}}{m!\,n!\,k!\,j!}\,\text{UU}\left[a^{n+k},\,c,\,w^{m+k}\right],\ \{m,0,d\},\ \{k,0,d-m\},\ \{n,0,d-m-k\},\ \{j,0,d-m-n-k\}\right]\ //\ \text{Simp};$$

$$\mu = 1 - (1-t)\,\delta;$$

$$\eta$$

$$\frac{(1+t)\,\epsilon}{-1+t}$$

```
(*Here we guess and verify the Logos coefficient by coefficient*)
```

$$\text{ToDegh}\Big[\text{Coefficient}\Big[\Big(-\text{LHS}+\text{RHS}+ \mu^{-4}(1-t)\,\eta\Big( (1-t)\left(\frac{1}{2}\delta^2\mu^2+\alpha\beta\delta\mu+\frac{\alpha^2\beta^2}{4}\right)\text{RHS}+ \mu^2(\delta\mu+\alpha\beta)\,\text{CRHS}+ \beta\left(\delta\mu+\frac{\alpha\beta}{2}\right)\text{U[a]}**\text{RHS}+ \alpha\left(\delta\mu+\frac{\alpha\beta}{2}\right)\text{RHS}**\text{U[w]}+ \beta\delta\mu^2\ \text{U[a]}**\text{CRHS}+ \alpha\delta\mu^2\ \text{CRHS}**\text{U[w]}+ \delta(1+\mu)(\mu\delta+\alpha\beta)\,\text{U[a]}**\text{RHS}**\text{U[w]}+ \delta^2\mu^2\,\text{U[a]}**\text{CRHS}**\text{U[w]}+ \frac{\beta^2\delta}{4}(1+\mu)\,\text{U[a, a]}**\text{RHS}+ \frac{\alpha^2\delta}{4}(1+\mu)\,\text{RHS}**\text{U[w, w]}+ \frac{\beta\ \delta^2}{2}(1+2\mu)\,\text{U[a, a]}**\text{RHS}**\text{U[w]}+ \frac{\alpha\ \delta^2}{2}(1+2\mu)\,\text{U[a]}**\text{RHS}**\text{U[w, w]}+ \frac{\delta^3}{4}(1+3\mu)\,\text{U[a, a]}**\text{RHS}**\text{U[w, w]}\Big) \Big),\ \text{U[]}\Big],\ 10\Big]$$

$O[h]^{11}$

```
(*Final check, put back ϵ.*)
ToDegh[(-LHS + RHS +
    -μ^-4 (1 + t) ϵ (
        (1 - t) (1/2 δ^2 μ^2 + α β δ μ + α^2 β^2/4) RHS +
        μ^2 (δ μ + α β) CRHS +
        β (δ μ + α β/2) U[a] ** RHS +
        α (δ μ + α β/2) RHS ** U[w] +
        β δ μ^2 U[a] ** CRHS +
        α δ μ^2 CRHS ** U[w] +
        δ (1 + μ) (μ δ + α β) U[a] ** RHS ** U[w] +
        δ^2 μ^2 U[a] ** CRHS ** U[w] +
        β^2 δ/4 (1 + μ) U[a, a] ** RHS +
        α^2 δ/4 (1 + μ) RHS ** U[w, w] +
        β δ^2/2 (1 + 2 μ) U[a, a] ** RHS ** U[w] +
        α δ^2/2 (1 + 2 μ) U[a] ** RHS ** U[w, w] +
        δ^3/4 (1 + 3 μ) U[a, a] ** RHS ** U[w, w])
  ), 10] // Simp
```

0

## Double Reverse

```
Δ[1, 2, 3][U@c₁] // Simp
Δ[1, 2, 3][U@w₁] // Simp
Δ[1, 2, 3][U@a₁] // Simp
```

$U[c_2] + U[c_3]$

$U[w_2] + U[w_3] + \epsilon\, U[c_3, w_2]$

$t_3\, U[a_2] + U[a_3] - \epsilon\, U[a_3, c_2]$

```
((Δ[1, 2, 3][U@w₁] // Simp) ** (Δ[1, 2, 3][U@w₁] // Simp) -
   (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) - ϵ U[w₂, w₃] - 2 ϵ U[c₃, w₂, w₃] - 2 ϵ U[c₃, w₂, w₂]) // Simp
```

0

```
((Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) - (t₃ U[a₂] + U[a₃]) **
    (t₃ U[a₂] + U[a₃]) + ϵ t₃ U[a₂, a₃] + 2 ϵ t₃ U[a₂, a₃, c₂] + 2 ϵ U[a₃, a₃, c₂]) // Simp
```

0

```
((Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) -
   (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) +
   3 ϵ t₃ (t₃ U[a₂] + U[a₃]) ** U[a₂, a₃] +
   3 ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** U[a₃, c₂]) // Simp
```

0

```
((Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) **
    (Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) -
   (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) +
   6 ϵ t₃ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** U[a₂, a₃] +
   4 ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** U[a₃, c₂]) // Simp
```

0

```
((((Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp) **
        (Δ[1, 2, 3][U@a₁] // Simp) ** (Δ[1, 2, 3][U@a₁] // Simp)) // Simp) **
    (((Δ[1, 2, 3][U@w₁] // Simp) ** (Δ[1, 2, 3][U@w₁] // Simp) **
        (Δ[1, 2, 3][U@w₁] // Simp) ** (Δ[1, 2, 3][U@w₁] // Simp)) // Simp) -
   (
    (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) **
      (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃])
     + ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) **
       (4 U[c₃, w₂]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃])
     + ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) **
       (t₃ U[a₂] + U[a₃]) ** (6 U[w₂, w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃])
     - ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (4 U[a₃, c₂]) **
       (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃])
     - ϵ (t₃ U[a₂] + U[a₃]) ** (t₃ U[a₂] + U[a₃]) ** (6 t₃ U[a₂, a₃]) **
       (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃]) ** (U[w₂] + U[w₃])

   ) // Expand // Simp
```

0

```
UU[c, a]
```

$U[a] + U[a, c]$

```
S[1]@U@w₁ - (-U[w₁] + ϵ UU[w₁, c₁]) // Simp
S[1]@U@c₁ // Simp
S[1]@U@a₁ - (-t₁^-1 (U[a₁] + ϵ UU[c₁, a₁])) // Expand // Simp
```

0

$-U[c_1]$

0

```
Δ[1, 2, 3][U@c1] // S[2]
Δ[1, 2, 3][U@w1] // S[2] // Simp
((Δ[1, 2, 3][U@a1] // S[2] // Simp) /. t_ → t) // Simp
```

$-U[c_2] + U[c_3]$

$(-1 + \epsilon)\, U[w_2] + U[w_3] + \epsilon\, U[c_2, w_2] - \epsilon\, U[c_3, w_2]$

$(-1 - \epsilon)\, U[a_2] + U[a_3] - \epsilon\, U[a_2, c_2] + \epsilon\, U[a_3, c_2]$

```
U[w] ** ((U[c] + U[]) ** U[w]) // Simp
```

$2\, U[w, w] + U[c, w, w]$

```
(S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
(S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) ** (S[1]@U[w1]) // Simp
```

$(-1 + \epsilon)\, U[w_1] + \epsilon\, U[c_1, w_1]$

$(1 - 3\epsilon)\, U[w_1, w_1] - 2\epsilon\, U[c_1, w_1, w_1]$

$(-1 + 6\epsilon)\, U[w_1, w_1, w_1] + 3\epsilon\, U[c_1, w_1, w_1, w_1]$

$(1 - 10\epsilon)\, U[w_1, w_1, w_1, w_1] - 4\epsilon\, U[c_1, w_1, w_1, w_1, w_1]$

```
(S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
(S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) ** (S[1]@U[a1]) // Simp
```

$$\frac{(-1 - \epsilon)\, U[a_1]}{t_1} - \frac{\epsilon\, U[a_1, c_1]}{t_1}$$

$$\frac{(1 + 3\epsilon)\, U[a_1, a_1]}{t_1^2} + \frac{2\epsilon\, U[a_1, a_1, c_1]}{t_1^2}$$

$$\frac{(-1 - 6\epsilon)\, U[a_1, a_1, a_1]}{t_1^3} - \frac{3\epsilon\, U[a_1, a_1, a_1, c_1]}{t_1^3}$$

$$\frac{(1 + 10\epsilon)\, U[a_1, a_1, a_1, a_1]}{t_1^4} + \frac{4\epsilon\, U[a_1, a_1, a_1, a_1, c_1]}{t_1^4}$$

```
(U[c] - 4 U[]) ** UU[a^4] - UU[a^4, c]
```

0