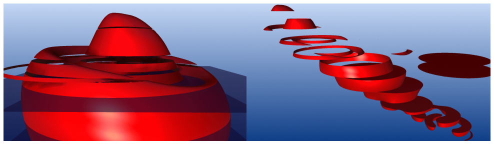
Canonical Decomposition of 3-Manifolds

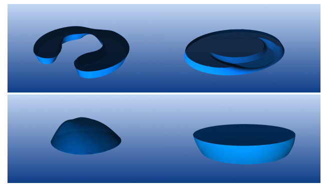
in Hatcher's *Notes on Basic 3-Manifold Topology* katlas.math.toronto.edu/caldermf/3manifolds/cheatsheet.pdf

Alexander's Theorem

Theorem (Alexander). *Every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.*
Proof. • Partition the z-axis such that h, the height function, has one or zero critical points in each section.



• Surger S along each partition plane, capping the resulting holes with disks. • Each resulting piece is isotopic to one of seven model surfaces, all of which bound balls.

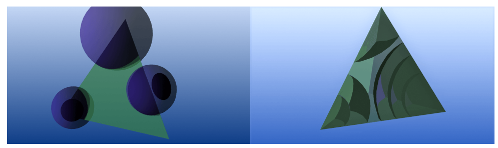


• Reversing the surgery, the property of bounding a ball is preserved. • We reverse each surgery, gluing these model surfaces along disks until we arrive at S, and so S must bound a ball. □

Prime Decomposition

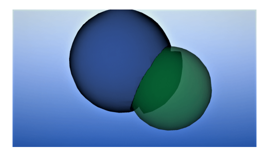
Definition. A *prime* M is a 3-manifold such that $M = A\#B$ implies that either A or B is S^3.
Definition. An *irreducible* M has the property that any sphere in M bounds a ball.
Theorem. *Let M be compact, connected, and orientable. Then there is a unique decomposition of M into a connected sum of primes.*
Proof (Existence). • Nonseparating spheres split off copies of $S^1 \times S^2$. • Want to show there is a bound on the number of spheres in a system S satisfying (*) No component of the splitting is trivial (a punctured 3-sphere).

• Splitting a sphere in S along a disk preserves (*).• Triangulate. For each 3-simplex τ, make the $S \cap \tau$ into disks by splitting spheres along disks close to $\partial\tau$. • For each 2-simplex σ, eliminate arcs of $S \cap \sigma$ having endpoints on the same edge of σ by pushing across.



• Now for each 3-simplex, the number of components which are not I-bundles is at most 4.
• Each I-bundle contributes to $H_1(M)$, and so their quantity must also be bounded. □
Proof (Uniqueness). • Given two systems P and Q of spheres, construct a new disjoint system P' by splitting Q along P.

This system "contains" both other systems and splits M into some collection of primes, and so the original two systems must have given rise to the same decomposition. □

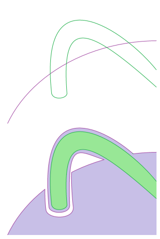


Torus Decomposition (Existence)

Definition. A 2-sided surface S is *incompressible* if $D \subset M$ with $D \cap S = \partial D$ implies that there exists $D' \subset S$ with $\partial D' = \partial D$.

Facts. • If $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective, S is incompressible. • No incompressible surfaces in $\mathbb{R}^3$. • A 2-sided torus T in an irreducible M is compressible if and only if it bounds a solid torus or lies in a ball.

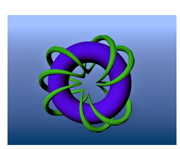
• If S is a collection of incompressible surfaces, M irreducible if and only if $M|S$ irreducible. • If S is a collection of incompressible surfaces, a surface $T \subset M|S$ is incompressible if and only if it is incompressible in M.



Definitions. A properly embedded surface $S \subset M$ is *∂-parallel* if it is isotopic, fixing ∂S, to a subsurface of ∂M. An irreducible manifold M is *atoroidal* if every incompressible torus in M is ∂-parallel.

Existence. *In a compact connected irreducible M there exists a finite collection T of disjoint incompressible tori such that each component of $M|T$ is atoroidal.*

Proof. • First we show a bound on the number of components in a disjoint system of incompressible surfaces S such that no component of $M|S$ is the product of I with a closed surface. • For this, perturb and split to start with the same setup as in prime decompositions, leaving only I-bundles to handle. • Replace I-bundles with one-sided surfaces S'. • Show that $H_3(M, S'; Z_2) \to H_2(S'; Z_2) \to H_2(M; Z_2)$ is exact, and first and last terms do not depend on S', meaning middle term is bounded, showing number of components are bounded. • If the system S is composed of non-∂-parallel tori, the desired existence follows. □
This decomposition is not unique.



We need to introduce Seifert manifolds and include them as possible components of $M|S$ to obtain uniqueness.

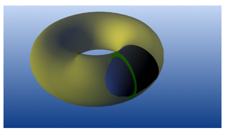
Torus Decomposition (Uniqueness)

Definition. A *Seifert manifold* is a 3-manifold which can be decomposed into fibres with neighbourhoods homeomorphic to a neighbourhood of a fibre in a gluing of twisted tori.
We wish to prove the following:

Uniqueness. For a compact irreducible orientable 3-manifold M there exists a collection $T \subset M$ of disjoint incompressible tori such that each component of $M|T$ is either atoroidal or a Seifert manifold, and a minimal such collection T is unique up to isotopy.

Definition. A non-disk S is *∂-incompressible* if for each disk $D \subset M$ such that $\partial D \cap S$ is an arc α in ∂D and the rest of ∂D lies in ∂M, there is a disk $D' \subset S$ with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial S$.

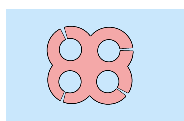
Definition. A properly embedded surface is *essential* if it is a sphere that does not bound a ball, a disk which does not split off a ball, or both incompressible and ∂-incompressible.



Example. The only essential surfaces in $S^1 \times D^2$ are meridian disks.

Proposition. *If M is a connected compact irreducible Seifert-fibered manifold, then any essential surface in M is either* vertical, *(a union of regular fibers), or* horizontal, *(transverse to all fibers).*

Proof. • Form a circle bundle M_0 by deleting neighbourhoods of multiple fibres in M, then split along annuli A to split M_0 into a solid torus M_1.



• Circles of ∂S cannot be trivial, and so can be isotoped to be vertical or horizontal, then circles bounding disks and trivial arcs can be eliminated in the usual way. • Let $S_1 = S|A$, then use our example to classify components of S_1 as either ∂-parallel annuli or meridian disks. Isotope the boundaries of S_1 components to be vertical or horizontal. • Extend this isotopy to the original surface S to conclude that it must be horizontal or vertical. □

Definition. A *semi-bundle* $p : M \to I$ is the union of two twisted I-bundles $p^{-1}[0, \frac{1}{2}]$ and $p^{-1}[\frac{1}{2}, 1]$ glued together at $p^{-1}\left(\frac{1}{2}\right)$. Horizontal surfaces in 3-manifolds are bundles or semi-bundles.

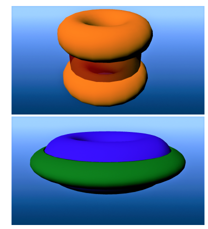
Proposition. *A compact connected Seifert-fibered manifold M is irreducible unless it is $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

Proof. • Use the same argument as in the last proposition to show that in a reducible M, there exists a horizontal sphere not bounding a ball. • If S is a horizontal sphere in M, then M is a sphere bundle or semi-bundle - our only options for M are then the manifolds listed above.

Proposition. In a Seifert manifold, all 2-sided horizontal surfaces are essential. All 2-sided vertical surfaces are essential except a torus bounding a solid twisted torus containing at most one multiple fiber, or an annulus which splits a solid torus from M.

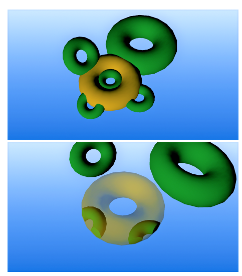
Lemma. *If M is compact, connected, orientable, irreducible, and atoroidal, and M contains an incompressible and ∂-incompressible annulus meeting torus components of ∂M, then M is a Seifert manifold.*

Proof. • If A is an annulus as in the hypothesis, there are three possibilities for how A can meet ∂M.



• Let N be a neighbourhood of A's union with the relevant components of ∂M. Then N is a circle bundle with A vertical. • Working case-by-case, the possibilities for components of $\partial N - \partial M$ can either be isotoped into N, or they are solid tori. • This means M is N with solid tori attached to tori of $\partial N - \partial M$. We can extend the circle bundle structure of N to these tori to get a Seifert fibering of M. □

Proof of Uniqueness. • Let T and T' be two minimal collections splitting M into manifolds M_j and M'_j respectively. We can assume no torus in T is isotopic to one in T' since if this were true, we could split along this torus and work by induction.



• Eliminate circles bounding disks in $T \cap T'$ and ∂-parallel annuli in $M_j \cap T'$ by usual methods. • Circles that do not bound disks induce fiberings on T_i which agree on M_j and M'_j, meaning T_i could be deleted from the collection (a contradiction). • We arrive at a situation where $T \cap T' = \varnothing$. • Then show that all T_i and T'_i are vertical and all M_j and M'_j are Seifert-fibered. • The T_i and T'_i together cut M into pieces we call N_p. • We can isotope N_p so its fiberings agree on T_i except in certain exceptional cases: • $N_p = S^1 \times D^2$, which cannot occur because T_i would be its compressible boundary. • $N_p = S^1 \times S^2 \times I$, which cannot occur because its boundary components are T_i and either T_j or T'_i, which can be isotoped to agree and contradict minimality. • $M_j \cap M'_k = S^1 \tilde{\times} S^1 \tilde{\times} I$, which has only one boundary component so $N_p = M_j \subset M'_j$, so we could change the fibering of M_j to match the restriction of M'_j. • We conclude that fiberings of M_j and M_k agree on T_i since they agree with the fibering from M'_j, meaning we can omit T_i from T, contradicting minimality. □