A Visual Companion to Hatcher's *Notes on Basic 3-Manifold Topology*

These notes, originally written in the 1980's, were intended as the beginning of a book on 3-manifolds, but unfortunately that project has not progressed very far since then. A few small revisions have been made in 1999 and 2000, but much more remains to be done, both in improving the existing sections and in adding more topics. The next topic to be added will probably be Haken manifolds in Section 3.2. For any subsequent updates which may be written, the interested reader should check my webpage:

http://www.math.cornell.edu/~hatcher

The three chapters here are to a certain extent independent of each other. The main exceptions are that the beginning of Chapter 1 is a prerequisite for almost everything else, while some of the later parts of Chapter 1 are used in Chapter 2.

This version of Allen Hatcher's *Notes on Basic 3-Manifold Topology* is intended as a visual companion to his text. The notes, images, and links to animations which accompany the original text are intended to aid in visualizing some of the key steps in the text's proofs and examples. Content which appears in the original text is in serif font while my additions are in sans-serif font. All black-and-white figures are Professor Hatcher's, while I created all of the colour images and animations. A copy of this text along with all of the animations which are referenced within can be found at http://katlas.math.toronto.edu/caldermf/3manifolds; links are also provided throughout the text. A brief summary of the results in the first section of this book are compiled at http://katlas.math.toronto.edu/caldermf/3manifolds/cheatsheet.pdf. All errors and inconsistencies which may appear in this version of the text are mine.

I would like to thank Professor Hatcher for his generosity in sharing his text and providing me with his source files, as well as for his encouragement during the process of creating this work. In addition, I would like to thank Professor Dror Bar-Natan, who had the idea for this project as well as supervised its creation while providing advice and aid in both the design of this work and the mathematics I learned along the way.

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Contents

Chapter 1

Canonical Decomposition

This chapter begins with the first general result on 3-manifolds, Kneser's theorem that every compact orientable 3-manifold M decomposes uniquely as a connected sum $M = P_1 \# \cdots \# P_n$ of 3-manifolds P_i which are prime in the sense that they can be decomposed as connected sums only in the trivial way $P_i = P_i \# S^3$.

After the prime decomposition, we turn in the second section to the canonical torus decomposition due to Jaco-Shalen and Johannson.

We will work in the C^∞ category throughout. Some basic results about differentiable manifolds will be needed. In particular this includes:

- Tubular neighborhoods: submanifolds have neighborhoods that are diffeomorphic to their normal bundles, and these neighborhoods are unique up to isotopy.
- Isotopy extension: an isotopy of a submanifold can be extended to an isotopy of the ambient manifold.
- Transversality: a submanifold of codimension p can be perturbed by a small isotopy to intersect another submanifold of codimension q transversely, and then the intersection is a submanifold of codimension $p + q$.
- Triangulability: a smooth manifold is homeomorphic to a simplicial complex whose simplices are smoothly embedded.

All 3-manifolds in this chapter are assumed to be connected, orientable, and compact, possibly with boundary, unless otherwise stated or constructed.

1.1 Prime Decomposition

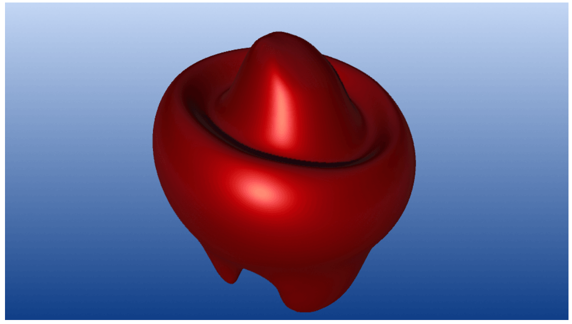
Implicit in the prime decomposition theorem is the fact that S^3 is prime, otherwise one could only hope for a prime decomposition modulo invertible elements, as in algebra. The fact that S^3 is prime is a consequence of Alexander's theorem, our first topic.

Alexander's Theorem

This quite fundamental result was one of the earliest theorems in the subject:

Theorem 1.1. *Every embedded 2-sphere in $\mathbb{R}^3$ bounds an embedded 3-ball.*

Figure 1.1 – An embedded 2-sphere.



katlas.math.toronto.edu/caldermf/3manifolds/1-1/manifold.mp4

The version of this that Alexander proved, in the 1920s, was slightly different: a piecewise linearly embedded sphere in $\mathbb{R}^3$ bounds a topological ball. It would not have been difficult for him to improve the conclusion to say that the ball was piecewise linear as well. The famous example of the Alexander horned sphere, which he constructed at about the same time, shows that a topological sphere need not bound a topological ball. The proof we give for smooth spheres follows the same general strategy as Alexander's proof for piecewise linear spheres, namely, to cut the given sphere along horizontal planes to produce simpler spheres and apply an induction argument. Alexander cut along horizontal planes passing through vertices of the triangulated sphere. In the smooth category one cuts instead along horizontal planes that are transverse to the sphere. In order to have a nice starting point for the induction we will do a preliminary isotopy of the sphere to arrange that the projection of the sphere onto the z-axis is a morse function, so we will assume the reader knows a little Morse theory. This could be avoided by using a more direct construction as at the beginning of [Hatcher 1983], but this would slow down the exposition.

The proof will also use the analogous result in one lower dimension, that a smooth circle in $\mathbb{R}^2$ bounds a smooth disk. This can be proved by a similar but simpler inductive argument, which it would be a good exercise for the reader to work out. In this dimension it is even true that a topologically embedded circle in $\mathbb{R}^2$ bounds a topological disk, the Schoenflies theorem, whose proof is more difficult since a simple inductive argument is not possible.

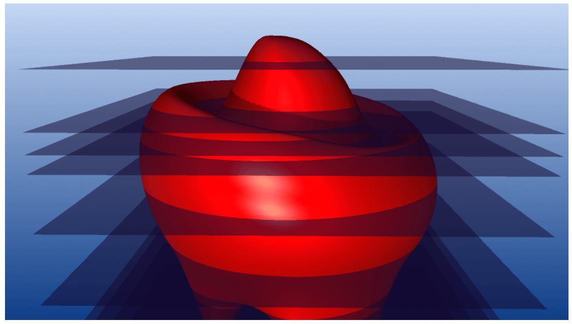
There is a simple proof due to M. Brown in 1960 that a smoothly embedded S^{n-1} in $\mathbb{R}^n$ bounds a topological ball, for arbitrary n. The ball is known to be a smooth ball for all n except $n = 4$ where this question remains open. For $n \geq 5$ this follows from the h-cobordism theorem (plus surgery theory in the case $n = 5$).

Proof. Let $S \subset \mathbb{R}^3$ be an embedded closed surface, with $h : S \to \mathbb{R}$ the height function given by the z-coordinate. The first step is to arrange that h is a morse function, as follows. We can approximate h arbitrarily closely by a morse function, and the linear path between h and the approximation gives a small homotopy of h. Keeping the same x and y coordinates for S, this gives a small homotopy of S in $\mathbb{R}^3$. Since embeddings are open in the space of all maps (with the C^∞ topology), if this homotopy is small enough, it will be an isotopy. By a further small isotopy we can also assume the finitely many critical points of h (local maxima, minima, and saddles) all have distinct critical values.

Let $a_1 < \cdots < a_n$ be noncritical values of h such that each interval $(-\infty, a_1)$, (a_1, a_2), $\cdots$,(a_n, ∞) contains just one critical value. For each i, $h^{-1}(a_i)$ consists of a number of disjoint circles in the level $z = a_i$.

By the two-dimensional version of Alexander's theorem, each circle of $h^{-1}(a_i)$ bounds a disk

Figure 1.2

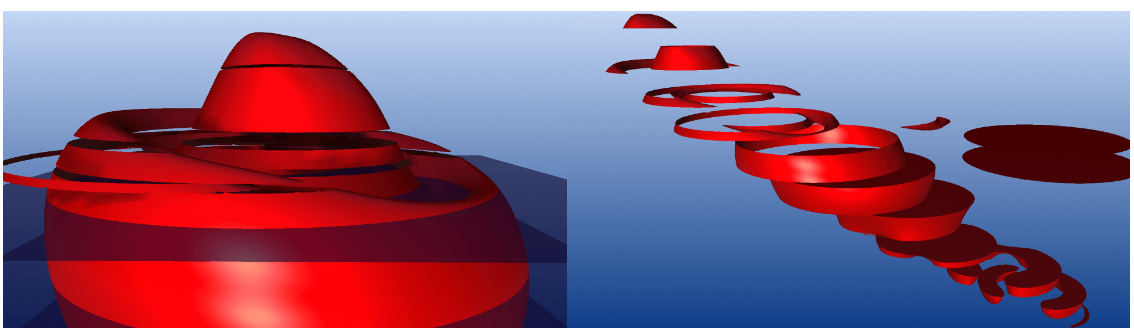


katlas.math.toronto.edu/caldermf/3manifolds/1-1/planes.mp4

in the plane $z = a_i$. Let C be a circle of $h^{-1}(a_i)$ which is innermost in the plane $z = a_i$, so the disk D it bounds in this plane is disjoint from all the other circles of $h^{-1}(a_i)$. We can use D to *surger* S along C. This means that for some small $\epsilon > 0$ we first remove from S the open annulus A consisting of points near C between the two planes $z = a_i \pm \epsilon$, then we cap off the resulting pair of boundary circles of $S - A$ by adding to $S - A$ the disks in $z = a_i \pm \epsilon$ which these circles bound. The result of this surgery is thus a new embedded surface, with perhaps one more component than S, if C separated S.

This surgery process can now be iterated, taking at each stage an innermost remaining circle of $h^{-1}(a_i)$, and choosing ϵ small enough so that the newly introduced horizontal cap disks intersect the previously constructed surface only in their boundaries (see Figure 1.2). After surgering all the circles of $h^{-1}(a_i)$ for all i, the original surface S becomes a disjoint union of closed surfaces S_j, each consisting of a number of horizontal caps together with a connected subsurface S'_j of S containing at most one critical point of h.

Figure 1.3 – Iterative surgery and capping each component with discs.



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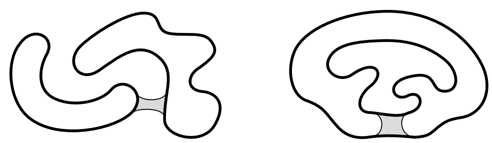
.../1-1/caps.mp4

Lemma 1.2. *Each S_j is isotopic to one of seven models: the four shown in Figure 1.5 plus three more obtained by turning these upside down. Hence each S_j bounds a ball.*

Proof. Consider the case that S_j has a saddle, say in the level $z = a$. First isotope S_j in a neighborhood of this level $z = a$ so that for some $\delta > 0$ the subsurface S_j^δ of S_j lying in $a - \delta \leq z \leq a + \delta$ is vertical, i.e., a union of vertical line segments, except in a neighborhood $N \subset \mathrm{Int}(S_j^\delta)$ of the saddle, where S_j has the standard form of the saddles in the models. Next, isotope S_j so that its subsurface S'_j (the complement of the horizontal caps) lies in S^δ_j. This is done by pushing its horizontal caps, innermost ones first, to lie near $z = a$, keeping the caps horizontal throughout the deformation.

After this has been done, S_j is entirely vertical except for the standard saddle and the horizontal caps. Viewed from above, S_j minus its horizontal caps then looks like two smooth circles, possibly nested, joined by a 1-handle, a neighborhood of an arc joining the two circles, as in Figure 1.4.

Figure 1.4



Since these circles bound disks, they can be isotoped to the standard position of one of the models, yielding an isotopy of S_j to one of the models.

lemma

The remaining cases, when S'_j has a local maximum or minimum, or no critical points, are similar but simpler, so we leave them as exercises. □

Now we assume the given surface S is a sphere. Since every circle in a sphere separates the sphere into two components, each surgery splits one sphere into two spheres. Reversing the sequence of surgeries, we then start with a collection of spheres S_j bounding balls. The inductive assertion is that at each stage of the reversed surgery process, we have a collection of spheres each bounding a ball.

For the inductive step we have two balls A and B bounded by the spheres ∂A and ∂B resulting from a surgery. Letting the ϵ for the surgery go to 0 isotopes A and B so that $\partial A \cap \partial B$ equals the horizontal surgery disk D. There are two cases, up to changes in notation:

(i) $A \cap B = D$, with pre-surgery sphere denoted $\partial(A + B)$

(ii) $B \subset A$, with pre-surgery sphere denoted $\partial(A - B)$.

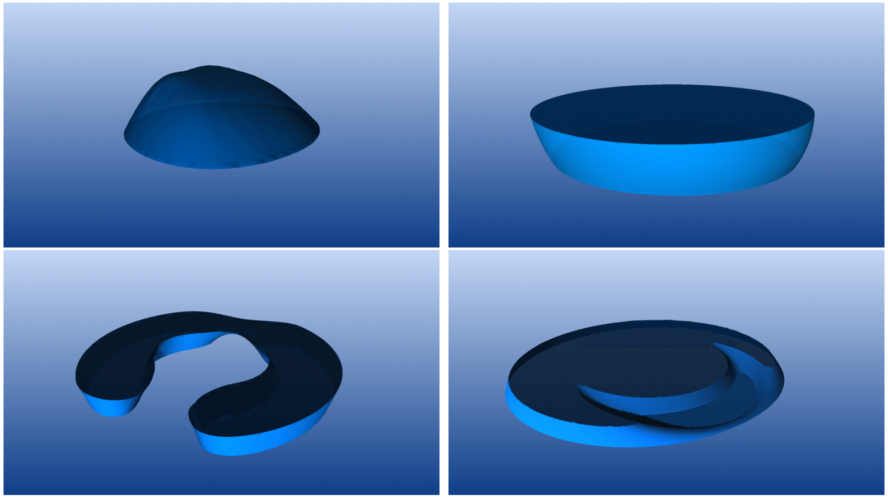
Since B is a ball, the lemma below implies that A and $A \pm B$ are diffeomorphic. Since A is a ball, so is $A \pm B$, and the inductive step is completed. □

Lemma 1.3. *Given an n-manifold M and a ball $B^{n-1} \subset \partial M$, let the manifold N be obtained from M by attaching a ball B^n via an identification of a ball $B^{n-1} \subset \partial B^n$ with the ball $B^{n-1} \subset \partial M$. Then M and N are diffeomorphic.*

Proof. Any two codimension-zero balls in a connected manifold are isotopic. Applying this fact to the given inclusion $B^{n-1} \subset \partial B^n$ and using isotopy extension, we conclude that the pair (B^n, B^{n-1}) is diffeomorphic to the standard pair. So there is an isotopy of ∂N to ∂M in N, fixed outside B^n, pushing $\partial N - \partial M$ across B^n to $\partial M - \partial N$. By isotopy extension, M and N are then diffeomorphic. □

One technical point that we have been somewhat lax about in the preceding arguments is the issue of corners. The surgery construction as we have described it produces surfaces in $\mathbb{R}^3$ that are not smooth along the curves bounded by the horizontal caps, since they have two distinct tangent planes at each point of these curves, one tangent plane being horizontal and the other one not horizontal. It would be easy to modify the definition of surgery to round these corners and make all the surgeries produce genuinely smooth surfaces. Similar considerations apply for the Lemma.

Figure 1.5 – *Each of these figures arises as a component from the surgery of a sphere.*



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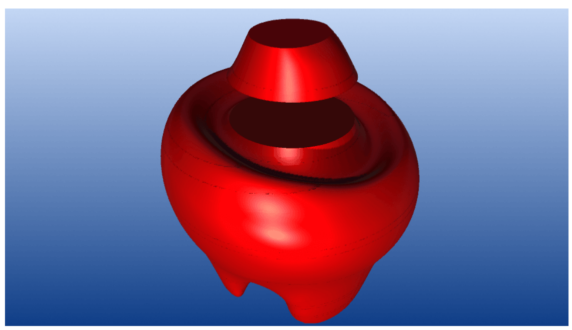
.../1-2/cylinder.mp4, .../1-2/arch.mp4, .../1-2/cup.mp4

Figure 1.6



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Existence and Uniqueness of Prime Decompositions

Let M be a 3-manifold and $S \subset M$ a surface which is properly embedded, i.e., $S \cap \partial M = \partial S$, a transverse intersection. For the moment we do not assume S is connected. Deleting a small open tubular neighborhood $N(S)$ of S from M, we obtain a 3-manifold $M|S$ which we say is obtained from M by *splitting* along S. The neighborhood $N(S)$ is an interval-bundle over S, so if M is orientable, $N(S)$ is a product $S \times (-\epsilon, \epsilon)$ iff S is orientable.

Now suppose that M is connected and S is a sphere such that $M|S$ has two components, M_1' and M_2'. Let M_i be obtained from M_i' by filling in its boundary sphere corresponding to S with a ball. In this situation we say M is the *connected sum* $M_1 \# M_2$. We remark that M_i is uniquely determined by M_i' since any two ways of filling in a ball B^3 differ by a diffeomorphism of ∂B^3, and any diffeomorphism of ∂B^3 extends to a diffeomorphism of B^3. This last fact follows from the stronger assertion that any diffeomorphism of S^2 is isotopic to either the identity or a reflection (orientation-reversing), and each of these two diffeomorphisms extends over a ball.

The connected sum operation is commutative by definition and has S^3 as an identity since a decomposition $M = M \# S^3$ is obtained by choosing the sphere S to bound a ball in M. The connected sum operation is also associative, since in a sequence of connected sum decompositions, e.g., $M_1\#(M_2\#M_3)$, the later splitting spheres can be pushed off the balls filling in earlier splitting spheres, so one may assume all the splitting spheres are disjointly embedded in the original manifold M. Thus $M = M_1\#\cdots\#M_n$ means there is a collection S consisting of $n-1$ disjoint spheres such that $M|S$ has n components M_i', with M_i obtained from M_i' by filling in with balls its boundary spheres corresponding to spheres of S.

A connected 3-manifold M is called *prime* if $M = P\#Q$ implies $P = S^3$ or $Q = S^3$. For example, Alexander's theorem implies that S^3 is prime, since every 2-sphere in S^3 bounds a 3-ball. The latter condition, stronger than primeness, is called irreducibility: M is *irreducible* if every 2-sphere $S^2 \subset M$ bounds a ball $B^3 \subset M$. The two conditions are in fact very nearly equivalent:

Side Note: A 2D Analogue

Proposition 1.4 and Theorem 1.5 (The Prime Decomposition Theorem) have analogues in the case of orientable 2-manifolds, which provide some motivation for the 3-manifold versions and may be easier to visualize. See Appendix A for the full argument.

Proposition 1.4*. *The only orientable prime 2-manifold which is not irreducible is* $S^1 \times S^1$.

Proof. If M is prime, every circle in M which separates M into two components bounds a disk. So if M is prime but not irreducible, there must exist a nonseparating circle in M. For a nonseparating circle C in an orientable manifold M, the union of a product neighbourhood $C \times I$ of C with a tubular neighbourhood of an arc joining $C \times \{0\}$ to $C \times \{1\}$ in the complement of $C \times I$ is a manifold diffeomorphic to $S^1 \times S^1$ minus a ball (see Figure 1.7 in Proposition 1.4). Thus M has $S^1 \times S^2$ as a connected summand (the boundary of this missing ball is the circle along which we split in order to obtain this connected summand). Assuming M is prime, then $M = S^1 \times S^1$.

... continued in Appendix A on page 30

Proposition 1.4. *The only orientable prime 3-manifold which is not irreducible is* $S^1 \times S^2$.

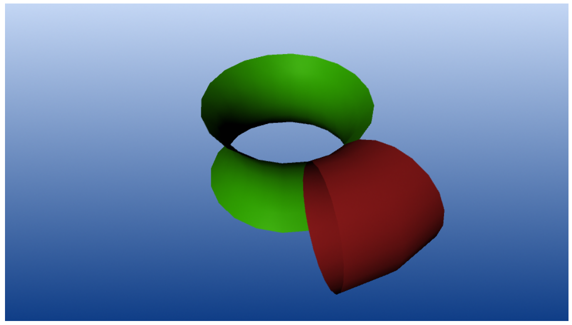
Proof. If M is prime, every 2-sphere in M which separates M into two components bounds a ball. So if M is prime but not irreducible there must exist a nonseparating sphere in M. For a nonseparating sphere S in an orientable manifold M the union of a product neighborhood $S \times I$ of S with a tubular neighborhood of an arc joining $S \times \{0\}$ to $S \times \{1\}$ in the complement of $S \times I$ is a manifold diffeomorphic to $S^1 \times S^2$ minus a ball. Thus M has $S^1 \times S^2$ as a connected summand. Assuming M is prime, then $M = S^1 \times S^2$.

It remains to show that $S^1 \times S^2$ is prime. Let $S \subset S^1 \times S^2$ be a separating sphere, so $S^1 \times S^2|S$ consists of two compact 3-manifolds V and W each with boundary a 2-sphere. We have $\mathbb{Z} = \pi_1(S^1 \times S^2) \cong \pi_1 V * \pi_1 W$, so either V or W must be simply-connected, say V is simply-connected. The universal cover of $S^1 \times S^2$ can be identified with $\mathbb{R}^3 - \{0\}$, and V lifts to a diffeomorphic copy $\tilde{V}$ of itself in $\mathbb{R}^3 - \{0\}$. The sphere $\partial\tilde{V}$ bounds a ball in $\mathbb{R}^3$ by Alexander's theorem. Since $\partial\tilde{V}$ also bounds $\tilde{V}$ in $\mathbb{R}^3$ we conclude that $\tilde{V}$ is a ball, hence also V. Thus every separating sphere in $S^1 \times S^2$ bounds a ball, so $S^1 \times S^2$ is prime. □

Theorem 1.5. *Let M be compact, connected, and orientable. Then there is a decomposition $M = P_1\#\cdots\#P_n$ with each P_i prime, and this decomposition is unique up to insertion or deletion of S^3's.*

Proof. The existence of prime decompositions is harder, and we tackle this first. If M contains a nonseparating S^2, this gives a decomposition $M = N\#S^1 \times S^2$, as we saw in the proof of

Figure 1.7 – To visualize this argument one dimension down, consider a product $S^1 \times I$ (red) along with a tubular neighbourhood connecting $S^1 \times \{0\}$ with $S^1 \times \{1\}$ (green). This is $S^1 \times S^1$ minus a 2-ball.



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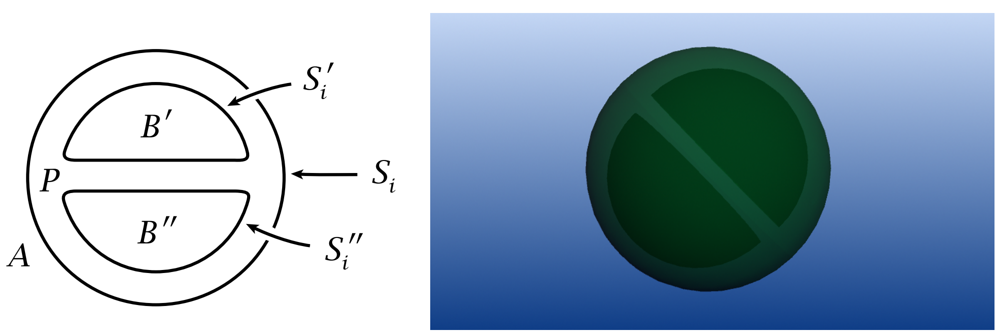
Proposition 1.4. We can repeat this step of splitting off an $S^1 \times S^2$ summand as long as we have nonseparating spheres, but the process cannot be repeated indefinitely since each $S^1 \times S^2$ summand gives a $\mathbb{Z}$ summand of $H_1(M)$, which is a finitely generated abelian group since M is compact. Thus we are reduced to proving existence of prime decompositions in the case that each 2-sphere in M separates. Each 2-sphere component of ∂M corresponds to a B^3 summand of M, so we may also assume ∂M contains no 2-spheres.

We shall prove the following assertion, which clearly implies the existence of prime decompositions: There is a bound on the number of spheres in a system S of disjoint spheres satisfying:

$(*)$ No component of $M|S$ is a punctured 3-sphere, i.e., a compact manifold obtained from S^3 by deleting finitely many open balls with disjoint closures. Before proving this we make a preliminary observation: If S satisfies $(*)$ and we do surgery on a sphere S_i of S using a disk $D \subset M$ with $D \cap S = \partial D \subset S_i$, then at least one of the systems S', S'' obtained by replacing S_i with the spheres S_i' and S_i'' resulting from the surgery satisfies $(*)$. To see this, first perturb S_i' and S_i'' to be disjoint from S_i and each other, so that S_i, S_i', and S_i'' together bound a 3-punctured sphere P, as in Figure 1.8.

Figure 1.8 – Surgering along a disk and perturbing the two remaining components inwards, we get a 3-punctured 3-sphere when the two surgered spheres are taken together with the original sphere.



katlas.math.toronto.edu/caldermf/3manifolds/1-5/perturb.mp4

On the other side of S_i from P we have a component A of $M|S$, while the spheres S_i' and S_i'' split the component of $M|S$ containing P into pieces B', B'', and P. If both B' and B'' were punctured spheres, then $B' \cup B'' \cup P$, a component of $M|S$, would be a punctured sphere, contrary to hypothesis. So one of B' and B'', say B', is not a punctured sphere. If $A \cup P \cup B''$ were a punctured sphere, this would force A to be a punctured sphere, by Alexander's theorem. This is also contrary to hypothesis. So we conclude that neither component of $M|S'$ adjacent to S'_i is a punctured sphere, hence the sphere system S' satisfies $(*)$.

Now we prove the assertion that the number of spheres in a system S satisfying $(*)$ is bounded. Choose a smooth triangulation $\mathcal{T}$ of M. This has only finitely many simplices since M is compact. The given system S can be perturbed to be transverse to all the simplices of $\mathcal{T}$. This perturbation can be done inductively over the skeleta of $\mathcal{T}$: First make S disjoint from vertices, then transverse to edges, meeting them in finitely many points, then transverse to 2-simplices, meeting them in finitely many arcs and circles.

For a 3-simplex τ of $\mathcal{T}$, we can make the components of $S \cap \tau$ all disks, as follows. Such a component must meet $\partial\tau$ by Alexander's theorem and condition $(*)$. Consider a circle C of $S \cap \partial\tau$ which is innermost in $\partial\tau$. If C bounds a disk component of $S \cap \tau$ we may isotope this disk to lie near $\partial\tau$ and then proceed to a remaining innermost circle C. If an innermost remaining C does not bound a disk component of $S \cap \tau$ we may surger S along C using a disk D lying near $\partial\tau$ with $D \cap S = \partial D = C$, replacing S by a new system S' satisfying $(*)$, in which either C does bound a disk component of $S' \cap \tau$ or C is eliminated from $S' \cap \tau$. After finitely many such steps we arrive at a system S with $S \cap \tau$ consisting of disks, for each τ. In particular, note that no component of the intersection of S with a 2-simplex of $\mathcal{T}$ can be a circle, since this would bound disks in both adjacent 3-simplices, forming a sphere of S bounding a ball in the union of these two 3-simplices, contrary to $(*)$.

Figure 1.9 – Surgering the sphere along a disk inside the simplex near its boundary, we either remove the component intersecting the simplex's boundary or remove the other component, leaving a sphere component intersecting the boundary of the simplex in a disk.

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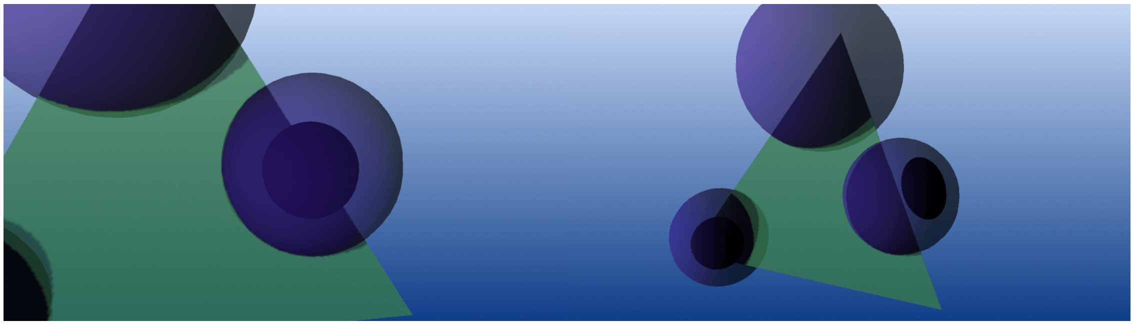
.../1-5/chop2.mp4

Next, for each 2-simplex σ we eliminate arcs α of $S \cap \sigma$ having both endpoints on the same edge of σ. Such an α cuts off from σ a disk D which meets only one edge of σ. We may choose α to be 'edgemost,' so that D contains no other arcs of $S \cap \sigma$, and hence $D \cap S = \alpha$ since circles of $S \cap \sigma$ have been eliminated in the previous step. By an isotopy of S supported near α we then push the intersection arc α across D, eliminating α from $S \cap \sigma$ and decreasing by two the number of points of intersection of S with the 1-skeleton of $\mathcal{T}$.

After such an isotopy decreasing the number of points of intersection of S with the 1-skeleton of $\mathcal{T}$ we repeat the first step of making S intersect all 3-simplices in disks. This does not increase the number of intersections with the 1-skeleton, so after finitely many steps, we arrive at the situation where S meets each 2-simplex only in arcs connecting adjacent sides, and S meets 3-simplices only

Figure 1.10



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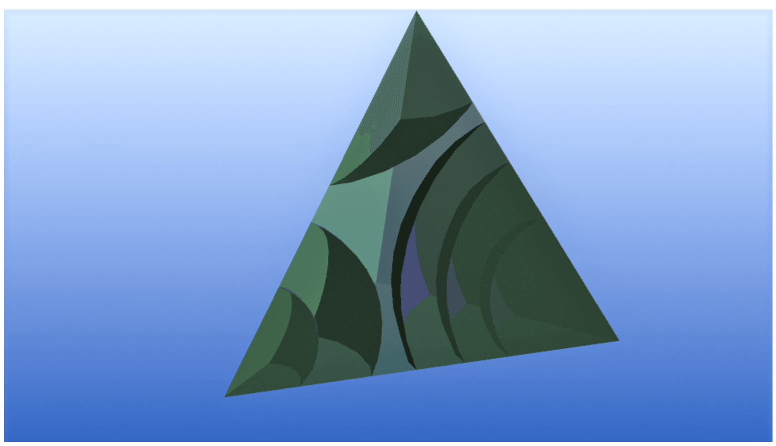
in disks.

Now consider the intersection of S with a 2-simplex σ. With at most four exceptions the complementary regions of $S \cap \sigma$ in σ are rectangles with two opposite sides on $\partial\sigma$ and the other two opposite sides arcs of $S \cap \sigma$, as in Figure 1.7. Thus if $\mathcal{T}$ has t 2-simplices, then all but at most $4t$ of the components of $M|S$ meet all the 2-simplices of $\mathcal{T}$ only in such rectangles.

Figure 1.11 – *Pictured are three rectangular regions, with the four rectangular regions corresponding to the front-facing 2-simplex lying at the corners and in the middle.*



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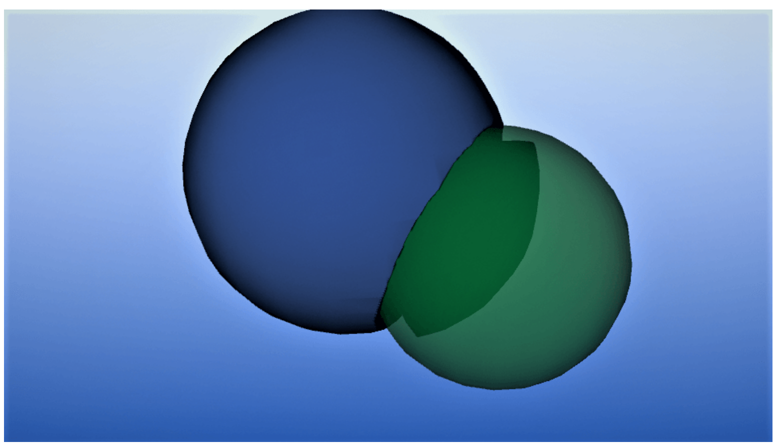
Let R be a component of $M|S$ meeting all 2-simplices only in rectangles. For a 3-simplex τ, each component of $R \cap \partial\tau$ is an annulus A which is a union of rectangles. The two circles of ∂A bound disks in τ, and A together with these two disks is a sphere bounding a ball in τ, a component of $R \cap \tau$ which can be written as $D^2 \times I$ with $\partial D^2 \times I = A$. The I-fiberings of all such products $D^2 \times I$ may be assumed to agree on their common intersections, the rectangles, to give R the structure of an I-bundle. Since ∂R consists of sphere components of S, R is either the product $S^2 \times I$ or the twisted I-bundle over $\mathbb{R}P^2$. (R is the mapping cylinder of the associated ∂I-subbundle, a union of spheres which is a two-sheeted covering space of a connected base surface.) The possibility $R = S^2 \times I$ is excluded by $(*)$. Each I-bundle R is thus the mapping cylinder of the covering space $S^2 \to \mathbb{R}P^2$. This is just $\mathbb{R}P^3$ minus a ball, so each I-bundle R gives a connected summand $\mathbb{R}P^3$ of M, hence a $\mathbb{Z}_2$ direct summand of $H_1(M)$. Thus the number of such components R of $M|S$ is bounded. Since the number of other components was bounded by $4t$, the number of components of $M|S$ is bounded. Since every 2-sphere in M separates, the number of components of $M|S$ is one more than the number of spheres in S. This finishes the proof of the existence of prime decompositions.

For uniqueness, suppose the nonprime M has two prime decompositions $M = P_1\#\cdots\#P_k\#\ell(S^1\times S^2)$ and $M = Q_1\#\cdots\#Q_m\#n(S^1\times S^2)$ where the P_i's and Q_i's are irreducible and not S^3. Let S be a disjoint union of 2-spheres in M reducing M to the P_i's, i.e., the components of $M|S$ are the manifolds $P_1,\cdots,P_k$ with punctures, plus possibly some punctured S^3's. Such a system S exists: Take for example a collection of spheres defining the given prime decomposition $M = P_1\#\cdots\#P_k\#\ell(S^1\times S^2)$ together with a nonseparating S^2 in each $S^1\times S^2$. Note that if S reduces M to the P_i's, so does any system S' containing S.

Similarly, let T be a system of spheres reducing M to the Q_i's. If $S\cap T\neq\varnothing$, we may assume this is a transverse intersection, and consider a circle of $S\cap T$ which is innermost in T, bounding a disk $D\subset T$ with $D\cap S=\partial D$. Using D, surger the sphere S_j of S containing ∂D to produce two spheres S_j' and S_j'', which we may take to be disjoint from S_j, so that S_j, S_j', and S_j'' together bound a 3-punctured 3-sphere P. By an earlier remark, the enlarged system $S\cup S_j'\cup S_j''$ reduces M to the P_i's. Deleting S_j from this enlarged system still gives a system reducing M to the P_i's since this affects only one component of $M|S\cup S_j'\cup S_j''$, by attaching P to one of its boundary spheres, which has the net effect of simply adding one more puncture to this component.

Figure 1.12



katlas.math.toronto.edu/caldermf/3manifolds/1-5/spheres.mp4

The new system S' meets T in one fewer circle, so after finitely many steps of this type we produce a system S disjoint from T and reducing M to the P_i's. Then $S\cup T$ is a system reducing M both to the P_i's and to the Q_i's. Hence $k=m$ and the P_i's are just a permutation of the Q_i's.

Finally, to show $l=n$, we have $M = N\#l(S^1\times S^2) = N\#n(S^1\times S^2)$, so $H_1(M)=H_1(N)\oplus\mathbb{Z}^l = H_1(N)\oplus\mathbb{Z}^n$, hence $l=n$. □

The proof of the Prime Decomposition Theorem applies equally well to manifolds which are not just orientable, but oriented. The advantage of working with oriented manifolds is that the operation of forming $M_1\#M_2$ from M_1 and M_2 is well-defined: Remove an open ball from M_1 and M_2 and then identify the two resulting boundary spheres by an orientation-reversing diffeomorphism, so the orientations of M_1 and M_2 fit together to give a coherent orientation of $M_1\#M_2$. The gluing map $S^2\to S^2$ is then uniquely determined up to isotopy, as we remarked earlier.

Thus to classify oriented compact 3-manifolds it suffices to classify the irreducible ones. In particular, one must determine whether each orientable irreducible 3-manifold possesses an orientation-reversing self-diffeomorphism.

To obtain a prime decomposition theorem for nonorientable manifolds requires very little more work. In Proposition 1.4 there are now two prime reducible manifolds, $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$, the nonorientable S^2 bundle over S^1, which can also arise from a nonseparating 2-sphere. Existence of prime decompositions then works as in the orientable case. For uniqueness, one observes that $N \# S^1 \times S^2 = N \# S^1 \tilde{\times} S^2$ if N is nonorientable. This is similar to the well-known fact in one lower dimension that connected sum of a nonorientable surface with the torus and with the Klein bottle give the same result. Uniqueness of prime decomposition can then be restored by replacing all the $S^1 \times S^2$ summands in nonorientable manifolds with $S^1 \tilde{\times} S^2$'s.

A useful criterion for recognizing irreducible 3-manifolds is the following:

Proposition 1.6. *If $p : \tilde{M} \to M$ is a covering space and $\tilde{M}$ is irreducible, then so is M.*

Proof. A sphere $S \subset M$ lifts to spheres $\tilde{S} \subset \tilde{M}$. Each of these lifts bounds a ball in $\tilde{M}$ since $\tilde{M}$ is irreducible. Choose a lift $\tilde{S}$ bounding a ball B in $\tilde{M}$ such that no other lifts of S lie in B, i.e., $\tilde{S}$ is an innermost lift. We claim that $p : B \to p(B)$ is a covering space. To verify the covering space property, consider first a point $x \in p(B) - S$, with U a small ball neighborhood of x disjoint from S. Then $p^{-1}(U)$ is a disjoint union of balls in $\tilde{M} - p^{-1}(S)$, and the ones of these in B provide a uniform covering of U. On the other hand, if $x \in S$, choose a small ball neighborhood U of x meeting S in a disk. Again $p^{-1}(U)$ is a disjoint union of balls, only one of which, $\tilde{U}$ say, meets B since we chose $\tilde{S}$ innermost and p is one-to-one on $\tilde{S}$. Therefore p restricts to a homeomorphism of $\tilde{U} \cap B$ onto a neighborhood of x in $p(B)$, and the verification that $p : B \to p(B)$ is a covering space is complete. This covering space is single-sheeted on $\tilde{S}$, hence on all of B, so $p : B \to p(B)$ is a homeomorphism with image a ball bounded by S. $\square$

The converse of Proposition 1.6 will be proved in Section 3.1.

By the proposition, manifolds with universal cover S^3 are irreducible. This includes $\mathbb{R}P^3$, and more generally each 3-dimensional lens space $L_{p/q}$, which is the quotient space of S^3 under the free $\mathbb{Z}_q$ action generated by the rotation $(z_1, z_2) \mapsto (e^{2\pi i/q} z_1, e^{2p\pi i/q} z_2)$, where S^3 is viewed as the unit sphere in $\mathbb{C}^2$.

For a product $M = S^1 \times F^2$, or more generally any surface bundle $F^2 \to M \to S^1$, with F^2 a compact connected surface other than S^2 or $\mathbb{R}P^2$, the universal cover of $M - \partial M$ is $\mathbb{R}^3$, so such an M is irreducible.

Curiously, the analogous covering space assertion with 'irreducible' replaced by 'prime' is false, since there is a 2-sheeted covering $S^1 \times S^2 \to \mathbb{R}P^3 \# \mathbb{R}P^3$. Namely, $\mathbb{R}P^3 \# \mathbb{R}P^3$ is the quotient of $S^1 \times S^2$ under the identification $(x, y) \sim (\rho(x), -y)$ with ρ a reflection of the circle. This quotient can also be described as the quotient of $I \times S^2$ where (x, y) is identified with $(x, -y)$ for $x \in \partial I$. In this description the 2-sphere giving the decomposition $\mathbb{R}P^3 \# \mathbb{R}P^3$ is $\{\frac{1}{2}\} \times S^2$.

Exercises

1. Prove the (smooth) Schoenflies theorem in $\mathbb{R}^2$: An embedded circle in $\mathbb{R}^2$ bounds an embedded disk.
2. Show that for compact M^3 there is a bound on the number of 2-spheres S_i which can be embedded in M disjointly, with no S_i bounding a ball and no two S_i's bounding a product $S^2 \times I$.
3. Use the method of proof of Alexander's theorem to show that every torus $T \subset S^3$ bounds a solid torus $S^1 \times D^2 \subset S^3$ on one side or the other. (This result is also due to Alexander.)
4. Develop an analog of the prime decomposition theorem for splitting a compact irreducible 3-manifolds along disks rather than spheres. In a similar vein, study the operation of splitting nonorientable manifolds along $\mathbb{R}P^2$'s with trivial normal bundles.
5. Show: If $M^3 \subset \mathbb{R}^3$ is a compact submanifold with $H_1(M) = 0$, then $\pi_1(M) = 0$.

1.2 Torus Decomposition

Beyond the prime decomposition, there is a further canonical decomposition of irreducible compact orientable 3-manifolds, splitting along tori rather than spheres. This was discovered only in the mid 1970's, by Johannson and Jaco-Shalen, though in the simplified geometric version given here it could well have been proved in the 1930's. (A 1967 paper of Waldhausen comes very close to this geometric version.) Perhaps the explanation for this late discovery lies in the subtlety of the uniqueness statement. There are counterexamples to a naive uniqueness statement, involving a class of manifolds studied extensively by Seifert in the 1930's. The crucial observation, not made until the 1970's, was that these Seifert manifolds give rise to the only counterexamples. It then becomes possible to get a unique decomposition by treating the Seifert submanifolds in a given manifold as pieces that are to be left intact, and not decomposed.

Existence of Torus Decompositions

A properly embedded connected surface $S \subset M^3$ is called *2-sided* if its normal bundle is trivial, and *1-sided* if its normal bundle is nontrivial. (The 'sides' of S then correspond to the components of the complement of S in a tubular neighborhood.) A 2-sided connected surface S other than S^2 or D^2 is called *incompressible* if for each disk $D \subset M$ with $D \cap S = \partial D$ there is a disk $D' \subset S$ with $\partial D' = \partial D$. See Figure 1.8. Thus, surgery on S in M cannot produce a simpler surface, but only splits off an S^2 from S, leaving a diffeomorphic copy of S as the other piece resulting from the surgery.

A disk D with $D \cap S = \partial D$ will sometimes be called a *compressing disk* for S, whether or not a disk $D' \subset S$ with $\partial D' = \partial D$ exists.

Figure 1.13 – If M is $\mathbb{R}^3$ with tubular neighbourhoods of both the unit circle and the z-axis removed and S is a torus embedded around the removed circle, we get that S is incompressible in M; the disk in S bounded by a compressing disk D is shown in blue.

katlas.math.toronto.edu/caldermf/3manifolds/1-7/incompressible.mp4

Here are some preliminary facts about incompressible surfaces:

(1) A connected 2-sided surface S which is not a sphere or disk is incompressible if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective. This is because if $D \subset M$ is a compressing disk, then ∂D is nullhomotopic in M, hence also in S if the map $\pi_1(S) \to \pi_1(M)$ is injective. Then it is a standard fact in surface theory that a nullhomotopic embedded circle in a surface must bound a disk in the surface. Note that it is suffices to assume that the two inclusions of S into $M|S$ on either side of S are injective on π_1.

Side Note: A Nullhomotopic Circle in a Surface Bounds a Disk

A proof of the fact that a nullhomotopic embedded circle in a surface must bound a disk using universal covers is included in Appendix B.

Proposition. *A nullhomotopic circle in a compact surface S bounds a disk in S.*

Proof. If $S = S^2$, this follows by the 2-dimensional version of Alexander's theorem. If S is any surface other than S^2, then its universal cover must be $\mathbb{R}^2$. Let C be a nullhomotopic circle in S. Since C is nullhomotopic, it lifts to a set of disjoint circles in $\mathbb{R}^2$; we choose one of these circles and denote it by $\tilde{C}$, and denote by $\tilde{D}$ the disk in $\mathbb{R}^2$ bounded by $\tilde{C}$. We would like to show that the restricting the covering map from $\mathbb{R}^2$ to S to the disk $\tilde{D}$ gives us an injective map $p : \tilde{D} \to S$.

. . . continued in Appendix B on page 33

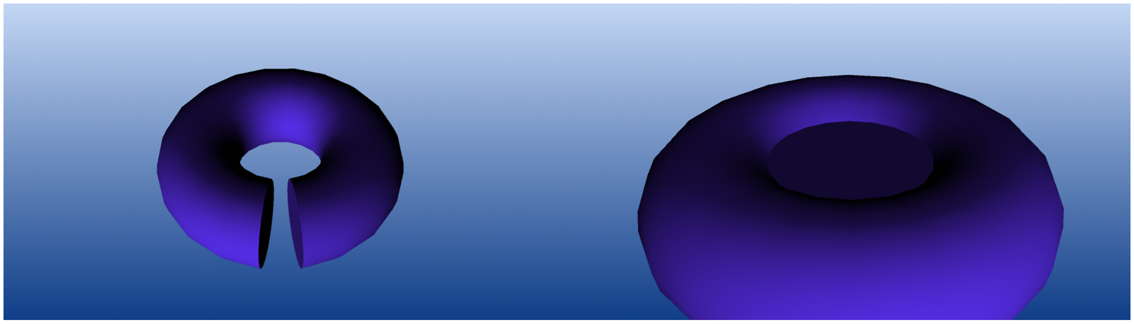
The converse of (1) is also true, but this is a more difficult result, proved in Chapter 3 as Corollary 3.3. For 1-sided surfaces these two conditions for incompressibility are no longer equivalent, π_1-injectivity being strictly stronger in general; see the exercises. Since the definition we have given requires incompressible surfaces to be 2-sided, this potential source of confusion should be avoided.

(2) There are no incompressible surfaces in $\mathbb{R}^3$ or, equivalently, in S^3. This is immediate from the converse to (1), but can also be proved directly, as follows. As we saw in the proof of Alexander's theorem, after isotopically perturbing a surface $S \subset \mathbb{R}^3$ to make the height function a morse function with all critical points lying in different levels, there is a sequence of surgeries on S along horizontal disks converting S into a disjoint union of spheres. If S is incompressible, each successive surgery splits S into two surfaces one of which is a sphere. This sphere bounds balls on each side in S^3, and we can use one of these balls to isotope S in S^3 to the other surface produced by the surgery. At the end of the sequence of surgeries we have isotoped S to a sphere, but the definition of incompressibility does not allow spheres.

(3) A 2-sided torus T in an irreducible M is compressible iff T either bounds a solid torus $S^1 \times D^2 \subset M$ or lies in a ball in M. (Figure 1.13 depicts an example of a torus for which neither of these conditions hold, and so is incompressible.) For if T is compressible there is a surgery of T along some disk D which turns T into a sphere. This sphere bounds a ball $B \subset M$ by the assumption that M is irreducible. There are now two cases: If $B \cap D = \varnothing$ then reversing the surgery glues together two disks in ∂B to create a solid torus bounded by T. The other possibility is that $D \subset B$, and then $T \subset B$. Note that if $M = S^3$ the ball B can be chosen disjoint from D, so the alternative $D \subset B$ is not needed. Thus using statement (2) above we obtain the result, due to Alexander, that a torus in S^3 bounds a solid torus on one side or the other.

(4) If $S \subset M$ is a finite collection of disjoint incompressible surfaces, then M is irreducible iff $M|S$ is irreducible. For suppose first that M is irreducible. Then a 2-sphere in $M|S$ bounds a ball in M, and this ball must be disjoint from S by statement (2) above, so the sphere bounds a ball in $M|S$ and $M|S$ is irreducible. Conversely, given a sphere $S^2 \subset M$ which we may assume is transverse to S, consider a circle of $S \cap S^2$ which is innermost in S^2, bounding a disk $D \subset S^2$ with $D \cap S = \partial D$. By incompressibility of the components of S, ∂D bounds a disk $D' \subset S$. The sphere $D \cup D'$ bounds a ball $B \subset M$ if $M|S$ is irreducible. We must have

Figure 1.14 – If D is the disk along which the torus is surgered, then ball B resulting from the surgery either has the property that $D \cap B = \varnothing$ (example shown on the left), or that $D \subset B$ (example shown on the right). Reversing the surgery in the first case leaves us with a solid torus, and in the second case we have $T \subset B$.



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.../1-7/compressible-torus2.mp4

$B \cap S = D'$, otherwise a component of S would be contained in B, contrary to statement (2). Isotoping S by pushing D' across B to D and slightly beyond eliminates the circle ∂D from $S \cap S^2$, along with any other circles of $S \cap S^2$ which happen to lie in D'. For the new S it is still true that $M|S$ is irreducible since it is diffeomorphic to the old $M|S$ by isotopy extension. Thus we can repeat this process of eliminating circles of $S \cap S^2$ until we eventually get $S \cap S^2 = \varnothing$, or in other words $S^2 \subset M|S$. Since $M|S$ is irreducible, S^2 then bounds a ball in $M|S$ and therefore also in M.

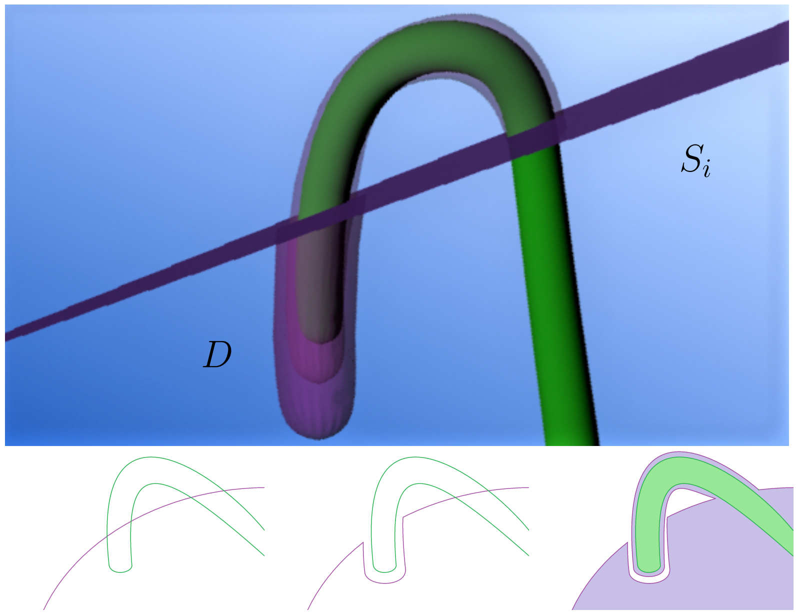
(5) If $S \subset M$ is a finite collection of disjoint, properly embedded surfaces that are either incompressible or spheres or disks, then a surface $T \subset M|S$ is incompressible in M iff it is incompressible in $M|S$. It is obvious that incompressibility in M implies incompressibility in $M|S$. For the less trivial converse, let $D \subset M$ be a compressing disk for T. If this intersects S, we can assume the intersection is transverse, so $D \cap S$ consists of circles in the interior of D. If any of these circles bound disks in S, we can take such a circle that is innermost in S, bounding a disk $D_0 \subset S$ with $D_0 \cap D = \partial D_0$. We can use D_0 to surger D, producing a new disk D intersecting S in fewer circles. After repeating this step a finite number of times, we may assume $D \cap S$ contains no circles that bound disks in S.

If there are any remaining circles in $D \cap S$, choose one that is innermost in D, bounding a disk $D_0 \subset D$ with $D_0 \cap S = \varnothing$. Since the components of S are either incompressible or spheres or disks, the circle ∂D_0 must bound a disk in S, contrary to what we have arranged by the earlier surgery. Thus we must have $D \cap S = \varnothing$. Since T is incompressible in $M|S$, the circle ∂D then bounds a disk in T. This shows T is incompressible in M.

Proposition 1.7. *For a compact irreducible M there is a bound on the number of components in a system $S = S_1 \cup \cdots \cup S_n$ of disjoint closed incompressible surfaces $S_i \subset M$ such that no component of $M|S$ is a product $T \times I$ with T a closed surface.*

Proof. This follows the scheme of the proof of existence of prime decompositions. First, perturb S to be transverse to a triangulation of M and perform the following two steps repeatedly to simplify the intersections of S with 2-simplices σ^2 and 3-simplices σ^3: (1) Make all components of $S \cap \sigma^3$ disks. In the proof of prime decomposition, this was done by surgery, but now the surgeries can be

Figure 1.15 – In red, an example of an incompressible component S_i of S intersects D, the plane shown in purple. To eliminate this intersection, disks bounded in S_i are perturbed and adjoined to D through surgery; two iterations of this process are shown, starting with the innermost circle in $S_i \cap D$ and ending with a new disk D which does not intersect S_i.



achieved by isotopy. Namely, given a surgery disk $D \subset M$ with $D \cap S = \partial D$, incompressibility gives a disk $D' \subset S$ with $\partial D' = \partial D$. The sphere $D \cup D'$ bounds a ball $B \subset M$ since M is irreducible. We have $B \cap S = D'$, otherwise a component of S would lie in B. Then isotoping S by pushing D' across B to D and a little beyond replaces S by one of the two surfaces produced by the surgery.

Note that Step (1) eliminates circles of $S \cap \sigma^2$, since such a circle would bound disks in both adjacent σ^3's, producing a sphere component of S. (2) Eliminate arcs of $S \cap \sigma^2$ with both endpoints on the same edge of σ^2. This can be done by isotopy of S just as in the prime decomposition theorem. After these simplifications, components of $M|S$ meeting 2-simplices only in rectangles are I-bundles (disjoint from ∂M), as before. Trivial I-bundles are ruled out by hypothesis. The nontrivial I-bundles are bounded by surfaces S_i and are tubular neighborhoods of 1-sided surfaces S_i'. Replacing these S_i's by the corresponding surfaces S_i' produces a new system S'. Splitting M along S' produces a manifold M' which is $M|S$ with the nontrivial I-bundles deleted. The number of components of M' is thus bounded by four times the number of 2-simplices in the triangulation of M. Consider now the exact sequence

$$H_3(M, S'; \mathbb{Z}_2) \to H_2(S'; \mathbb{Z}_2) \to H_2(M; \mathbb{Z}_2)$$

By excision the first group can be replaced by $H_3(M', \partial M' - \partial M; \mathbb{Z}_2)$. This is a vector space over $\mathbb{Z}_2$ of dimension bounded by the number of components of M' since, after triangulating M', there is at most one nontrivial simplicial relative 3-cycle with $\mathbb{Z}_2$ coefficients in each component of M'. Thus the first and third terms of the exact sequence have dimensions bounded by numbers depending only on M, not S. By exactness the dimension of the middle term $H_2(S'; \mathbb{Z}_2) \cong H_2(S; \mathbb{Z}_2)$ is also bounded, so the number of components of S is bounded. $\square$

A properly embedded surface $S \subset M$ is called *∂-parallel* if it is isotopic, fixing ∂S, to a subsurface of ∂M. By isotopy extension this is equivalent to saying that S splits off a product $S \times [0,1]$ from

M with $S = S \times \{0\}$. An irreducible manifold M is called *atoroidal* if every incompressible torus in M is ∂-parallel.

Corollary 1.8. *In a compact connected irreducible M there exists a (possibly empty) finite collection T of disjoint incompressible tori such that each component of $M|T$ is atoroidal.*

Proof. If M is atoroidal we take $T = \varnothing$. Otherwise, let T_1 be an incompressible torus in M that is not ∂-parallel. If $M|T_1$ is atoroidal we take $T = T_1$, and otherwise we let T_2 be an incompressible torus in $M|T_1$ that is not ∂-parallel. This process can be repeated as long as we do not obtain a splitting into atoroidal components, but this cannot be done infinitely often, otherwise we would have an arbitrarily large collection T of disjoint incompressible tori in M with no component of $M|T$ a product of a torus with I, contradicting the previous proposition. ☐

Now we describe an example of an irreducible M where this torus decomposition into atoroidal pieces is not unique, the components of $M|T$ for the two splittings being in fact nonhomeomorphic.

Example. For $i = 1, 2, 3, 4$, let M_i be a solid torus whose boundary torus is decomposed as the union of two annuli A_i and A_i' each winding $q_i > 1$ times around the S^1 factor of M_i. The union of these four solid tori, with each A_i' glued to A_{i+1} (subscripts mod 4), is the manifold M. This contains two tori $T_1 = A_1 \cup A_3$ and $T_2 = A_2 \cup A_4$. The components of $M|T_1$ are $M_1 \cup M_2$ and $M_3 \cup M_4$, and the components of $M|T_2$ are $M_2 \cup M_3$ and $M_4 \cup M_1$. The fundamental group of $M_i \cup M_{i+1}$ has presentation $\langle x_i, x_{i+1} | x_i^{q_i} = x_{i+1}^{q_{i+1}} \rangle$. The center of this amalgamated free product is cyclic, generated by the element $x_i^{q_i} = x_{i+1}^{q_{i+1}}$. Factoring out the center gives quotient $\mathbb{Z}_{q_i} * \mathbb{Z}_{q_{i+1}}$, with abelianization $\mathbb{Z}_{q_i} \oplus \mathbb{Z}_{q_{i+1}}$. Thus if the q_i's are for example distinct primes, then no two of the manifolds $M_i \cup M_{i+1}$ are homeomorphic.

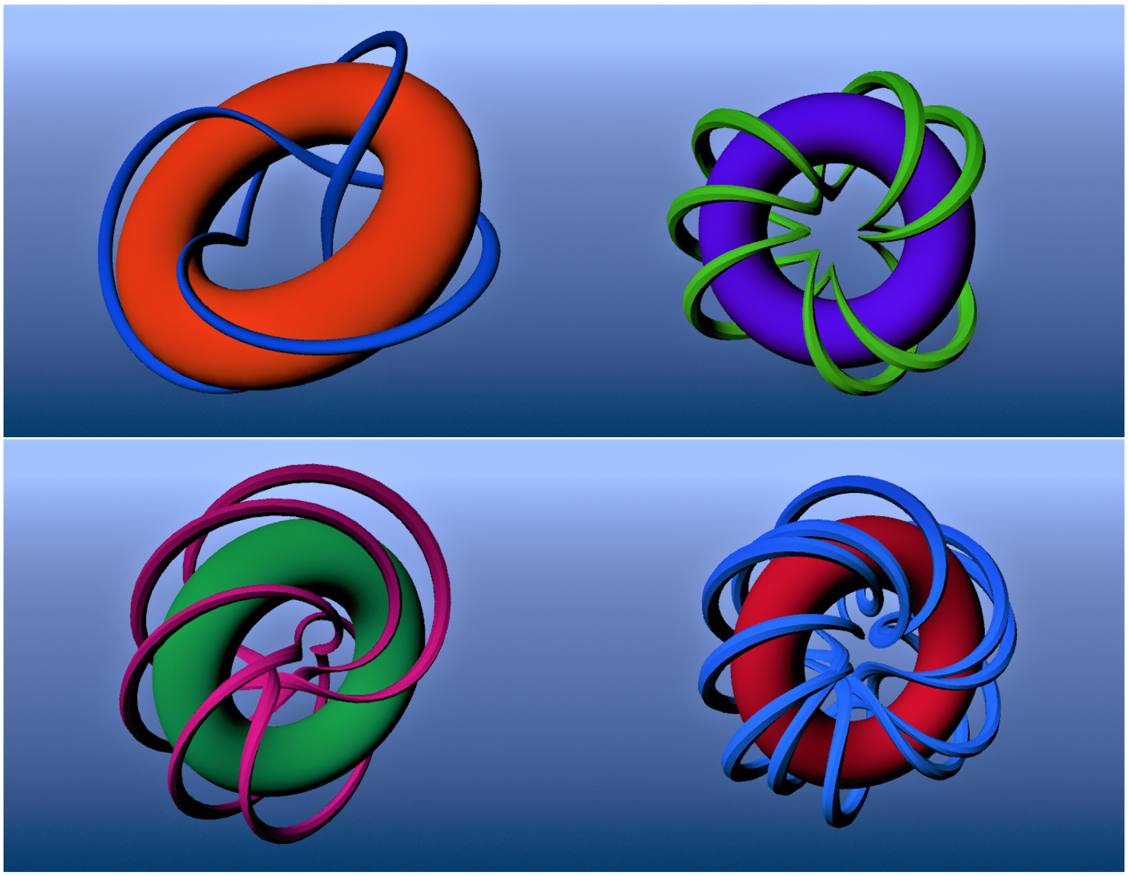
Results from later in this section will imply that M is irreducible, T_1 and T_2 are incompressible, and the four manifolds $M_i \cup M_{i+1}$ are atoroidal. So the splittings $M|T_1$ and $M|T_2$, though quite different, both satisfy the conclusions of the Corollary.

Manifolds like this M which are obtained by gluing together solid tori along noncontractible annuli in their boundaries belong to a very special class of manifolds called Seifert manifolds, which we now define. A *model Seifert fibering* of $S^1 \times D^2$ is a decomposition of $S^1 \times D^2$ into disjoint circles, called *fibers*, constructed as follows. Starting with $[0,1] \times D^2$ decomposed into the segments $[0,1] \times \{x\}$, identify the disks $\{0\} \times D^2$ and $\{1\} \times D^2$ via a $2\pi p/q$ rotation, for $p/q \in \mathbb{Q}$ with p and q relatively prime. The segment $[0,1] \times \{0\}$ then becomes a fiber $S^1 \times \{0\}$, while every other fiber in $S^1 \times D^2$ is made from q segments $[0,1] \times \{x\}$. A *Seifert fibering* of a 3-manifold M is a decomposition of M into disjoint circles, the *fibers*, such that each fiber has a neighborhood diffeomorphic, preserving fibers, to a neighborhood of a fiber in some model Seifert fibering of $S^1 \times D^2$. A *Seifert manifold* is one which possesses a Seifert fibering.

Each fiber circle C in a Seifert fibering of a 3-manifold M has a well-defined *multiplicity*, the number of times a small disk transverse to C meets each nearby fiber. For example, in the model Seifert fibering of $S^1 \times D^2$ with $2\pi p/q$ twist, the fiber $S^1 \times \{0\}$ has multiplicity q while all other fibers have multiplicity 1. Fibers of multiplicity 1 are *regular* fibers, and the other fibers are *multiple* (or *singular*, or *exceptional*). The multiple fibers are isolated and lie in the interior of M. The quotient space B of M obtained by identifying each fiber to a point is a surface, compact if M is compact, as is clear from the model Seifert fiberings. The projection $\pi : M \to B$ is an ordinary fiber bundle on the complement of the multiple fibers. In particular, $\pi : \partial M \to \partial B$ is a circle bundle, so ∂M consists of tori and Klein bottles, or just tori if M is orientable.

The somewhat surprising fact is that Seifert manifolds account for *all* the non-uniqueness in torus splittings, according to the following theorem, which is the main result of this section.

Figure 1.16 – *Visual representations of various model Seifert fiberings of $S^1 \times D^2$, with the outer helix around each solid torus T representing a fibering of a point x near the edge of T's D^2 component. From left to right, top to bottom, the tori shown have been twisted by angles of $2\pi\frac{3}{2}$,$2\pi\frac{7}{2}$, $2\pi\frac{5}{4}$, and $2\pi\frac{11}{4}$ respectively. The linked animation shows the twisting from the standard fibering of the torus to a rotation of $2\pi\frac{9}{5}$.*



katlas.math.toronto.edu/caldermf/3manifolds/1-7/seifert-follow.mp4

Theorem 1.9. *For a compact irreducible orientable 3-manifold M there exists a collection $T \subset M$ of disjoint incompressible tori such that each component of $M|T$ is either atoroidal or a Seifert manifold, and a minimal such collection T is unique up to isotopy.*

Here 'minimal' means minimal with respect to inclusions of such collections. Note the strength of the uniqueness: up to isotopy, not just up to homeomorphism of M, for example. The orientability assumption can be dropped if splittings along incompressible Klein bottles are also allowed, and the definition of 'atoroidal' is modified accordingly. For simplicity we will stick to the orientable case, however.

We have already proved the existence statement in the theorem, where Seifert manifolds are unnecessary. We will prove the uniqueness statement assuming four facts about Seifert manifolds, which will be proved in the next subsection. To state the four facts we first need another definition. A properly embedded surface $S \subset M$ which is not a disk or S^2 is *∂-incompressible* if for each disk $D \subset M$ such that $\partial D \cap S$ is an arc α in ∂D and the rest of ∂D lies in ∂M (such a D is called a *∂-compressing disk* for S) there is a disk $D' \subset S$ with $\alpha \subset \partial D'$ and $\partial D' - \alpha \subset \partial S$. If S is a disk, it is called ∂-incompressible if it is both ∂-incompressible in the definition above as well as not ∂-parallel.

We call a connected 2-sided surface S which is properly embedded in a 3-manifold M *essential* if any of the following hold:

- S is a sphere that does not bound a ball.
- S is a disk which does not split off a ball from M (a disk such that $M|S$ does not have two components, one of which is a ball.)
- S is both incompressible and ∂-incompressible.

We leave it as an exercise to show that a surface S is essential iff each component of S is essential. Also, as in the absolute case, S is ∂-incompressible if $\pi_1(S, \partial S) \to \pi_1(M, \partial M)$ is injective for all choices of basepoint in ∂S.

Incompressible Surfaces in Seifert Manifolds

Example. Let us show that the only essential surfaces in the manifold $M = S^1 \times D^2$ are disks isotopic to meridian disks $\{x\} \times D^2$. For let S be a connected essential surface in M. We may isotope S so that all the circles of ∂S are either meridian circles $\{x\} \times \partial D^2$ or are transverse to all meridian circles. By a small perturbation S can also be made transverse to a fixed meridian disk D_0. Circles of $S \cap D_0$ can be eliminated, innermost first, by isotopy of S using incompressibility of S and irreducibility of M. After this has been done, consider an edgemost arc α of $S \cap D_0$. This cuts off a ∂-compressing disk D from D_0, so α also cuts off a disk D' from S, meeting ∂M in an arc γ. The existence of D' implies that the two ends of γ lie on the same side of the meridian arc $\beta = D \cap \partial M$ in ∂M. But this is impossible since γ is transverse to all meridians and therefore proceeds monotonically through the meridian circles of ∂M. Thus we must have S disjoint from D_0, so ∂S consists of meridian circles. Moreover, S is incompressible in $M|D_0$, a 3-ball, so S must be a disk since each of its boundary circles bounds a disk in the boundary of the ball, and pushing such a disk slightly into the interior of the ball yields a compressing disk for S. It follows from Alexander's theorem that any two disks in a ball having the same boundary are isotopic fixing the boundary, so S is isotopic to a meridian disk in M.

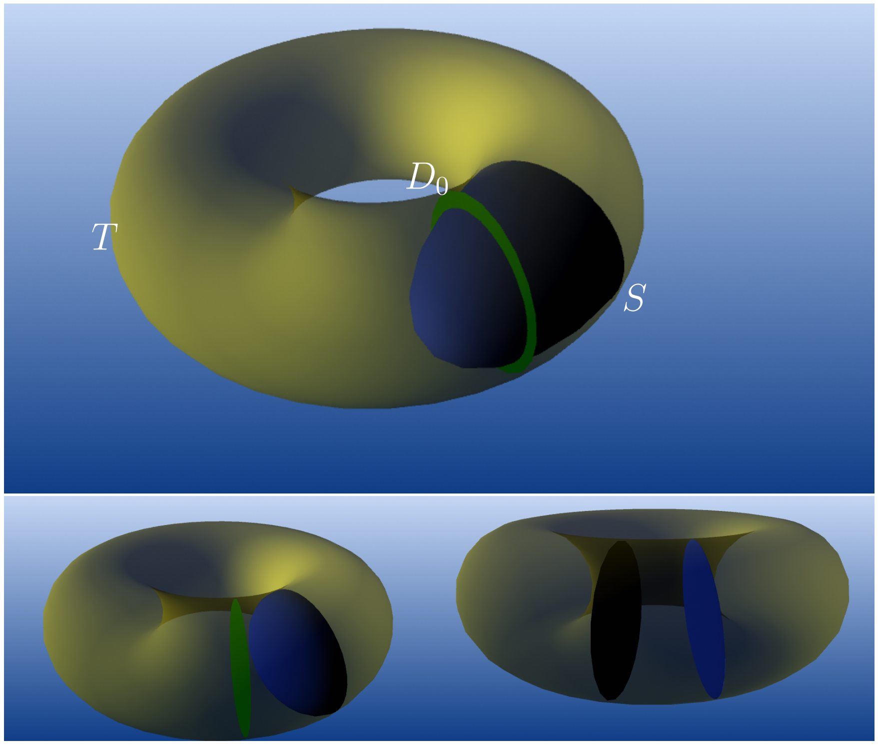
Lemma 1.10. *Let S be a connected incompressible surface in the irreducible 3-manifold M, with ∂S contained in torus boundary components of M. Then either S is essential or it is a ∂-parallel annulus.*

Proof. Suppose S is ∂-compressible, with a ∂-compressing disk D meeting S in an arc α which does not cut off a disk from S. Let β be the arc $D \cap \partial M$, lying in a torus component T of ∂M. The circles of $S \cap T$ do not bound disks in T, otherwise incompressibility of S would imply S was a disk, but disks are ∂-incompressible. Thus β lies in an annulus component A of $T|\partial S$. If β were trivial in A, cutting off a disk D', incompressibility applied to the disk $D \cup D'$ would imply that α cuts off a disk from S, contrary to assumption; see Figure 1.10(a).

So β joins the two components of ∂A. If both of these components are the same circle of ∂S, i.e., if $S \cap T$ consists of a single circle, then S would be 1-sided. For consider the normals to S pointing into D along α. At the two points of $\partial\alpha$ these normals point into β, hence point to opposite sides of the circle $S \cap T$.

Thus the endpoints of β must lie in different circles of ∂S, and we have the configuration in Figure 1.10(b). Let N be a neighborhood of $\partial A \cup \alpha$ in S, a 3-punctured sphere. The circle $\partial N - \partial S$ bounds an obvious disk in the complement of S, lying near $D \cup A$, so since S is incompressible this boundary circle also bounds a disk in S. Thus S is an annulus. Surgering the torus $S \cup A$ via D

Figure 1.17 – An example of the case where the boundary circle of S (blue) is a meridian circle. First, S is isotoped to be disjoint from D_0 (green), then when $S \cap D_0 = \varnothing$, S is isotoped to the meridian disk bounded by the circle in its boundary.



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yields a sphere, which bounds a ball in M since M is irreducible. Hence $S \cup A$ bounds a solid torus and S is ∂-parallel, being isotopic to A rel ∂S. □

Proposition 1.11. *If M is a connected compact irreducible Seifert-fibered manifold, then any essential surface in M is isotopic to a surface which is either* vertical*, i.e., a union of regular fibers, or* horizontal*, i.e., transverse to all fibers.*

Proof. Let $C_1, \cdots, C_n$ be fibers of the Seifert fibering which include all the multiple fibers together with at least one regular fiber if there are no multiple fibers.e o Let M_0 be M with small fibered open tubular neighborhoods of all the C_i's deleted. Thus M_0 is a circle bundle $M_0 \to B_0$ over a compact connected surface B_0 with nonempty boundary.

Choose disjoint arcs in B_0 whose union splits B_0 into a disk, and let A be the pre-image in M_0 of this collection of arcs, a union of disjoint vertical annuli $A_1, \cdots, A_m$ in M_0 such that the manifold $M_1 = M_0|A$ is a solid torus.

For an essential surface S in M, the circles of ∂S are nontrivial in ∂M since S is incompressible and M is irreducible. Hence S can be isotoped so that the circles of ∂S are either vertical or horizontal in each component torus or Klein bottle of ∂M. Vertical circles of S may be perturbed to be disjoint from A. We may assume S meets the fibers C_i transversely, and hence meets the neighborhoods of these fibers in disks transverse to fibers. So the surface $S_0 = S \cap M_0$ also has each its boundary circles horizontal or vertical.

Circles of $S \cap A$ bounding disks in A can be eliminated by isotopy of S in the familiar way, using incompressibility of S and irreducibility of M. Arcs of $S \cap A$ with both endpoints on the same

Figure 1.18 – The (impossible) case where $S \cap T$ consists of a single circle.

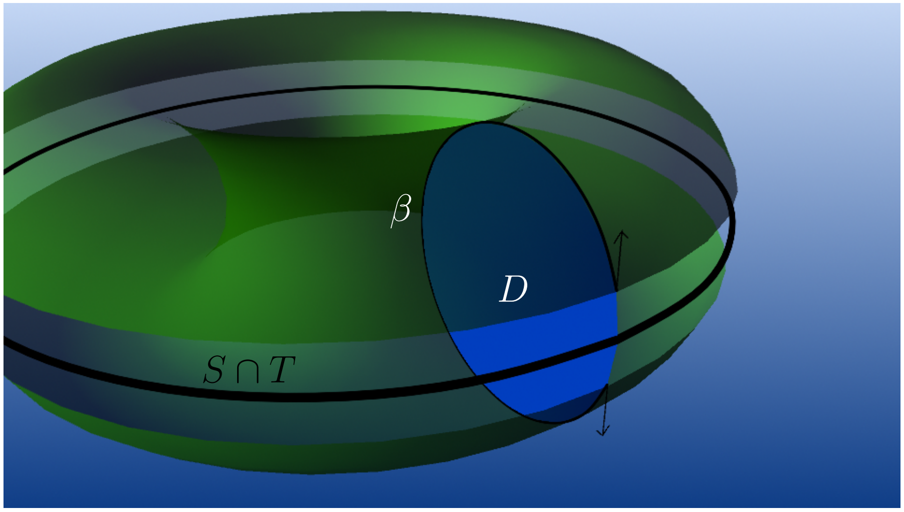
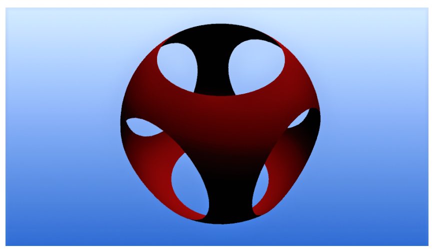


Figure 1.19 – An example of a possible base surface over which M_0 is a circle bundle.



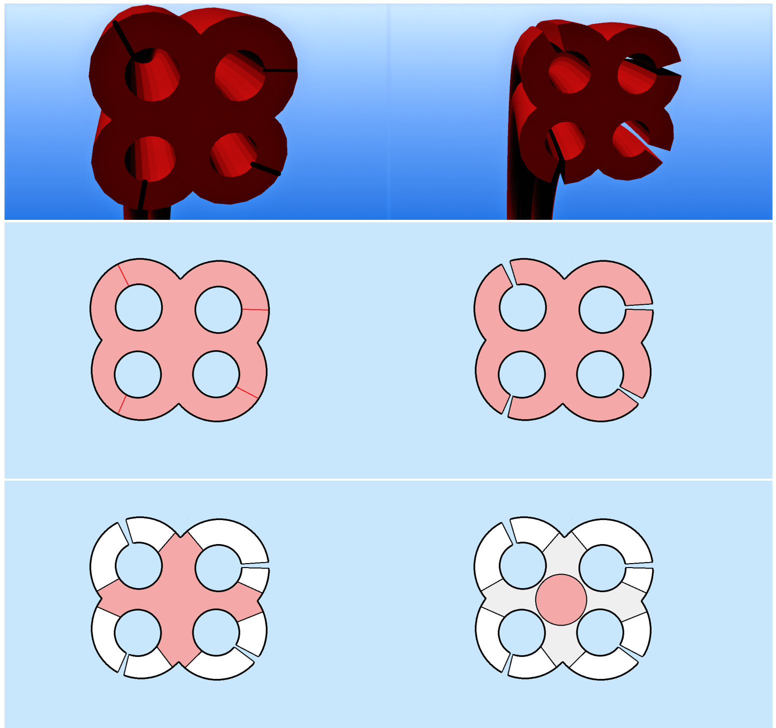
component of ∂A can be eliminated as follows. An edgemost such arc α cuts off a disk D from A. If the two endpoints of α lie in a component of $\partial M_0 - \partial M$, then S can be isotoped across D to eliminate two intersection points with a fiber C_i. The other possibility, that the two endpoints of α lie in ∂M, cannot actually occur, for if it did, the disk D would be a ∂-compressing disk for S in M, a configuration ruled out by the monotonicity argument in the Example preceding Lemma 1.10, with the role of meridians in that argument now played by vertical circles.

So we may assume the components of $S \cap A$ are either vertical circles or horizontal arcs. If we let $S_1 = S_0|A$ in $M_0|A = M_1$, it follows that ∂S_1 consists entirely of horizontal or vertical circles in the torus ∂M_1. We may assume S_1 is incompressible in M_1. For let $D \subset M_1$ be a compressing disk for S_1. Since S is incompressible, ∂D bounds a disk $D' \subset S$. If this does not lie in S_1, we can isotope S by pushing D' across the ball bounded by $D \cup D'$, thereby eliminating some components of $S \cap A$.

Since S_1 is incompressible, its components are either ∂-parallel annuli or are essential in the solid torus M_1, hence are isotopic to meridian disks by the Example before Lemma 1.10. If S_1 contains a ∂-parallel annulus with horizontal boundary, then this annulus has a ∂-compressing disk D with $D \cap \partial M_1$ a vertical arc in ∂M_0. As in the earlier step when we eliminated arcs of $S \cap A$ with endpoints on the same component of ∂A, this leads to an isotopy of S removing intersection points with a fiber C_i. So we may assume all components of S_1 are either ∂-parallel annuli with vertical boundary or disks with horizontal boundary.

Since vertical circles in ∂M_1 cannot be disjoint from horizontal circles, S_1 is either a union of ∂-parallel annuli with vertical boundary, or a union of disks with horizontal boundary. In the

Figure 1.20 – On the left, a cross-section of a neighbourhood of M_0 with A in black. On the right, $M_1 = M_0|A$. Below, we see that once B_0 (light red) is split along a union of disjoint arcs, it can be deformed into a circle, making $M_0|A$ a solid torus.



former case S_1 can be isotoped to be vertical, staying fixed on ∂S_1 where it is already vertical. This isotopy gives an isotopy of S to a vertical surface. In the opposite case that S_1 consists of disks with horizontal boundary, isotopic to meridian disks in M_1, we can isotope S_1 to be horizontal fixing ∂S_1, and this gives an isotopy of S to a horizontal surface. □

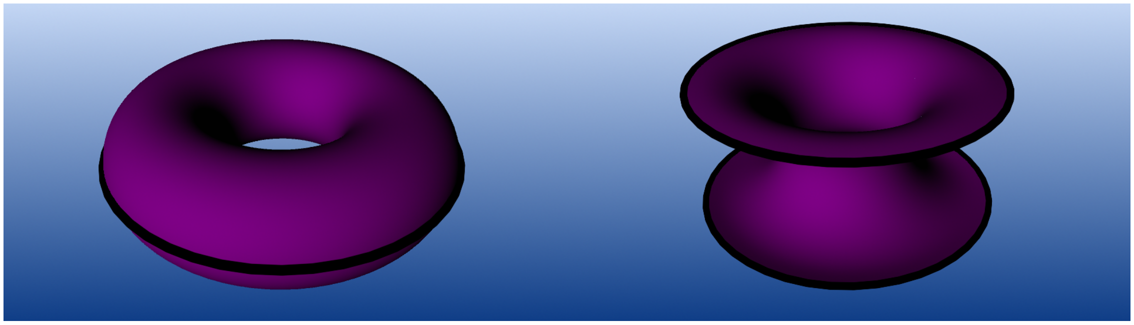
Vertical surfaces are easy to understand: They are circle bundles since they are disjoint from multiple fibers by definition, hence they are either annuli, tori, or Klein bottles.

Horizontal surfaces are somewhat more subtle. For a horizontal surface S the projection $\pi : S \to B$ onto the base surface of the Seifert fibering is a branched covering, with a branch point of multiplicity q for each intersection of S with a singular fiber of multiplicity q. (To see this, look in a neighborhood of a fiber, where the map $S \to B$ is equivalent to the projection of a number of meridian disks onto B, clearly a branched covering.) For this branched covering $\pi : S \to B$ there is a useful formula relating the Euler characteristics of S and B,

$$\chi(B) - \chi(S)/n = \sum_i (1 - 1/q_i)$$

where n is the number of sheets in the branched cover and the multiple fibers of M have multiplicities $q_1, \cdots, q_m$. To verify this formula, triangulate B so that the images of the multiple fibers are vertices, then lift this to a triangulation of S. Counting simplices would then yield the usual formula $\chi(S) = n\chi(B)$ for an n-sheeted unbranched cover. In the present case, however, a vertex in B which is the image of a fiber of multiplicity q_i has n/q_i pre-images in S, rather than n. This yields a modified formula $\chi(S) = n\chi(B) + \sum_i(-n + n/q_i)$, which is equivalent to the one above.

Figure 1.21 – One dimension down, an example of a circle C lying in a surface S (which is, in this case, $S^1 \times S^1$) such that $S|C$ is connected. This shows that $S|C = S^1 \times I$, meaning S is a bundle over S^1 with fiber S^1.



There is further structure associated to a horizontal surface S in a Seifert-fibered manifold M. Assume S is connected and 2-sided. (If S is 1-sided, it has an I-bundle neighborhood whose boundary is a horizontal 2-sided surface.) Since $S \to B$ is onto, S meets all fibers of M, and $M|S$ is an I-bundle. The local triviality of this I-bundle is clear if one looks in a model-fibered neighborhood of a fiber. The associated ∂I-subbundle consists of two copies of S, so the I-bundle is the mapping cylinder of a 2-sheeted covering projection $S \sqcup S \to T$ for some surface T. There are two cases, according to whether S separates M or not:

(i) If $M|S$ is connected, so is T, and $S \sqcup S \to T$ is the trivial covering $S \sqcup S \to S$, so $M|S = S \times I$ and hence M is a bundle over S^1 with fiber S. The surface fibers of this bundle are all horizontal surfaces in the Seifert fibering.

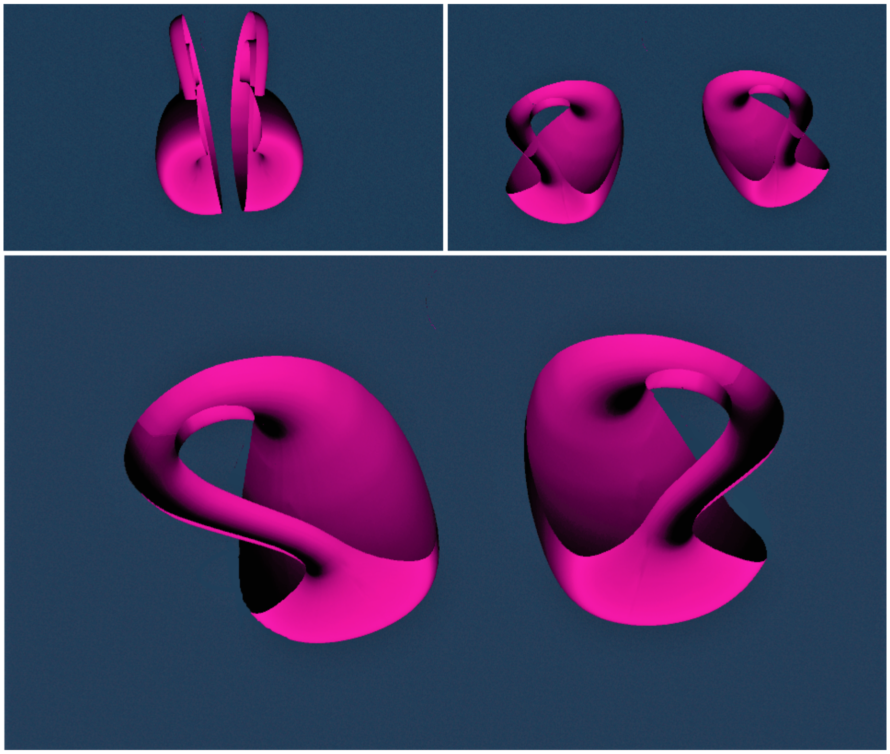
(ii) If $M|S$ has two components, each is a twisted I-bundle over a component T_i of T, the mapping cylinder of a nontrivial 2-sheeted covering $S \to T_i$, $i = 1, 2$. The parallel copies of S in these mapping cylinders, together with T_1 and T_2, are the leaves of a foliation of M. These leaves are the 'fibers' of a natural projection $p : M \to I$, with T_1 and T_2 the pre-images of the endpoints of I. This 'fiber' structure on M is not exactly a fiber bundle, so let us give it a new name: a *semi-bundle*. Thus a semi-bundle $p : M \to I$ is the union of two twisted I-bundles $p^{-1}\left[0, \frac{1}{2}\right]$ and $p^{-1}\left[\frac{1}{2}, 1\right]$ glued together by a homeomorphism of the fiber $p^{-1}(\frac{1}{2})$. For example, in one lower dimension, the Klein bottle is a semi-bundle with fibers S^1, since it splits as the union of two Möbius bands. More generally, one could define semi-bundles with base any manifold with boundary.

The techniques we have been using can also be applied to determine which Seifert manifolds are irreducible:

Proposition 1.12. *A compact connected Seifert-fibered manifold M is irreducible unless it is $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.*

Proof. We begin by observing that if M is reducible then there is a horizontal sphere in M not bounding a ball. This is proved by imitating the argument of the preceding proposition, with S now a sphere not bounding a ball in M. The only difference is that when incompressibility was used before, e.g., to eliminate trivial circles of $S \cap A$, we must now use surgery rather than isotopy. Such surgery replaces S with a pair of spheres S' and S''. If both S' and S'' bounded balls, so would S, as we saw in the proof of Alexander's theorem, so we may replace S by one of S', S'' not

Figure 1.22 – One dimension down again, we now have an example of a circle C lying in S (a Klein bottle) such that $S|C$ has two connected components (two Möbius bands). In this case, S is the two-dimensional semi-bundle $S^1 \tilde{\times} S^1$.



bounding a ball. With these modifications in the proof, we eventually get a sphere which is either horizontal or vertical, but the latter cannot occur since S^2 is not a circle bundle.

If S is a horizontal sphere in M, then as we have seen, M is either a sphere bundle or a sphere semi-bundle. The only two sphere bundles are $S^1 \times S^2$ and $S^1 \tilde{\times} S^2$. A sphere semi-bundle is two copies of the twisted I-bundle over $\mathbb{R}P^2$ glued together via a diffeomorphism of S^2. Such a diffeomorphism is isotopic to either the identity or the antipodal map. The antipodal map extends to a diffeomorphism of the I-bundle $\mathbb{R}P^2 \tilde{\times} I$, so both gluings produce the same manifold, $\mathbb{R}P^3 \# \mathbb{R}P^3$. □

Note that the three manifolds $S^1 \times S^2$, $S^1 \tilde{\times} S^2$, and $\mathbb{R}P^3 \# \mathbb{R}P^3$ do have Seifert fiberings. Namely, $S^1 \tilde{\times} S^2$ is $S^2 \times I$ with the two ends identified via the antipodal map, so the I-bundle structure on $S^2 \times I$ gives $S^1 \tilde{\times} S^2$ a circle bundle structure; and the I-bundle structures on the two halves $\mathbb{R}P^2 \tilde{\times} I$ of $\mathbb{R}P^3 \# \mathbb{R}P^3$, which are glued together by the identity, give it a circle bundle structure.

Now we can give a converse to Proposition 1.11:

Proposition 1.13. *Let M be a compact irreducible Seifert-fibered 3-manifold. Then every 2-sided horizontal surface $S \subset M$ is essential. The same is true of every connected 2-sided vertical surface except:*

(i) a torus bounding a solid torus with a model Seifert fibering, containing at most one multiple fiber, or

(ii) an annulus cutting off from M a solid torus with the product fibering.

Proof. For a 2-sided horizontal surface S we have noted that the Seifert fibering induces an I-bundle structure on $M|S$, so $M|S$ is the mapping cylinder of a 2-sheeted covering $S \sqcup S \to T$ for some surface T. Being a covering space projection, this map is injective on π_1, so the inclusion of the ∂I-subbundle into the I-bundle is also injective on π_1. Therefore S is incompressible. (In case S is a disk, $M|S$ is $D^2 \times I$, so S is clearly not ∂-parallel.) Similarly, ∂-incompressibility follows from injectivity of relative π_1's.

Now suppose S is a compressible 2-sided vertical surface, with a compressing disk D which does not cut off a disk from S. Then D is incompressible in $M|S$, and can therefore be isotoped to be horizontal. The Euler characteristic formula in the component of $M|S$ containing D takes the form $\chi(B) - 1/n = \sum_i (1 - 1/q_i)$. The right-hand side is non-negative and $\partial B \neq \varnothing$, so $\chi(B) = 1$ and B is a disk. Each term $1 - 1/q_i$ is at least $\frac{1}{2}$, so there can be at most one such term, and so at most one multiple fiber. Therefore this component of $M|S$ is a solid torus with a model Seifert fibering and S is its torus boundary. (If S were a vertical annulus in its boundary, S would be incompressible in this solid torus.)

Similarly, if S is a ∂-compressible vertical annulus there is a ∂-compressing disk D with horizontal boundary, and D may be isotoped to be horizontal in its interior as well. Again D must be a meridian disk in a solid torus component of $M|S$ with a model Seifert fibering. In this case there can be no multiple fiber in this solid torus since ∂D meets S in a single arc. $\square$

Note that the argument just given shows that the only Seifert fiberings of $S^1 \times D^2$ are the model Seifert fiberings.

Uniqueness of Torus Decompositions

We need three preliminary lemmas:

Lemma 1.14. *An incompressible, ∂-incompressible annulus in a compact connected Seifert-fibered M can be isotoped to be vertical, after possibly changing the Seifert fibering if $M = S^1 \times S^1 \times I$, $S^1 \times S^1 \tilde{\times} I$ (the twisted I-bundle over the torus), $S^1 \tilde{\times} S^1 \times I$ (the Klein bottle cross I), or $S^1 \tilde{\times} S^1 \tilde{\times} I$ (the twisted I-bundle over the Klein bottle).*

Proof. Suppose S is a horizontal annulus in M. If S does not separate M then $M|S$ is the product $S \times I$, and so M is a bundle over S^1 with fiber S, the mapping torus $S \times I/\{(x, 0) \sim (\phi(x), 1)\}$ of a diffeomorphism $\phi\, S \to S$. There are only four isotopy classes of diffeomorphisms of $S^1 \times I$, obtained as the composition of either the identity or a reflection in each factor, so ϕ may be taken to preserve the S^1 fibers of $S = S^1 \times I$. This S^1-fibering of S then induces a circle bundle structure on M in which S is vertical. The four choices of ϕ give the four exceptional manifolds listed.

If S is separating, $M|S$ is two twisted I-bundles over a Möbius band, each obtained from a cube by identifying a pair of opposite faces by a 180 degree twist. Each twisted I-bundle is thus a model Seifert fibering with a multiplicity 2 singular fiber. All four possible gluings of these two twisted I-bundles yield the same manifold M, with a Seifert fibering over D^2 having two singular fibers of multiplicity 2, with S vertical. This manifold is easily seen to be $S^1 \tilde{\times} S^1 \tilde{\times} I$. $\square$

Lemma 1.15. *Let M be a compact connected Seifert manifold with ∂M orientable. Then the restrictions to ∂M of any two Seifert fiberings of M are isotopic unless M is $S^1 \times D^2$ or one of the four exceptional manifolds in Lemma 1.14.*

Proof. Let M be Seifert-fibered, with $\partial M \neq \varnothing$. First we note that M contains an incompressible, ∂-incompressible vertical annulus A unless $M = S^1 \times D^2$. Namely, take $A = \pi^{-1}(\alpha)$ where α is an arc in the base surface B which is either nonseparating (if $B \neq D^2$) or separates the images of multiple fibers (if $B = D^2$ and there are at least two multiple fibers). This guarantees incompressibility and ∂-incompressibility of A by Proposition 1.13. Excluding the exceptional cases in Lemma 1.14, A is then isotopic to a vertical annulus in any other Seifert fibering of M, so the two Seifert fiberings can be isotoped to agree on ∂A, hence on the components of ∂M containing ∂A. Since α could be chosen to meet any component of ∂B, the result follows. □

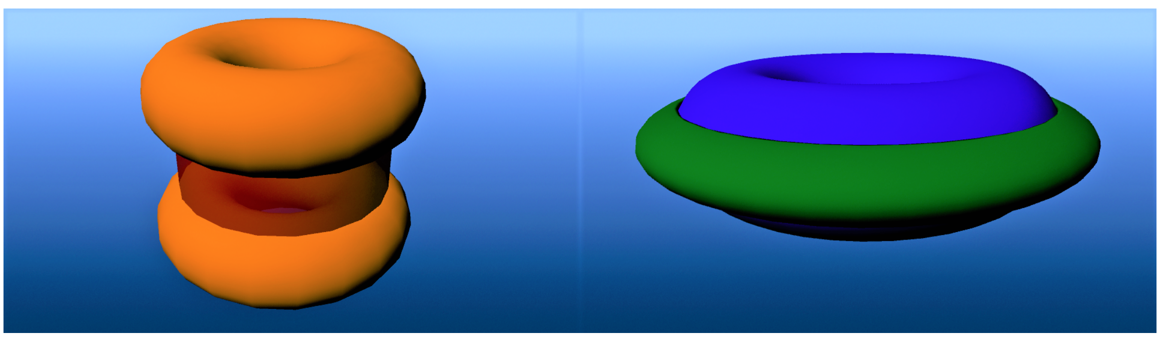
Lemma 1.16. *If M is compact, connected, orientable, irreducible, and atoroidal, and M contains an incompressible, ∂-incompressible annulus meeting only torus components of ∂M, then M is a Seifert manifold.*

Proof. Let A be an annulus as in the hypothesis. There are three possibilities, indicated in Figure 1.11 below:

(a) A meets two different tori T_1 and T_2 in ∂M, and $A \cup T_1 \cup T_2$ has a neighborhood N which is a product of a 2-punctured disk with S^1.

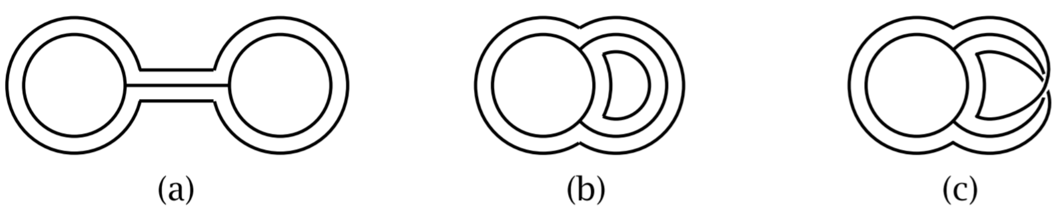
(b) A meets only one torus T_1 in ∂M, the union of A with either annulus of $T_1 | \partial A$ is a torus, and $A \cup T_1$ has a neighborhood N which is a product of a 2-punctured disk with S^1.

Figure 1.23 – The situations described in (a) and (b) respectively.



(c) A meets only one torus T_1 in ∂M, the union of A with either annulus of $T_1 | \partial A$ is a Klein bottle, and $A \cup T_1$ has a neighborhood N which is an S^1 bundle over a punctured Möbius band.

Figure 1.24



In all three cases N has the structure of a circle bundle $N \to B$ with A vertical.

By hypothesis, the tori of $\partial N - \partial M$ must either be compressible or ∂-parallel in M. Suppose D is a nontrivial compressing disk for $\partial N - \partial M$ in M, with ∂D a nontrivial loop in a component torus T of $\partial N - \partial M$. If $D \subset N$, then N would be a solid torus $S^1 \times D^2$ by Proposition 1.13, which is impossible since N has more than one boundary torus. So $D \cap N = \partial D$. Surgering T along D yields a 2-sphere bounding a ball $B^3 \subset M$. This B^3 lies on the opposite side of T from

N, otherwise we would have $N \subset B^3$ with T the only boundary component of N. Reversing the surgery, B^3 becomes a solid torus outside N, bounded by T.

The other possibility for a component T of $\partial N - \partial M$ is that it is ∂-parallel in M, cutting off a product $T \times I$ from M. This $T \times I$ cannot be N since $\pi_1 N$ is nonabelian, the map $\pi_1 N \to \pi_1 B$ induced by the circle bundle $N \to B$ being a surjection to a free group on two generators. So $T \times I$ is an external collar on N, and hence can be absorbed into N.

Thus M is N with solid tori perhaps attached to one or two tori of $\partial N - \partial M$. The meridian circles $\{x\} \times \partial D^2$ in such attached $S^1 \times D^2$'s are not isotopic in ∂N to circle fibers of N, otherwise A would be compressible in M (recall that A is vertical in N). Thus the circle fibers wind around the attached $S^1 \times D^2$'s a nonzero number of times in the S^1 direction. Hence the circle bundle structure on N extends to model Seifert fiberings of these $S^1 \times D^2$'s, and so M is Seifert-fibered. $\square$

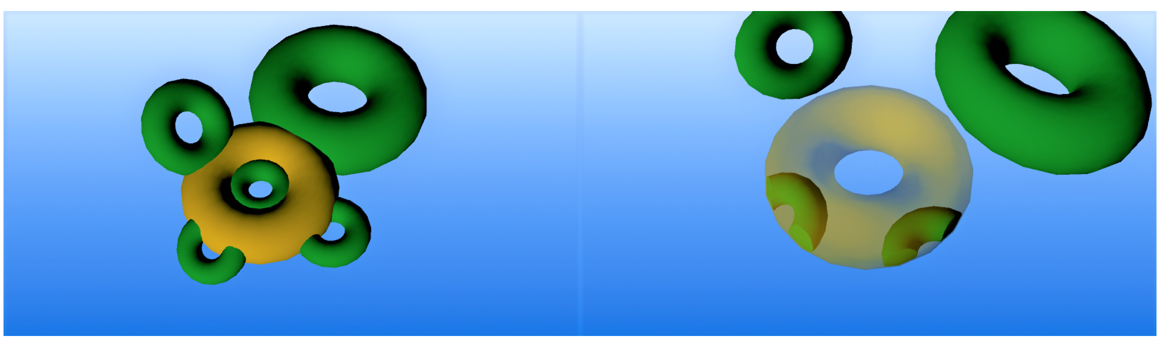
Proof of Theorem 1.9. Only the uniqueness statement remains to be proved. So let $T = T_1 \cup \cdots \cup T_m$ and $T' = T'_1 \cup \cdots \cup T'_n$ be two minimal collections of disjoint incompressible tori splitting M into manifolds M_j and M'_j, respectively, which are either atoroidal or Seifert-fibered. We may suppose T and T' are nonempty, otherwise the theorem is trivial since if T is empty for example, M itself would be Seifert-fibered or atoroidal and the minimality of T' would force it to be empty as well. We can also assume no torus of T is isotopic to a torus of T', otherwise we could extend this isotopy to an ambient isotopy of M and in particular of T, to make the two tori agree, then split M along this torus and appeal to induction on the number of tori in T and T'. Symmetrically, we can assume no torus of T' is isotopic to a torus of T.

After perturbing T to meet T' transversely, we can isotope T or T' to eliminate circles of $T \cap T'$ which bound disks in T or T', by the usual argument using incompressibility and irreducibility.

For each M_j the components of $T' \cap M_j$ are then tori or annuli.

The annulus components are incompressible in M_j since they are noncontractible in T' and T' is incompressible. Annuli of $T' \cap M_j$ which are ∂-compressible are then ∂-parallel, by Lemma 1.10, so they can be eliminated by isotopy of T'.

Figure 1.25 – *Components of T' (green) intersect a component T_j (yellow) of T. Circle components of $T' \cap T$ which bound disks are eliminated, and then ∂-parallel annulus components of $T' \cap M_j$ are eliminated after M is split along T, leaving only annuli that are not ∂-parallel.*



katlas.math.toronto.edu/caldermf/3manifolds/1-9/torus-intersections.mp4

A circle C of $T \cap T'$ lies in the boundary of annulus components A_j of $T' \cap M_j$ and A_k of $T' \cap M_k$ (possibly $A_j = A_k$ or $M_j = M_k$). By Lemma 1.16 M_j and M_k are Seifert-fibered. If $M_j \neq M_k$ Lemma 1.14 implies that we can choose Seifert fiberings of M_j and M_k in which A_j and A_k are vertical. In particular, the two fiberings of the torus component T_i of T containing C induced from the Seifert fiberings of M_j and M_k have a common fiber C. Therefore these two fiberings of T_i can be isotoped to agree. After extending the isotopies to M_j and M_k, their Seifert-fiberings agree on T_i so the collection T is not minimal since T_i can be deleted from it.

Essentially the same argument works if $M_j = M_k$: If we are not in the exceptional cases in Lemma 1.14, then the circle C is isotopic in T_i to fibers of each of the two fiberings of T_i induced from M_j, so these two fiberings are isotopic, and after extending to an isotopy of the Seifert-fibering of M_j, we can delete T_i from T. In the exceptional case $M_j = S^1 \times S^1 \times I$, if we have to rechoose the Seifert fibering to make A_j vertical, then as we saw in the proof of Lemma 1.14, the new fibering is simply the trivial circle bundle over $S^1 \times I$. The annulus A_j, being vertical, incompressible, and ∂-incompressible, must then join the two boundary tori of M_j, since its projection to the base surface $S^1 \times I$ must be an arc joining the two boundary components of $S^1 \times I$. The two boundary circles of A_j in T_i either coincide or are disjoint, hence isotopic, so once again the two induced fiberings of T_i are isotopic and T_i can be deleted from T. The other exceptional cases in Lemma 1.14 cannot arise since M is orientable and M_j has at least two boundary tori.

Thus $T \cap T' = \varnothing$. If any component T_i of T lies in an atoroidal M'_j it must be isotopic to a component T'_i of T', a situation we have already excluded. Thus we may assume each T_i lies in a Seifert-fibered M'_j, and similarly, each T'_i lies in a Seifert-fibered M_j. These Seifert-fibered manifolds all have nonempty boundary, so they contain no horizontal tori. Thus we may assume all the tori $T_i \subset M'_j$ and $T'_i \subset M_j$ are vertical.

We can also assume all the M_j's and M'_j's are Seifert-fibered by the following argument. An atoroidal M_j, for example, would have to lie entirely within an M'_j, as M_j contains no T'_i's. This M'_j would have to be Seifert-fibered since it contains the T_i's in ∂M_j. Then since the T_i's in ∂M_j are vertical in M'_j, the Seifert-fibering of M'_j restricts to a Seifert-fibering of M_j.

The T_i's and T'_i's together cut M into pieces N_p. Each N_p has two possibly different Seifert-fiberings, the one from the M_j which contains it, the other from the M'_j which contains it. Consider a torus T_i. This has four fiberings from the two Seifert-fiberings on the N_p and N_q on either side of it (possibly $N_p = N_q$). Two of these fiberings of T_i must be equal since they come from the same M'_j containing T_i. We will show that the Seifert-fiberings of the M_j containing N_p and the M_k containing N_q (possibly $M_j = M_k$) can be made to agree on T_i. But this would contradict the minimality of the collection T since T_i could be deleted from it.

In most cases the two Seifert-fiberings of N_p can be made to agree on T_i by an isotopy of the Seifert-fibering of M_j supported near T_i, by Lemma 1.15. Since we assume M is orientable, the exceptional cases are:

- $N_p = S^1 \times D^2$. This would have T_i as its compressible boundary, so this case cannot occur.
- $N_p = S^1 \times S^1 \times I$. One boundary component of this is T_i. If the other is a T'_i, then T_i and T'_i would be isotopic, contradicting an earlier assumption. Thus both tori of ∂N_p come from tori in T. If these tori in T are distinct then N_p gives an isotopy between them so one of them could be omitted from T, contradicting the minimality of T. There remains the possibility that both tori of ∂N_p must come from the same T_i, but this would mean T' is empty, a case excluded at the beginning. Thus the case $N_p = S^1 \times S^1 \times I$ cannot occur.
- $M_j \cap M'_k = S^1 \tilde{\times} S^1 \tilde{\times} I$. This has only one boundary component, so $N_p = M_j \subset M'_j$ and we can change the Seifert-fibering of M_j to be the restriction of the Seifert-fibering of M'_j.

The same reasoning applies with N_q and M_k in place of N_p and M_j. The conclusion is that we have Seifert-fiberings of M_j and M_k that agree on T_i since they agree with the fibering from M'_j. As noted before, this means T_i can be omitted from T, contrary to minimality, and the proof is complete. $\square$

Exercises

1. Show: If $S \subset M$ is a 1-sided connected surface, then $\pi_1 S \to \pi_1 M$ is injective iff $\partial N(S)$ is incompressible, where $N(S)$ is a twisted I-bundle neighborhood of S in M.

2. Call a 1-sided surface $S \subset M$ *g*eometrically incompressible if for each disk $D \subset M$ with $D \cap S = \partial D$ there is a disk $D' \subset S$ with $\partial D' = \partial D$. Show that if $H_2 M = 0$ but $H_2(M; \mathbb{Z}_2) \neq 0$ then M contains a 1-sided geometrically incompressible surface which is nonzero in $H_2(M; \mathbb{Z}_2)$. This applies for example if M is a lens space $L_{p/2q}$. Note that if $q > 1$, the resulting geometrically incompressible surface $S \subset L_{p/2q}$ cannot be S^2 or $\mathbb{R}P^2$, so the map $\pi_1 S \to \pi_1 L_{p/2q}$ is not injective.

3. S is ∂-incompressible if $\pi_1(S, \partial S) \to \pi_1(M, \partial M)$ is injective for all choices of basepoint in ∂S.

4. Develop a canonical torus and Klein bottle decomposition theorem for irreducible nonorientable 3-manifolds.

Appendix A

A 2D Analogue of the Prime Decomposition Theorem

The following claims marked with a * are 2-manifold analogues of the prime decomposition theorem for 3-manifolds as presented in *Notes on Basic 3-Manifold Topology*. For these claims only, the definitions of **splitting** and **connected sums** are taken one dimension down (connected sums are obtained by splitting along circles, not spheres.)

Proposition 1.4*. *The only orientable prime 2-manifold which is not irreducible is $S^1 \times S^1$.*

Proof. If M is prime, every circle in M which separates M into two components bounds a disk. So if M is prime but not irreducible, there must exist a nonseparating circle in M. For a nonseparating circle C in an orientable manifold M, the union of a product neighbourhood $C \times I$ of C with a tubular neighbourhood of an arc joining $C \times \{0\}$ to $C \times \{1\}$ in the complement of $C \times I$ is a manifold diffeomorphic to $S^1 \times S^1$ minus a ball (see Figure 1.7 in Proposition 1.4). Thus M has $S^1 \times S^2$ as a connected summand (the boundary of this missing ball is the circle along which we split in order to obtain this connected summand). Assuming M is prime, then $M = S^1 \times S^1$.

It remains to show that $S^1 \times S^1$ is prime. Let $C \subset S^1 \times S^1$ be a separating circle, so $S^1 \times S^1 | C$ consists of two compact 2-manifolds V and W each with boundary a circle. We have $\mathbb{Z} \times \mathbb{Z} = \pi_1(S^1 \times S^1) = \pi_1 V * \pi_1 W$, so either V or W must be simply-connected, say V is simply connected. The universal cover of $S^1 \times S^1$ is $\mathbb{R}^2$, and V lifts do a diffeomorphic copy $\tilde{V}$ of itself in $\mathbb{R}^2$. The circle $\partial \tilde{V}$ bounds a disk in $\mathbb{R}^2$ by the Schoenflies theorem. Since $\partial \tilde{V}$ also bounds $\tilde{V}$ in $\mathbb{R}^2$, we conclude that $\tilde{V}$ is a disk, hence also V. Thus every separating circle in $S^1 \times S^1$ bounds a disk, so $S^1 \times S^1$ is prime. □

Theorem 1.5*. *Let M be compact, connected, and orientable. Then there is a decomposition $M = P_1 \# \cdots \# P_n$ with each P_i prime, and this decomposition is unique up to insertion or deletion of S^2's.*

Proof. We begin with existence. If M contains a nonseparating S^1, this gives a decomposition $M = N \# S^1 \times S^1$, as we saw in the proof of Proposition 1.4*. We can repeat this step of splitting off an $S^1 \times S^1$ summand as long as we have nonseparating circles, but the process cannot be repeated indefinitely since each $S^1 \times S^1$ summand gives a $\mathbb{Z} \times \mathbb{Z}$ summand of $H_1(M)$, which is a finitely generated abelian group since M is compact. Thus we are reduced to proving existence of prime decompositions in the case that each circle in M separates. Each circle component of ∂M corresponds to a disk summand of M, so we may also assume ∂M contains no circles.

We shall prove the following assertion, which clearly implies the existence of prime decompositions: There is a bound on the number of circles in a system C of disjoint circles satisfying:

(*) No component of $M|C$ is a punctured 2-sphere, i.e., a compact manifold obtained from S^2 by deleting finitely many open balls with disjoint closures.

Before proving this we make a preliminary observation: If C satisfies $(*)$ and we do surgery on a circle C_i of C using a line segment $L \subset M$ with $L \cap C = \partial L \subset C_i$, then at least one of the systems C', C'' obtained by replacing C_i with the spheres C_i' and C_i'' resulting from the surgery satisfies $(*)$. To see this, first perturb C_i' and C_i'' to be disjoint from C_i and each other, so that C_i, C_i', and C_i'' together bound a 3-punctured sphere P.

On the other side of C_i from P we have a component A of $M|C$, while the spheres C_i' and C_i'' split the component of $M|C$ containing P into pieces B', B'', and P. If both B' and B'' were punctured spheres, then $B' \cup B'' \cup P$, a component of $M|C$, would be a punctured sphere, contrary to hypothesis. So one of B' and B'', say B', is not a punctured sphere. If $A \cup P \cup B''$ were a punctured sphere, this would force A to be a punctured sphere, by the Schoenflies theorem. This is also contrary to hypothesis. So we conclude that neither component of $M|C'$ adjacent to C_i' is a punctured sphere, hence the sphere system C' satisfies $(*)$.

Now we prove the assertion that the number of circles in a system C satisfying $(*)$ is bounded. Choose a smooth triangulation $\mathcal{T}$ of M. This has only finitely many simplices since M is compact. The given system C can be perturbed to be transverse to all the simplices of $\mathcal{T}$. This perturbation can be done inductively over the skeleta of $\mathcal{T}$: First make C disjoint from vertices, then transverse to edges, meeting them in finitely many points.

First note that any component of $M|C$ must meet $\partial\tau$ by the Schoenflies theorem and condition $(*)$ (if it didn't meet the boundary, the circle would bound a disk, which is a punctured sphere).

Next, for each 2-simplex σ we eliminate arcs α of $C \cap \sigma$ having both endpoints on the same edge of σ. Such an α cuts off from σ a disk D which meets only one edge of σ. We may choose α to be 'edgemost,' so that D contains no other arcs of $C \cap \sigma$, and hence $D \cap C = \alpha$ since circles of $C \cap \sigma$ have been eliminated in the previous step. By an isotopy of C supported near α we then push the intersection arc α across D, eliminating α from $C \cap \sigma$ and decreasing by two the number of points of intersection of C with the 1-skeleton of $\mathcal{T}$. We continue this process until after finitely many steps, we arrive at the situation where C meets each 2-simplex only in arcs connecting adjacent sides.

Now consider the intersection of C with a 2-simplex σ. With at most four exceptions the complementary regions of $C \cap \sigma$ in σ are rectangles with two opposite sides on $\partial\sigma$ and the other two opposite sides arcs of $C \cap \sigma$. Thus if $\mathcal{T}$ has t 2-simplices, then all but at most $4t$ of the components of $M|C$ meet all the 2-simplices of $\mathcal{T}$ only in such rectangles. However, we now show that each 2 simplex can have a maximum of 4 disjoint regions, as unions of these rectangular regions constitute a punctured sphere.

To see this, let R be a component of $M|C$ meeting all 2-simplices only in rectangles. For a 2-simplex τ, each component of $R \cap \tau$ is diffeomorphic to a circle, the skeleton of a rectangular region A. This region is diffeomorphic to a disk in τ, a component of $R \cap \tau$ which can be written as $D^1 \times I$ with $\partial D^1 \times I = A$. The I-fiberings of all such products $D^1 \times I$ may be assumed to agree on their common intersections, the rectangles, to give R the structure of an I-bundle. Since ∂R consists of circle components of C, R is the product $S^1 \times I$. However, this is a 2-punctured sphere, and so this possibility is excluded by $(*)$. Since every circle in M separates, the number of components of $M|C$ is one more than the number of spheres in C. This finishes the proof of the existence of prime decompositions.

For uniqueness, suppose the nonprime M has two prime decompositions $M = P_1 \# \cdots \# P_k \# \ell(S^1 \times S^1)$ and $M = Q_1 \# \cdots \# Q_m \# n(S^1 \times S^1)$ where the P_i's and Q_i's are irreducible and not S^2. Let C be a disjoint union of circles in M reducing M to the P_i's, i.e., the components of $M|C$ are the manifolds $P_1, \cdots, P_k$ with punctures, plus possibly some punctured S^2's. Such a system C exists: Take for example a collection of circles defining the given prime decomposition $M = P_1 \# \cdots \# P_k \# \ell(S^1 \times S^1)$

together with a nonseparating circle in each $S^1 \times S^1$. Note that if C reduces M to the P_i's, so does any system C' containing C.

Similarly, let T be a system of circles reducing M to the Q_i's. If $C \cap T \neq \varnothing$, we may assume this is a transverse intersection, and consider a line segment $L \subset T$ with $L \cap C = \partial L$. Using L, surger the circle C_j of C containing ∂L to produce two circles C_j' and C_j'', which we may take to be disjoint from C_j, so that C_j, C_j', and C_j'' together bound a 3-punctured 2-sphere P. By an earlier remark, the enlarged system $C \cup C_j' \cup C_j''$ reduces M to the P_i's. Deleting C_j from this enlarged system still gives a system reducing M to the P_i's since this affects only one component of $M|C \cup C_j' \cup C_j''$, by attaching P to one of its boundary circles, which has the net effect of simply adding one more puncture to this component.

The new system C' meets T in one fewer circle, so after finitely many steps of this type we produce a system C disjoint from T and reducing M to the P_i's. Then $C \cup T$ is a system reducing M both to the P_i's and to the Q_i's. Hence $k = m$ and the P_i's are just a permutation of the Q_i's.

Finally, to show $l = n$, we have $M = N \# l(S^1 \times S^1) = N \# n(S^1 \times S^1)$, so $H_1(M) = H_1(N) \oplus (\mathbb{Z} \times \mathbb{Z})^l = H_1(N) \oplus (\mathbb{Z} \times \mathbb{Z})^n$, hence $l = n$.

□

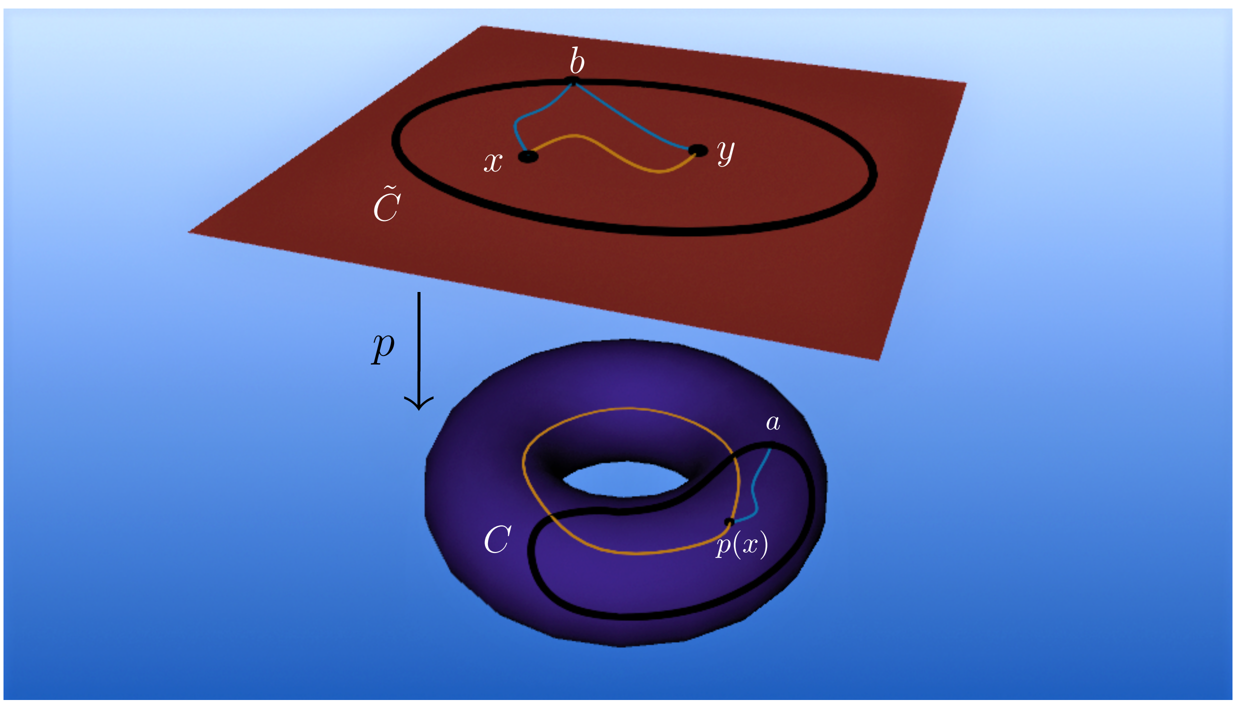
The proof above follows the structure of the proof of Proposition 1.4 (in the 3-manifold case) very closely, but a stronger result holds – in fact, once all of the $S^1 \times S^1$ summands are split off, the only possible surface that can remain is a 2-sphere; this is the classification theorem for orientable surfaces, that every orientable 2-manifold is the connected sum of 0 or more tori. This result does not hold one dimension up – the study of prime 3-manifolds is extensive, and much of *Notes on Basic 3-Manifold Topology* is devoted to this.

Appendix B

A Nullhomotopic Circle in a Surface Bounds a Disk

We include a proof of the following proposition, which is stated as a standard fact in surface theory and used in Section 1.2 to show that a connected 2-sided surface S which is not a sphere or disk is incompressible if the map $\pi_1(S) \to \pi_1(M)$ induced by inclusion is injective.

Figure B.1



Proposition. *A nullhomotopic circle in a compact surface S bounds a disk in S.*

Proof. If $S = S^2$, this follows by the 2-dimensional version of Alexander's theorem. If S is any surface other than S^2, then its universal cover must be $\mathbb{R}^2$, and so we have a covering map $p : \mathbb{R}^2 \to S$.

Let C be a nullhomotopic circle in S. Since C is nullhomotopic, it lifts to a set of disjoint circles in $\mathbb{R}^2$; we choose one of these circles and denote it by $\tilde{C}$, and denote by $\tilde{D}$ the disk in $\mathbb{R}^2$ bounded by $\tilde{C}$. We would like to show that the restricting the covering map p to the disk $\tilde{D}$ gives us an injective map

$p' : \tilde{D} \to S$. Suppose for contradiction that this map p' is not injective; then there exist distinct points x and y in $\tilde{D}$ such that $p'(x) = p'(y)$. A path x and y in $\tilde{D}$ is not a closed loop, and so its projection in is a loop in S centred at $p'(x)$ which is not nullhomotopic. Adjoining to this loop a path from $p'(x)$ to some point $a \in C$ does not change the fact that this new loop, which we call γ, is nullhomotopic.

Now γ lifts to two paths in $\mathbb{R}^2$, one which travels from x to some point $b \in C$ and one which travels from y to some point $b' \in C$. We immediately note that $b = b'$, or else we would have that the projection from $\tilde{C}$ to C is not injective, since we have $p'(b) = p'(b') = a$.

We form a loop l in $\mathbb{R}^2$ by composing these two paths with the path between x and y. This is a closed loop based at x, and so its projection must be nullhomotopic. Note, however, that its projection is a path from a to $p'(x)$ composed with to a loop around $p'(x)$ which is not nullhomotopic. This means the loop $p'(l)$ is both not nullhomotopic and nullhomotopic, a contradiction. This means the map $p' : \tilde{D} \to S$ must be injective, as desired.

From this, we can now conclude that $\tilde{D}$ and $p(\tilde{D})$ are diffeomorphic. Since C bounds $p(\tilde{D})$, we can conclude that C bounds a disk in S.