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Vita
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Notes
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Vita

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- October 2002: Doctorate, University of Nantes. Advisor: Christian Blanchet.

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SPIN BORROMEAN SURGERIES

GWÉNAËL MASSUYEAU

Abstract. In 1986, Matveev defined the notion of Borromean surgery for closed oriented 3-manifolds and showed that the equivalence relation generated by this move is characterized by the pair (first betti number, linking form up to isomorphism).
We explain how this extends for 3-manifolds with spin structure if we replace the linking form by the quadratic form defined by the spin structure. We then show that the equivalence relation among closed spin 3-manifolds generated by spin Borromean surgeries is characterized by the triple (first betti number, linking form up to isomorphism, Rochlin invariant modulo 8).

Introduction

The notion of Borromean surgery was introduced by Matveev in [Ma] as an example of what he called a $\mathcal{V}$-surgery. Since then, this transformation has become the elementary move of Goussarov-Habiro finite type invariants theory for oriented 3-manifolds ([Ha], [Go], [GGP]). Matveev showed that the equivalence relation, among closed oriented 3-manifolds, generated by Borromean surgery is characterized by the pair:

$$(\beta_1(M),\ \text{isomorphism class of } \lambda_M),$$

where $\beta_1(M)$ is the first Betti number of a 3-manifold M and

$$TH_1(M;\mathbf{Z})\otimes TH_1(M;\mathbf{Z}) \xrightarrow{\lambda_M} \mathbf{Q}/\mathbf{Z}$$

is its torsion linking form. This result gives a characterization of degree 0 invariants in Goussarov-Habiro theory for closed oriented 3-manifolds.

As mentioned by Habiro and Goussarov, their finite type invariants theory (in short: "FTI theory") makes sense also for 3-manifolds with spin structure because Borromean surgeries work well with spin structures (see §2). So, the question is: *what is the "spin" analogue of Matveev's theorem?*
For each closed spin 3-manifold (M,σ), a quadratic form

$$TH_1(M;\mathbf{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbf{Q}/\mathbf{Z}$$

can be defined by many ways (see [LL], [MS], and also [Tu], [Gi]). The bilinear form associated to $\phi_{M,\sigma}$ is λ_M. Its Gauss-Brown invariant is equal to $-R_M(\sigma)$ modulo 8, where

$$Spin(M) \xrightarrow{R_M} \mathbf{Z}_{16}$$

is the Rochlin function of M, sending a spin structure σ of M to the modulo 16 signature of a spin 4-manifold which spin-bounds (M,σ). The main result of this paper is the following refinement of Matveev's theorem:

Theorem 1. *Let (M,σ) and (M',σ') be connected closed spin 3-manifolds. Then, the following assertions are equivalent:*

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Key words and phrases. 3-manifolds, finite type invariants, spin structures, Y-graphs.

CHARACTERIZATION OF Y_2-EQUIVALENCE FOR HOMOLOGY CYLINDERS

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ABSTRACT

For Σ a compact connected oriented surface, we consider homology cylinders over Σ: these are homology cobordisms with an extra homological triviality condition. When considered up to Y_2-equivalence, which is a surgery equivalence relation arising from the Goussarov-Habiro theory, homology cylinders form an Abelian group.
In this paper, when Σ has one or zero boundary component, we define a surgery map from a certain space of graphs to this group. This map is shown to be an isomorphism, with inverse given by some extensions of the first Johnson homomorphism and Birman-Craggs homomorphisms.

Keywords: homology cylinder, finite type invariant, clover, clasper.

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1. Introduction

1.1. *Homology cylinders*

Homology cylinders are important objects in the theory of finite type invariants of Goussarov-Habiro: they have thus appeared in both [6] and [4]. Let us recall the definition of these objects.

Let Σ be a compact connected oriented surface. A *homology cobordism* over Σ is a triple (M, i^+, i^-) where M is a compact oriented 3-manifold and $i^\pm : \Sigma \longrightarrow M$ are oriented embeddings with images $\Sigma^\pm$, such that:

(i) $i^\pm$ are homology isomorphisms;
(ii) $\partial M = \Sigma^+ \cup (-\Sigma^-)$ and $\Sigma^+ \cap (-\Sigma^-) = \pm\partial\Sigma^\pm$;
(iii) $i^+|_{\partial\Sigma} = i^-|_{\partial\Sigma}$.

Homology cobordisms are considered up to orientation-preserving diffeomorphisms. When $(i^-)_*^{-1} \circ (i^+)_* : H_1(\Sigma; \mathbf{Z}) \longrightarrow H_1(\Sigma; \mathbf{Z})$ is the identity, M is said to be a *homology cylinder*. The set of homology cobordisms is denoted here by $\mathcal{C}(\Sigma)$, and $\mathcal{HC}(\Sigma)$ denotes the subset of homology cylinders. If $M = (M, i^+, i^-)$ and

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Reidemeister–Turaev torsion modulo one of rational homology three–spheres

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Abstract

Given an oriented rational homology 3–sphere M, it is known how to associate to any Spin^c–structure σ on M two quadratic functions over the linking pairing. One quadratic function is derived from the reduction modulo 1 of the Reidemeister–Turaev torsion of (M,σ), while the other one can be defined using the intersection pairing of an appropriate compact oriented 4–manifold with boundary M.

In this paper, using surgery presentations of the manifold M, we prove that those two quadratic functions coincide. Our proof relies on the comparison between two distinct combinatorial descriptions of Spin^c–structures on M: Turaev's charges vs Chern vectors.

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Secondary: 57Q10, 57R15

Keywords Rational homology 3–sphere, Reidemeister torsion, complex spin structure, quadratic function

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QUADRATIC FUNCTIONS ON TORSION GROUPS

FLORIAN DELOUP AND GWÉNAËL MASSUYEAU

Abstract. We investigate classification results for general quadratic functions on torsion abelian groups. Unlike the previously studied situations, general quadratic functions are allowed to be inhomogeneous or degenerate. We study the discriminant construction which assigns, to an integral lattice with a distinguished characteristic form, a quadratic function on a torsion group. When the associated symmetric bilinear pairing is fixed, we construct an affine embedding of a quotient of the set of characteristic forms into the set of all quadratic functions and determine explicitly its cokernel. We determine a suitable class of torsion groups so that quadratic functions defined on them are classified by the stable class of their lift. This refines results due to A.H. Durfee, V. Nikulin, C.T.C. Wall and E. Looijenga – J. Wahl. Finally, we show that on this class of torsion groups, two quadratic functions q, q' are isomorphic if and only if they have equal associated Gauss sums and there is an isomorphism between the associated symmetric bilinear pairings b_q and $b_{q'}$ which sends d_q to $d_{q'}$, where d_q is the homomorphism defined by $d_q(x) = q(x) - q(-x)$. This generalizes a classical result due to V. Nikulin. Our results are elementary in nature and motivated by low-dimensional topology.

A quadratic function q on an abelian group G is a map, with values in an abelian group, such that the map $b : (x, y) \mapsto q(x + y) - q(x) - q(y)$ is $\mathbb{Z}$-bilinear. Such a map q satisfies $q(0) = 0$. If, in addition, q satisfies the relation $q(nx) = n^2q(x)$ for all $n \in \mathbb{Z}$ and $x \in G$, then q is homogeneous. In general, a quadratic function cannot be recovered from the associated bilinear pairing b. Homogeneous quadratic functions on torsion groups first appeared as quadratic enhancements of the linking pairing on the torsion subgroup of the $(2n-1)$-th homology group of an oriented $(4n-1)$-manifold. Typically, these quadratic enhancements appear in topology when the manifold is equipped with a framing [BM] [MS] [LL]. They were used as a fundamental ingredient in the classification up to regular homotopy of immersed surfaces in $\mathbb{R}^3$ [Pi]. They were extensively studied from the algebraic viewpoint of Witt and Grothendieck groups, see for instance [Du] [Ka] [La]. However, there are topological motivations to consider inhomogeneous enhancements of the linking pairing [LW] [De2]. It is also convenient to consider possibly degenerate quadratic functions [De1]. The motivation for considering *general* quadratic functions stems from our work on closed Spin^c-manifolds of dimension 3 and their finite type invariants [DM].

This paper studies, and gives classification results for, quadratic functions on torsion abelian groups with values in $\mathbb{Q}/\mathbb{Z}$. Those results are relatively well known in the case of *nondegenerate* symmetric bilinear pairings and *homogeneous* quadratic functions. However, the authors have not succeeded in finding in the litterature the general results for quadratic functions.

To describe the first result, we review (§2.2) a construction, known as the discriminant construction. This construction assigns to a symmetric bilinear lattice

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Key words and phrases. quadratic function, quadratic form, torsion group, lattice, discriminant construction.

QUADRATIC FUNCTIONS AND COMPLEX SPIN STRUCTURES ON THREE–MANIFOLDS

FLORIAN DELOUP AND GWÉNAËL MASSUYEAU

Abstract. We show how the space of complex spin structures of a closed oriented three–manifold embeds naturally into a space of quadratic functions associated to its linking pairing. Besides, we extend the Goussarov–Habiro theory of finite type invariants to the realm of compact oriented three–manifolds equipped with a complex spin structure. Our main result states that two closed oriented three–manifolds endowed with a complex spin structure are undistinguishable by complex spin invariants of degree zero if, and only if, their associated quadratic functions are isomorphic.

Complex spin structures, or Spin^c–structures, are additional structures with which manifolds may be equipped. They are needed to define the Seiberg–Witten invariants of 4–manifolds, as well as the Heegaard–Floer homologies of 3–manifolds by Ozsváth and Szabó. Any closed oriented 3–manifold M can be endowed with a Spin^c–structure and, in that case, Spin^c–structures are in canonical correspondence with Euler structures. The latter are classes of nonsingular vector fields on M which have been introduced by Turaev in order to refine Reidemeister torsion.

In this paper, we investigate the rôle played by quadratic functions in the topology of closed oriented 3–manifolds equipped with a Spin^c–structure or, equivalently, an Euler structure.

Extending constructions from [LL, MS, LW], we associate, to any closed oriented 3–manifold M with a Spin^c–structure σ, its *linking quadratic function*

$$H_2(M;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\phi_{M,\sigma}} \mathbb{Q}/\mathbb{Z}.$$

The function $\phi_{M,\sigma}$ is quadratic in the sense that the symmetric pairing defined by $(x,y) \mapsto \phi_{M,\sigma}(x+y) - \phi_{M,\sigma}(x) - \phi_{M,\sigma}(y)$ is bilinear. Moreover, this symmetric bilinear pairing coincides with $L_M := \lambda_M \circ (B \times B)$ where

$$\mathrm{Tors}\, H_1(M;\mathbb{Z}) \times \mathrm{Tors}\, H_1(M;\mathbb{Z}) \xrightarrow{\lambda_M} \mathbb{Q}/\mathbb{Z}$$

is the linking pairing of M and B denotes the Bockstein homomorphism associated to the short exact sequence of coefficients $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. In contrast with $\phi_{M,\sigma}$, the bilinear pairing L_M does not depend on σ. Spin^c–structures on a given manifold M are determined by their corresponding quadratic functions.

Theorem 1. *Let M be a closed connected oriented 3–manifold. The map $\sigma \mapsto \phi_{M,\sigma}$ defines a canonical embedding*

$$\mathrm{Spin}^c(M) \overset{\phi_M}{\hookrightarrow} \mathrm{Quad}(L_M)$$

from the set of Spin^c–structures on M to the set of quadratic functions with L_M as associated bilinear pairing.

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Key words and phrases. Three–manifold, quadratic function, complex spin structure, Goussarov–Habiro theory.

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Cohomology rings, Rochlin function, linking pairing and the Goussarov–Habiro theory of three–manifolds

Gwénaël Massuyeau

Abstract We prove that two closed oriented 3–manifolds have isomorphic quintuplets (homology, space of spin structures, linking pairing, cohomology rings, Rochlin function) if, and only if, they belong to the same class of a certain surgery equivalence relation introduced by Goussarov and Habiro.

AMS Classification 57M27; 57R15

Keywords 3–manifold, surgery equivalence relation, calculus of claspers, spin structure

1 Introduction

Goussarov and Habiro have developed a theory of finite type invariants for compact oriented 3–manifolds [6, 7, 4]. Their theory is based on a new kind of 3–dimensional topological calculus, called *calculus of claspers*. In strong connection with their finite type invariants, some equivalence relations have been studied by Goussarov and Habiro. For any integer $k \geq 1$, the Y_k*–equivalence* is the equivalence relation among compact oriented 3–manifolds generated by positive diffeomorphisms and surgeries along graph claspers of degree k. The reader will find the precise definition of the Y_k–equivalence in Section 2 and, waiting for this, will be enlightened by the following characterization due to Habiro [7]. Two manifolds M and M' are Y_k–equivalent if, and only if, there exists a compact oriented connected surface Σ in M and an element h of the k–th lower central series subgroup of the Torelli group of Σ such that M' is diffeomorphic to the manifold obtained from M by cutting it along Σ and regluing it using h. In particular, we see that the Y_k–equivalence becomes finer and finer as k increases.

Thus, the problem of characterizing the Y_k–equivalence relation in terms of invariants of the manifolds naturally arises. In the case $k = 1$, this problem has been solved for manifolds without boundary. Indeed, a result of Matveev [14],

YANG–BAXTER OPERATORS ARISING FROM ALGEBRA STRUCTURES AND THE ALEXANDER POLYNOMIAL OF KNOTS

GWÉNAËL MASSUYEAU AND FLORIN F. NICHITA

ABSTRACT. In this paper, we consider the problem of constructing knot invariants from Yang–Baxter operators associated to algebra structures. We first compute the enhancements of these operators. Then, we conclude that Turaev's procedure to derive knot invariants from these enhanced operators, as modified by Murakami, invariably produces the Alexander polynomial of knots.

1. INTRODUCTION

The Yang–Baxter equation and its solutions, the Yang–Baxter operators, first appeared in theoretical physics and statistical mechanics. Later, this equation has emerged in other fields of mathematics such as quantum group theory. Some references on this topic are [5, 7].

The Yang–Baxter equation also plays an important role in knot theory. Indeed, Turaev has described in [12] a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". The Jones polynomial [4] and its two–variable extensions, namely the Homflypt polynomial [2, 10] and the Kauffman polynomial [6], can be obtained in that way by "enhancing" some Yang–Baxter operators obtained in [3]. Those solutions of the Yang–Baxter equation are associated to simple Lie algebras and their fundamental representations. The Alexander polynomial can be derived from a Yang–Baxter operator as well, using a slight modification of Turaev's construction [8].

More recently, Dăscălescu and Nichita have shown in [1] how to associate a Yang–Baxter operator to any algebra structure over a vector space, using the associativity of the multiplication. This method to produce solutions to the Yang–Baxter equation, initiated in [9], is quite simple.

In this paper, we consider the problem of applying Turaev's method to Yang–Baxter operators derived from algebra structures. In general, finding the enhancements of a given Yang–Baxter operator can be difficult or lengthy. In the case of Yang–Baxter operators associated to algebra structures, the simplicity of their definition makes the search for enhancements an easy task. We do this here in full generality. We conclude from this computation that the only invariant which can be obtained from those Yang–Baxter operators is the Alexander polynomial of knots. Thus, in a way, the Alexander polynomial is the knot invariant corresponding to the axioms of (unitary associative) algebras. Note that specializations of the Homflypt polynomial had to be expected from those Yang–Baxter operators since they have degree 2 minimal polynomials.

The paper is organized as follows. In §2, we recall how to associate to any (unitary associative) algebra a Yang–Baxter operator. Next, in §3, we review Turaev's procedure to derive a knot invariant from a Yang–Baxter operator as soon as this one can be

Date: January, 2004.

SOME FINITENESS PROPERTIES FOR THE REIDEMEISTER–TURAEV TORSION OF THREE-MANIFOLDS

GWÉNAËL MASSUYEAU

Abstract. We prove for the Reidemeister–Turaev torsion of closed oriented three-manifolds some finiteness properties in the sense of Goussarov and Habiro, that is, with respect to some cut-and-paste operations which preserve the homology type of the manifolds. In general, those properties require the manifolds to come equipped with an Euler structure and a homological parametrization.

Contents

1. Introduction

The theory of finite-type invariants of 3-manifolds aims at understanding how manifolds are related one to the other through cut-and-paste operations and, consequently, how their invariants behave with respect to such operations.

In the Goussarov–Habiro theory, manifolds are modified using surgery operations which preserve the homology type [9, 13, 6]. Given a closed oriented connected 3-manifold M, a handlebody $H \subset M$ and a Torelli automorphism h of ∂H (that is,

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Key words and phrases. 3-manifold, RT torsion, finite-type invariant.

FINITE-TYPE INVARIANTS OF THREE-MANIFOLDS AND THE DIMENSION SUBGROUP PROBLEM

GWÉNAËL MASSUYEAU

Abstract. For a certain class of compact oriented 3-manifolds, M. Goussarov and K. Habiro have conjectured that the information carried by finite-type invariants should be characterized in terms of "cut-and-paste" operations defined by the lower central series of the Torelli group of a surface. In this paper, we observe that this is a variation of a classical problem in group theory, namely the "dimension subgroup problem." This viewpoint allows us to prove, by purely algebraic methods, an analogue of the Goussarov–Habiro conjecture for finite-type invariants with values in a fixed field. We deduce that their original conjecture is true at least in a weaker form.

Contents

1. Introduction

The Goussarov–Habiro theory is aimed at understanding how 3-manifolds can be obtained one from the other by cut-and-paste operations of a certain kind [3, 6, 4, 1]. In particular, it applies to the study of finite-type invariants introduced by Ohtsuki [21]: The latter are invariants of 3-manifolds which, in a sense, behave polynomially with respect to certain surgery operations.

We will work with compact oriented 3-manifolds whose boundary, if any, is identified with a fixed abstract surface. Those manifolds are considered up to homeomorphisms that preserve the orientation and the boundary identification. The kind of surgery modifications that are used in the Goussarov–Habiro theory can be described as follows:

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2000 *Mathematics Subject Classification.* 57M27, 16S34.
Key words and phrases. 3-manifold, finite-type invariant, group ring, N-series, dimension subgroup.

A FUNCTORIAL LMO INVARIANT FOR LAGRANGIAN COBORDISMS

DORIN CHEPTEA, KAZUO HABIRO, AND GWÉNAËL MASSUYEAU

Abstract. Lagrangian cobordisms are three-dimensional compact oriented cobordisms between once-punctured surfaces, subject to some homological conditions. We extend the Le–Murakami–Ohtsuki invariant of homology three-spheres to a functor from the category of Lagrangian cobordisms to a certain category of Jacobi diagrams. We prove some properties of this functorial LMO invariant, including its universality among rational finite-type invariants of Lagrangian cobordisms. Finally, we apply the LMO functor to the study of homology cylinders from the point of view of their finite-type invariants.

Contents

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2000 *Mathematics Subject Classification*. 57M27, 57M25.
Key words and phrases. 3-manifold, finite-type invariant, LMO invariant, Kontsevich integral, cobordism category, Lagrangian cobordism, homology cylinder, bottom-top tangle, Jacobi diagram, clasper.

SYMPLECTIC JACOBI DIAGRAMS AND THE LIE ALGEBRA OF HOMOLOGY CYLINDERS

KAZUO HABIRO AND GWÉNAËL MASSUYEAU

Abstract. Let S be a compact connected oriented surface, whose boundary is connected or empty. A homology cylinder over the surface S is a cobordism between S and itself, homologically equivalent to the cylinder over S. The Y-filtration on the monoid of homology cylinders over S is defined by clasper surgery. Using a functorial extension of the Le–Murakami–Ohtsuki invariant, we show that the graded Lie algebra associated to the Y-filtration is isomorphic to the Lie algebra of "symplectic Jacobi diagrams". This Lie algebra consists of the primitive elements of a certain Hopf algebra whose multiplication is a diagrammatic analogue of the Moyal–Weyl product.

The mapping cylinder construction embeds the Torelli group into the monoid of homology cylinders, sending the lower central series to the Y-filtration. We give a combinatorial description of the graded Lie algebra map induced by this embedding, by connecting Hain's infinitesimal presentation of the Torelli group to the Lie algebra of symplectic Jacobi diagrams. This Lie algebra map is shown to be injective in degree two, and the question of the injectivity in higher degrees is discussed.

Contents

1. Introduction and statement of the results

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. The first homology group $H_1(\Sigma_{g,1};\mathbb{Z})$ is denoted by H and is equipped with the intersection pairing

$$\omega : H \otimes H \longrightarrow \mathbb{Z}.$$

This is a non-degenerate skew-symmetric form, the group of isometries of which is denoted by $\mathrm{Sp}(H)$. Similarly, $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ is equipped with the rational extension of ω and $\mathrm{Sp}(H_{\mathbb{Q}})$ denotes the group of isometries of the symplectic vector space $H_{\mathbb{Q}}$.

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2000 *Mathematics Subject Classification.* 57M27, 57R50, 20F12, 20F38, 20F40.

Key words and phrases. 3-manifold, monoid of homology cylinders, Torelli group, finite-type invariant, Jacobi diagram, clasper, LMO invariant, Malcev completion, Malcev Lie algebra.

INFINITESIMAL MORITA HOMOMORPHISMS AND THE TREE-LEVEL OF THE LMO INVARIANT

GWÉNAËL MASSUYEAU

Abstract. Let Σ be a compact connected oriented surface with one boundary component, and let π be the fundamental group of Σ. The Johnson filtration is a decreasing sequence of subgroups of the Torelli group of Σ, whose k-th term consists of the self-homeomorphisms of Σ that act trivially at the level of the k-th nilpotent quotient of π. Morita defined a homomorphism from the k-th term of the Johnson filtration to the third homology group of the k-th nilpotent quotient of π.

In this paper, we replace groups by their Malcev Lie algebras and we study the "infinitesimal" version of the k-th Morita homomorphism, which is shown to correspond to the original version by a canonical isomorphism. We provide a diagrammatic description of the k-th infinitesimal Morita homomorphism and, given an expansion of the free group π that is "symplectic" in some sense, we show how to compute it from Kawazumi's "total Johnson map".

Besides, we give a topological interpretation of the full tree-reduction of the LMO homomorphism, which is a diagrammatic representation of the Torelli group derived from the Le–Murakami–Ohtsuki invariant of 3-manifolds. More precisely, a symplectic expansion of π is constructed from the LMO invariant, and it is shown that the tree-level of the LMO homomorphism is equivalent to the total Johnson map induced by this specific expansion. It follows that the k-th infinitesimal Morita homomorphism coincides with the degree $[k, 2k[$ part of the tree-reduction of the LMO homomorphism. Our results also apply to the monoid of homology cylinders over Σ.

Introduction

Nilpotent homotopy types of 3-manifolds have been introduced by Turaev [42]. They are defined by elementary tools from algebraic topology as follows. We fix an integer $k \geq 1$ and an abstract group G of nilpotency class k, which means that commutators of length $(k+1)$ are trivial in G. Let M be a closed connected oriented 3-manifold, whose k-th nilpotent quotient of the fundamental group is parametrized by the group G:

$$\psi : G \xrightarrow{\simeq} \pi_1(M)/\Gamma_{k+1}\pi_1(M).$$

Then, the *k-th nilpotent homotopy type* of the pair (M, ψ) is the homology class

$$\mu_k(M, \psi) := f^{\psi}_*([M]) \in H_3(G; \mathbb{Z})$$

where $f^\psi : M \to \mathrm{K}(G, 1)$ induces the composition $\pi_1(M) \to \pi_1(M)/\Gamma_{k+1}\pi_1(M) \stackrel{\psi^{-1}}{\to} G$ at the level of fundamental groups. For example, for $k = 1$, we are considering the *abelian homotopy type* of 3-manifolds which, by the work of Cochran, Gerges and Orr [8], is very well understood: $\mu_1(M, \psi)$ determines the cohomology ring of M together with its linking pairing, and vice versa.

As suggested to the author by Turaev, one way to study the invariant μ_k for higher k is to study its behaviour under surgery. This method particularly applies if one wishes to understand nilpotent homotopy types from the point of view of finite-type invariants,

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CANONICAL EXTENSIONS OF MORITA HOMOMORPHISMS TO THE PTOLEMY GROUPOID

GWÉNAËL MASSUYEAU

Abstract. Let Σ be a compact connected oriented surface with one boundary component. We extend each of Johnson's and Morita's homomorphisms to the Ptolemy groupoid of Σ. Our extensions are canonical and take values into finitely generated free abelian groups. The constructions are based on the 3-dimensional interpretation of the Ptolemy groupoid, and a chain map introduced by Suslin and Wodzicki to relate the homology of a nilpotent group to the homology of its Malcev Lie algebra.

Introduction

Let Σ be a compact connected oriented surface of genus g, with one boundary component. The mapping class group $\mathcal{M} := \mathcal{M}(\Sigma)$ of the bordered surface Σ consists of self-homeomorphisms of Σ fixing the boundary pointwise, up to isotopy. A classical theorem by Dehn and Nielsen asserts that the action of $\mathcal{M}$ on the fundamental group $\pi := \pi_1(\Sigma, \star)$, whose base point $\star$ is chosen on $\partial\Sigma$, is faithful. Let $\mathcal{M}[k]$ be the subgroup of $\mathcal{M}$ acting trivially on the k-th nilpotent quotient $\pi/\Gamma_{k+1}\pi$, where $\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \Gamma_3\pi \supset \cdots$ denotes the lower central series of π. Thus, one obtains a decreasing sequence of subgroups

$$\mathcal{M} = \mathcal{M}[0] \supset \mathcal{M}[1] \supset \mathcal{M}[2] \supset \cdots$$

which is called the *Johnson filtration* of the mapping class group, and whose study started with Johnson's works and was then developed by Morita. (The reader is, for instance, refered to their surveys [19] and [26].)

The first term $\mathcal{M}[1]$ of this filtration is the subgroup of $\mathcal{M}$ acting trivially in homology, namely the *Torelli group* $\mathcal{I} := \mathcal{I}(\Sigma)$ of the bordered surface Σ. In their study of the Torelli group, Johnson [18, 19] and Morita [23] introduced two families of group homomorphims with values in some abelian groups. For every $k \geq 1$, the *k-th Johnson homomorphism*

$$\tau_k : \mathcal{M}[k] \longrightarrow \frac{\pi}{\Gamma_2\pi} \otimes \frac{\Gamma_{k+1}\pi}{\Gamma_{k+2}\pi}$$

is designed to record the action of $\mathcal{M}[k]$ on the $(k+1)$-st nilpotent quotient $\pi/\Gamma_{k+2}\pi$: its kernel is $\mathcal{M}[k+1]$. For every $k \geq 1$, the *k-th Morita homomorphism*

$$M_k : \mathcal{M}[k] \longrightarrow H_3\left(\frac{\pi}{\Gamma_{k+1}\pi}; \mathbb{Z}\right)$$

is stronger than τ_k: its kernel is $\mathcal{M}[2k]$ as shown by Heap [12].

The Ptolemy groupoid is a combinatorial object which has arisen from Teichmüller theory in Penner's work [28, 29]. In the case of a surface with one boundary component like Σ, the objects of the Ptolemy groupoid are decorated graphs of a certain kind (called "trivalent bordered fatgraphs") whose edges are colored with elements of π; its morphisms are sequences of elementary moves between those graphs (called "Whitehead

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Lefschetz fibrations, intersection numbers, and representations of the framed braid group

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28 July 2010

Abstract

We examine the action of the fundamental group Γ of a Riemann surface with m punctures on the middle dimensional homology of a regular fiber in a Lefschetz fibration, and describe to what extent this action can be recovered from the intersection numbers of vanishing cycles. Basis changes for the vanishing cycles result in a nonlinear action of the framed braid group $\widetilde{\mathcal{B}}$ on m strings on a suitable space of $m \times m$ matrices. This action is determined by a family of cohomologous 1-cocycles $\mathcal{S}_c : \widetilde{\mathcal{B}} \to \mathrm{GL}_m(\mathbb{Z}[\Gamma])$ parametrized by distinguished configurations c of embedded paths from the regular value to the critical values. In the case of the disc, we compare this family of cocycles with the Magnus cocycles given by Fox calculus and consider some abelian reductions giving rise to linear representations of braid groups. We also prove that, still in the case of the disc, the intersection numbers along straight lines, which conjecturally make sense in infinite dimensional situations, carry all the relevant information.

Contents

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EQUIVALENCE RELATIONS FOR HOMOLOGY CYLINDERS AND THE CORE OF THE CASSON INVARIANT

GWÉNAËL MASSUYEAU AND JEAN–BAPTISTE MEILHAN

Abstract. Let Σ be a compact oriented surface of genus g with one boundary component. Homology cylinders over Σ form a monoid $\mathcal{IC}$ into which the Torelli group $\mathcal{I}$ of Σ embeds by the mapping cylinder construction. Two homology cylinders M and M' are said to be *Y_k-equivalent* if M' is obtained from M by "twisting" an arbitrary surface $S \subset M$ with a homeomorphim belonging to the k-th term of the lower central series of the Torelli group of S. The *J_k-equivalence* relation on $\mathcal{IC}$ is defined in a similar way using the k-th term of the Johnson filtration. In this paper, we characterize the Y_3-equivalence with three classical invariants: (1) the action on the third nilpotent quotient of the fundamental group of Σ, (2) the quadratic part of the relative Alexander polynomial, and (3) a by-product of the Casson invariant. Similarly, we show that the J_3-equivalence is classified by (1) and (2). We also prove that the core of the Casson invariant (originally defined by Morita on the second term of the Johnson filtration of $\mathcal{I}$) has a unique extension (to the corresponding submonoid of $\mathcal{IC}$) that is preserved by Y_3-equivalence and the mapping class group action.

1. Introduction

Let Σ be a compact connected oriented surface with one boundary component, and let $g \geq 0$ be the genus of Σ. A *homology cobordism* of Σ is a pair (M, m) where M is a compact connected oriented 3-manifold and $m : \partial(\Sigma \times [-1,1]) \to \partial M$ is an orientation-preserving homeomorphism such that the inclusions $m_\pm : \Sigma \to M$ defined by $x \mapsto m(x, \pm 1)$ induce isomorphisms $H_*(\Sigma;\mathbb{Z}) \to H_*(M;\mathbb{Z})$. Thus the 3-manifold M is a cobordism (with corners) between $\partial_+ M := m_+(\Sigma)$ and $\partial_- M := m_-(\Sigma)$. It is convenient to denote the cobordism (M, m) simply by M, the convention being that the boundary parametrization is always denoted by the lower-case letter m. In particular, we shall denote the *trivial cobordism* $(\Sigma \times [-1,1], \mathrm{Id})$ simply by $\Sigma \times [-1,1]$. The set of homeomorphism classes of homology cobordisms of Σ is denoted by

$$\mathcal{C} := \mathcal{C}(\Sigma),$$

where two homology cobordisms M, M' are considered *homeomorphic* if there is an orientation-preserving homeomorphism $f : M \to M'$ such that $f|_{\partial M} \circ m = m'$. The *composition* of two cobordisms M and M' is defined by "stacking" M' on the top of M, i.e. we define

$$M \circ M' := M \cup_{m_+ \circ (m'_-)^{-1}} M'$$

with $\partial(M \circ M')$ parametrized in the obvious way. Hence a monoid structure on $\mathcal{C}$.

The *mapping class group* of Σ is the group of isotopy classes of self-homeomorphisms of Σ that leave the boundary pointwise invariant. We shall denote it by

$$\mathcal{M} := \mathcal{M}(\Sigma).$$

The mapping cylinder construction $\mathbf{c} : \mathcal{M} \to \mathcal{C}$ is defined in the usual way by

$$\mathbf{c}(s) := \big(\Sigma \times [-1,1], (\mathrm{Id} \times (-1)) \cup (\partial\Sigma \times \mathrm{Id}) \cup (s \times 1)\big). \tag{1.1}$$

Since the homomorphism $\mathbf{c}$ is injective, we shall sometimes consider the group $\mathcal{M}$ as a submonoid of $\mathcal{C}$ and remove $\mathbf{c}$ from our notation. A base point $\star$ being fixed on $\partial\Sigma$, the mapping class group acts on the fundamental group $\pi := \pi_1(\Sigma, \star)$. The resulting

Date: July 21, 2011.

FOX PAIRINGS AND GENERALIZED DEHN TWISTS

GWÉNAËL MASSUYEAU AND VLADIMIR TURAEV

Abstract. We introduce a notion of a Fox pairing in a group algebra and use Fox pairings to define automorphisms of the Malcev completions of groups. These automorphisms generalize to the algebraic setting the action of the Dehn twists in the group algebras of the fundamental groups of surfaces. This work is inspired by the Kawazumi–Kuno generalization of the Dehn twists to non-simple closed curves on surfaces.

1. Introduction

There is a simple and well-known construction producing families of automorphisms of modules from bilinear forms. Given a module H over a commutative ring $\mathbb{K}$ and a bilinear form $\bullet : H \times H \to \mathbb{K}$, one associates with any isotropic vector $a \in H$ and any $k \in \mathbb{K}$ a transvection $H \to H$ carrying each $h \in H$ to $h + k(a \bullet h)a$. We introduce in this paper a group-theoretic version of transvections. Note that any group π has a Malcev completion $\hat{\pi} = \hat{\pi}^{\mathbb{K}}$ formed by the group-like elements of the Hopf algebra $\widehat{\mathbb{K}[\pi]}$ which is the fundamental completion of the group algebra $\mathbb{K}[\pi]$, see [Qu]. Our main construction starts with a group π and a certain bilinear form, a Fox pairing, in $\mathbb{K}[\pi]$ and produces a family of group automorphisms of $\hat{\pi}$ which are in many respects similar to transvections.

Our original motivation comes from the study of diffeomorphisms of surfaces. Recall that simple closed curves on a connected oriented surface Σ give rise to diffeomorphisms $\Sigma \to \Sigma$ called the Dehn twists. The Dehn twists induce group automorphisms of $\pi_1(\Sigma)$ and algebra automorphisms of $\mathbb{K}[\pi_1(\Sigma)]$ and $\widehat{\mathbb{K}[\pi_1(\Sigma)]}$. When Σ is compact and $\partial\Sigma$ is a circle, N. Kawazumi and Y. Kuno [KK1] generalized the action of the Dehn twists on $\widehat{\mathbb{K}[\pi_1(\Sigma)]}$ to arbitrary (not necessarily simple) loops on Σ. Their definition uses so-called symplectic expansions of $\pi_1(\Sigma)$, see [Ma]. The present paper arose from our desire to avoid the use of symplectic expansions and to generalize the Kawazumi–Kuno automorphisms to all oriented surfaces. One simplification achieved here consists in replacing algebra automorphisms of the completed group algebras by group automorphisms of the Malcev completions.

The key ingredient in our approach is the homotopy intersection form on surfaces introduced by the second named author in [Tu1]. A version of this form was implicit already in the work of C. Papakyriakopoulos [Pa] who studied Reidemeister's equivariant intersection pairings on surfaces. Axiomatizing the homotopy intersection form, we introduce a notion of a Fox pairing in the group algebra $A = \mathbb{K}[\pi]$ of an arbitrary group π. Let $I \subset A$ be the fundamental ideal of A defined as the kernel of the augmentation homomorphism $\mathrm{aug} : A \to \mathbb{K}$ carrying $\pi \subset A$ to 1. A Fox pairing in A is a $\mathbb{K}$-bilinear pairing $\eta : A \times A \to A$ such that $1 \in A$ lies in both left and right annihilators and the restriction of η to $I \times I$ is left A-linear in the first variable and right A-linear in the second variable. Similar pairings were studied in

QUASI-POISSON STRUCTURES ON REPRESENTATION SPACES OF SURFACES

GWÉNAËL MASSUYEAU AND VLADIMIR TURAEV

ABSTRACT. Given an oriented surface Σ with base point $* \in \partial\Sigma$, we introduce for all $N \geq 1$, a canonical quasi-Poisson bracket on the space of N-dimensional linear representations of $\pi_1(\Sigma, *)$. Our bracket extends the well-known Poisson bracket on GL_N-invariant functions on this space. Our main tool is a natural structure of a quasi-Poisson double algebra (in the sense of M. Van den Bergh) on the group algebra of $\pi_1(\Sigma, *)$.

1. INTRODUCTION

The representation space $\mathcal{H} = \mathrm{Hom}(\pi, G)$ consisting of all homomorphisms from the fundamental group π of a compact oriented surface to a Lie group G is a rich source of geometry. The group G acts on $\mathcal{H}$ by conjugations and the quotient $\mathcal{H}/G$ can be identified with a moduli space of flat connections and with a moduli space of holomorphic vector bundles (for appropriate G). For closed surfaces, the space $\mathcal{H}/G$ carries symplectic geometry. The classical instances are the Weil–Petersson symplectic structure on the Teichmüller space (for $G = \mathrm{PSL}(2, \mathbb{R})$) and the Atiyah–Bott symplectic structure for compact G endowed with a nondegenerate $\mathrm{Ad}(G)$-invariant symmetric bilinear form on the corresponding Lie algebra. A systematic approach to the symplectic structure on $\mathcal{H}/G$ was introduced by W. Goldman [Go1, Go2] in extension of the work of S. Wolpert [Wo]. Goldman defined a Lie bracket in the free abelian group generated by the set of conjugacy classes of elements of π and used it to compute the Poisson structure on $\mathcal{H}/G$ induced by the symplectic structure. Surfaces with boundary have a canonical Poisson structure on the quotient $\mathcal{H}/G$. It was described in [FR] in terms of ciliated fat graphs and in [GHJW] in terms of group systems, see also [AMM], [La] and the surveys [Au], [Go3], [Hu].

In this paper we show that for surfaces with boundary and $G = \mathrm{GL}_N(\mathbb{R})$ with $N \geq 1$, there is a canonical Poisson-type structure on $\mathcal{H}$. More precisely, consider a compact oriented surface Σ with non-void boundary. Set $\pi = \pi_1(\Sigma, *)$ with $* \in \partial\Sigma$ and $\mathcal{H} = \mathrm{Hom}(\pi, G)$. Since π is a free group of a finite rank, n, a choice of a basis of π yields a bijection $\mathcal{H} \cong G^n$. This induces a structure of a smooth manifold on $\mathcal{H}$ independent of the choice of the basis. Moreover, the action of G on $\mathcal{H}$ by conjugations is smooth. A. Alekseev, Y. Kosmann-Schwarzbach, and E. Meinrenken [AKsM] introduced a notion of a quasi-Poisson structure on a smooth manifold endowed with a smooth action of a Lie group. We construct here a canonical quasi-Poisson structure on $\mathcal{H}$. One may think of this structure as of a skew-symmetric bracket $\{-,-\}$ in the algebra $C^\infty(\mathcal{H})$ of smooth $\mathbb{R}$-valued functions on $\mathcal{H}$ satisfying the Leibniz identity and the modified Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = \phi(f, g, h)$$

BRACKETS IN PONTRYAGIN ALGEBRAS OF MANIFOLDS

GWÉNAËL MASSUYEAU AND VLADIMIR TURAEV

Abstract. Given a smooth oriented manifold M with non-empty boundary, we study the Pontryagin algebra $A = H_*(\Omega)$ where Ω is the space of loops in M based at a distinguished point of ∂M. Using the ideas of string topology of Chas–Sullivan, we define a linear map $\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$ which is a double bracket in the sense of Van den Bergh satisfying a version of the Jacobi identity. For $\dim(M) \geq 3$, the double bracket $\{\!\{-,-\}\!\}$ induces Gerstenhaber brackets in the representation algebras associated with A. This extends our previous work on the case $\dim(M) = 2$ where $A = H_0(\Omega)$ is the group algebra of $\pi_1(M)$ and $\{\!\{-,-\}\!\}$ induces the classical Poisson brackets on the moduli spaces of representations of $\pi_1(M)$.

Contents

2010 *Mathematics Subject Classification.* 17B63, 55N33, 55P50, 57R19.

SPLITTING FORMULAS FOR THE LMO INVARIANT OF RATIONAL HOMOLOGY THREE-SPHERES

GWÉNAËL MASSUYEAU

Abstract. For rational homology 3-spheres, there exist two universal finite-type invariants: the Le–Murakami–Ohtsuki invariant and the Kontsevich–Kuperberg–Thurston invariant. These invariants take values in the same space of "Jacobi diagrams", but it is not known whether they are equal. In 2004, Lescop proved that the KKT invariant satisfies some "splitting formulas" which relate the variations of KKT under replacement of embedded rational homology handlebodies by others in a "Lagrangian-preserving" way. We show that the LMO invariant satisfies exactly the same relations. The proof is based on the LMO functor, which is a generalization of the LMO invariant to the category of 3-dimensional cobordisms, and we generalize Lescop's splitting formulas to this setting.

1. Introduction

A *rational homology* 3*-sphere* (or, $\mathbb{Q}$*-homology* 3*-sphere*) is a closed oriented 3-manifold S that has the same homology with rational coefficients as the standard 3-sphere S^3. Le, Murakami & Ohtsuki defined in [16] an invariant $Z(S)$ of rational homology 3-spheres S with values in the algebra $\mathcal{A}(\varnothing)$ of Jacobi diagrams. The LMO invariant $Z(S)$, which was originally denoted by $\hat{\Omega}(S)$ in [16], is multiplicative under connected sums. As shown in [4], it coincides with the Aarhus integral $\mathring{A}(S)$ introduced by Bar-Natan, Garoufalidis, Rozansky & Thurston [2, 3]. This paper is aimed at studying the behaviour of Z under a certain type of rational homology handlebody replacement, called "Lagrangian-preserving surgery" by Lescop [19] and whose definition we now recall.

A *rational homology handlebody* (or, $\mathbb{Q}$*-homology handlebody*) of *genus* g is a compact oriented 3-manifold C' that has the same homology with rational coefficients as the standard genus g handlebody. The *Lagrangian* of C' is the kernel $\mathbf{L}^{\mathbb{Q}}_{C'}$ of the homomorphism $\mathrm{incl}_*: H_1(\partial C';\mathbb{Q}) \to H_1(C';\mathbb{Q})$ induced by the inclusion: indeed, this is a Lagrangian subspace of $H_1(\partial C';\mathbb{Q})$ with respect to the intersection pairing. A $\mathbb{Q}$*-Lagrangian-preserving pair* (or, $\mathbb{Q}$*-LP pair*) is a pair $\mathsf{C} = (C', C'')$ of two rational homology handlebodies whose boundaries are identified $\partial C' \equiv \partial C''$ in such a way that $\mathbf{L}^{\mathbb{Q}}_{C'} = \mathbf{L}^{\mathbb{Q}}_{C''}$. The *total manifold* of the $\mathbb{Q}$-LP pair C is the closed oriented 3-manifold

$$C := (-C') \cup_\partial C''.$$

Note that the inclusion $C' \subset C$ induces a canonical isomorphism $H_1(C';\mathbb{Q}) \simeq H_1(C;\mathbb{Q})$. The form $H^1(C;\mathbb{Q})^{\otimes 3} \to \mathbb{Q}$ defined by triple-cup products $(x,y,z) \mapsto \langle x \cup y \cup z, [C]\rangle$ is skew-symmetric: we denote it by

$$\mu(C) \in \mathrm{Hom}_{\mathbb{Q}}\left(\Lambda^3 H^1(C;\mathbb{Q}), \mathbb{Q}\right) \simeq \Lambda^3 H_1(C;\mathbb{Q}).$$

Given a compact oriented 3-manifold M and a $\mathbb{Q}$-LP pair $\mathsf{C} = (C', C'')$ such that C' is embedded in the interior of M, one can replace the submanifold C' in M by C'' in order to obtain a new 3-manifold

$$M_{\mathsf{C}} := (M \setminus \mathrm{int}(C')) \cup_\partial C''.$$

The move $M \leadsto M_{\mathsf{C}}$ between compact oriented 3-manifolds is called a $\mathbb{Q}$*-LP surgery*.

Date: April 16, 2014.

A FUNCTORIAL EXTENSION OF THE ABELIAN REIDEMEISTER TORSIONS OF THREE-MANIFOLDS

VINCENT FLORENS AND GWÉNAËL MASSUYEAU

Abstract. Let $\mathbb{F}$ be a field and let $G \subset \mathbb{F} \setminus \{0\}$ be a multiplicative subgroup. We consider the category Cob_G of 3-dimensional cobordisms equipped with a representation of their fundamental group in G, and the category $\mathsf{Vect}_{\mathbb{F},\pm G}$ of $\mathbb{F}$-linear maps defined up to multiplication by an element of $\pm G$. Using the elementary theory of Reidemeister torsions, we construct a "Reidemeister functor" from Cob_G to $\mathsf{Vect}_{\mathbb{F},\pm G}$. In particular, when the group G is free abelian and $\mathbb{F}$ is the field of fractions of the group ring $\mathbb{Z}[G]$, we obtain a functorial formulation of an Alexander-type invariant introduced by Lescop for 3-manifolds with boundary; when G is trivial, the Reidemeister functor specializes to the TQFT developed by Frohman and Nicas to enclose the Alexander polynomial of knots. The study of the Reidemeister functor is carried out for any multiplicative subgroup $G \subset \mathbb{F} \setminus \{0\}$. We obtain a duality result and we show that the resulting projective representation of the monoid of homology cobordisms is equivalent to the Magnus representation combined with the relative Reidemeister torsion.

Contents

1. Introduction

Let Cob be the category of 3-dimensional cobordisms introduced by Crane and Yetter [CY99], and whose definition we briefly recall. The objects of Cob are integers $g \geq 0$, and correspond to compact connected oriented surfaces F_g of genus g with one boundary component. Indeed, we fix for every $g \geq 0$ a model surface F_g whose boundary is identified with S^1, and we also fix a base point $\star$ on $\partial F_g = S^1$. The morphisms $g_- \to g_+$ in the category Cob are the equivalence classes of cobordisms between the surfaces F_{g_-} and F_{g_+}. To be more specific, a *cobordism* from F_{g_-} to F_{g_+} is a pair (M, m) consisting of a compact connected oriented 3-manifold M and an orientation-preserving homeomorphism $m : F(g_-, g_+) \to \partial M$ where

$$F(g_-, g_+) := -F_{g_-} \cup_{S^1 \times \{-1\}} \left(S^1 \times [-1,1]\right) \cup_{S^1 \times \{1\}} F_{g_+};$$

Date: October 23, 2014.

AN INTRODUCTION TO THE ABELIAN REIDEMEISTER TORSION OF THREE-DIMENSIONAL MANIFOLDS

GWÉNAËL MASSUYEAU

ABSTRACT. These notes accompany some lectures given at the autumn school "*Tresses in Pau*" in October 2009. The abelian Reidemeister torsion for 3-manifolds, and its refinements by Turaev, are introduced. Some applications, including relations between the Reidemeister torsion and other classical invariants, are surveyed.

CONTENTS

1. INTRODUCTION

The Reidemeister torsion and the Alexander polynomial are among the most classical invariants of 3-manifolds. The first one was introduced in 1935 by Kurt Reidemeister [54] in order to classify lens spaces – those closed 3-manifolds which are obtained by gluing two copies of the solid torus along their boundary. The second invariant is even older since it dates back to 1928, when James Alexander defined it for knot and link complements [1]. Those two invariants can be extended to higher dimensions and refer by their very definition to the maximal *abelian* cover of the 3-manifold.

In 1962, John Milnor interpreted in [39] the Alexander polynomial of a link as a kind of Reidemeister torsion (with coefficients in a field of rational fractions, instead of a cyclotomic field as was the case for lens spaces). This new viewpoint on the Alexander polynomial clarified its properties, among which its symmetry. This approach to study the Alexander polynomial was systematized by Vladimir Turaev in [69, 72], where he re-proved most of the known properties of the Alexander polynomial of 3-manifolds and links using general properties of the Reidemeister torsion. Vladimir Turaev also defined a kind of "maximal" torsion, which is universal among Reidemeister torsions with abelian

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From mapping class groups to monoids of homology cobordisms: a survey

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Abstract. Let Σ be a compact oriented surface. A homology cobordism of Σ is a cobordism C between two copies of Σ, such that both the "top" inclusion and the "bottom" inclusion $\Sigma \subset C$ induce isomorphisms in homology. Homology cobordisms of Σ form a monoid, into which the mapping class group of Σ embeds by the mapping cylinder construction. In this chapter, we survey recent works on the structure of the monoid of homology cobordisms, and we outline their relations with the study of the mapping class group. We are mainly interested in the cases where $\partial\Sigma$ is empty or connected.

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Contents

UNE INTRODUCTION AUX INVARIANTS DE CHERN–SIMONS

GWÉNAËL

Résumé. Cette note, informelle et imprécise, accompagne un exposé donné le 30 janvier 2008 au « Séminaire G3 » à Strasbourg. Nous présentons l'article où Chern et Simons introduisent les formes et les invariants qui portent désormais leurs noms.

Table des matières

1. Introduction

Soit M une 3-variété lisse fermée orientée et soit $A \in \Omega^1(M; \mathfrak{su}(2))$, une 1-forme sur M à valeurs dans $\mathfrak{su}(2)$. On pose

$$\mathrm{CS}(A) := \frac{1}{8\pi^2} \int_M \mathrm{Tr}\left(\mathrm{d}A \wedge A + \frac{2}{3} A \wedge A \wedge A\right) \in \mathbb{R}. \tag{1.1}$$

UNE COURTE INTRODUCTION AUX INVARIANTS DE DIJKGRAAF–WITTEN

GWÉNAËL MASSUYEAU

RÉSUMÉ. Cette note, informelle et imprécise, accompagne un exposé donné durant la « Journée Géométrie, Topologie et Physique » à Strasbourg, le 28 Février 2008. Nous introduisons les invariants de Dijkgraaf–Witten et nous donnons leur formule par somme d'états.

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1. INTRODUCTION

Soit G un groupe fini et soit $\mathrm{K}(G,1)$ un espace d'Eilenberg–MacLane. Ainsi, $\mathrm{K}(G,1)$ est l'espace topologique pointé (unique à équivalence d'homotopie près) satisfaisant

$$\pi_1(\mathrm{K}(G,1),\centerdot) = G \quad \text{et} \quad \pi_i(\mathrm{K}(G,1),\centerdot) = 0 \text{ pour } i > 1.$$

On choisit aussi

$$\alpha \in H^d(G;\mathrm{U}(1)) = H^d(\mathrm{K}(G,1);\mathrm{U}(1)).$$

Définition 1.1. Soit M une d-variété fermée, orientée et connexe. L'*invariant de Dijkgraaf–Witten* de M, relatif à la classe de cohomologie α, est

$$Z_\alpha(M) := |G|^{-1} \sum_{\gamma\in \mathrm{Hom}(\pi_1(M,\centerdot),G)} \big\langle \alpha, (f_\gamma)_*([M])\big\rangle \in \mathbb{C}.$$

Ici, $\centerdot \in M$ est un point base et $f_\gamma : M \to \mathrm{K}(G,1)$ est l'application pointée (unique à homotopie près) telle que $(f_\gamma)_* = \gamma$ au niveau du $\pi_1(-)$: ces choix ne comptent pas.

Malgré leur définition très simple, les invariants de Dijkgraaf–Witten n'ont été formulés que récemment. Ils semblent être apparus pour la première fois en 1990 dans l'article [7], pour servir de « toy examples » aux invariants quantiques de Witten [20].

Date: 26 Février 2008.

A SHORT INTRODUCTION TO THE ALEXANDER POLYNOMIAL

GWÉNAËL MASSUYEAU

Abstract. These informal notes accompany a talk given in Grenoble for the workshop "*Représentations de $U_q(sl_2)$ et invariants d'Alexander*" (December 2008). We introduce the Alexander polynomial of links following Milnor and Turaev, who interpreted this classical invariant as a kind of Reidemeister torsion. As shown by Turaev, this approach allows for an intrinsic construction of the Conway function of links, which is a refinement of the Alexander polynomial.

Contents

1. Introduction

James Alexander introduced his link invariant in the paper [1] published in 1928. Since then, the Alexander polynomial has been extensively studied by hundreds of authors, and several fundamental properties of the Alexander polynomial have been shown. See any of the classical references in knot theory, including [16] and [3].

The Alexander polynomial of a link is defined up to some indeterminacy. In 1967, John Conway introduced in [5] a refinement of the Alexander polynomial and explained how to compute it recursively using some additive relations of a local nature (which are nowadays called "skein relations"). Conway's approach was fixed by Louis Kauffman in the one-variable case [10] and by Richard Hartley in the multi-variable case [9]. But, those two constructions of the Conway function are extrinsic. (Kauffman needs a Seifert surface for the link, while Hartley needs a diagram presentation of the link.)

In 1962, John Milnor noticed in [11] a very close connection between the Alexander polynomial of a link and a certain kind of Reidemeister torsion. This new viewpoint

Date: November 28, 2008.

A SHORT INTRODUCTION TO MAPPING CLASS GROUPS

GWÉNAËL MASSUYEAU

Abstract. These informal notes accompany a talk given in Strasbourg for the *Master Class on Geometry* (spring 2009). We introduce the mapping class group of a surface and its enigmating subgroup, the Torelli group. One hour and seventeen pages are certainly not enough to present this beautiful and rich subject. So, we recommend for further reading Ivanov's survey of the mapping class group [16] as well as Farb and Margalit's book [9], which we used to prepare this talk. Johnson's survey [20] gives a very nice introduction to the Torelli group, while Morita's paper [30] reports on more recent developments of the subject.

Contents

1. Definition and first examples

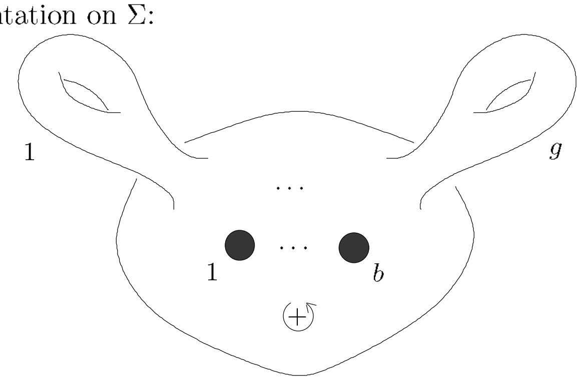
We consider a compact connected orientable surface Σ. By the classification theorem of surfaces, Σ is determined (up to homeomorphism) by the number of connected components of its boundary

$$b := |\pi_0(\partial\Sigma)|$$

and by its genus

$$g := \frac{1}{2} \cdot \left(\text{rank } H_1(\Sigma;\mathbb{Z}) - b + 1\right).$$

We also fix an orientation on Σ:



Date: May 4, 2009.
We are grateful to Sylvain Gervais for reading and commenting these notes.

Equivalence relations on three-dimensional manifolds defined by subgroups of the Torelli group & the core of the Casson invariant

GWÉNAËL MASSUYEAU

(joint work with Jean–Baptiste Meilhan)

Let Σ be a compact connected oriented surface of genus g with one boundary component. A *homology cylinder* over Σ is a compact oriented 3-manifold M with an orientation-preserving homeomorphism $m : \partial\left(\Sigma \times [-1,1]\right) \to \partial M$ such that

$$\begin{array}{ccc} H_*\left(\Sigma \times [-1,1];\mathbb{Z}\right) & \overset{\exists}{\underset{\simeq}{- - \to}} & H_*(M;\mathbb{Z}) \\ {\scriptstyle \mathrm{incl}_*}\uparrow & & \uparrow{\scriptstyle \mathrm{incl}_*} \\ H_*\left(\partial(\Sigma \times [-1,1]);\mathbb{Z}\right) & \underset{m_*}{\overset{\simeq}{\longrightarrow}} & H_*(\partial M;\mathbb{Z}). \end{array}$$

Homology cylinders over Σ can be regarded as cobordisms (with corners) between two copies of Σ, namely from $m\left(\Sigma \times \{+1\}\right)$ to $m\left(\Sigma \times \{-1\}\right)$. Thus homology cylinders can be "composed" in the usual way so that, if we consider them up to homeomorphisms (that preserve orientations and boundary parametrizations), we get a monoid $\mathcal{IC}(\Sigma)$. For instance, $\mathcal{IC}(\Sigma)$ is in genus $g = 0$ isomorphic to the monoid of homology 3-spheres. In genus $g > 0$, the mapping cylinder construction

$$\mathbf{c} : \mathcal{I}(\Sigma) \longrightarrow \mathcal{IC}(\Sigma),\ s \longmapsto \big(\Sigma \times [-1,1], (\mathrm{Id} \times \{-1\}) \cup (\partial\Sigma \times \mathrm{Id}) \cup (s \times \{1\})\big)$$

defines an embedding of the Torelli group of the surface Σ into the monoid $\mathcal{IC}(\Sigma)$.

Two homology cylinders M and M' over Σ are said to be *Y_k-equivalent* if M' can be obtained from M by "twisting" an arbitrary embedded surface E in the interior of M with an element of the k-th term $\Gamma_k\mathcal{I}(E)$ of the lower central series of the Torelli group $\mathcal{I}(E)$ of E. (The surface E has an arbitrary position in M, but it is assumed to be compact connected oriented with one boundary component.) The *J_k-equivalence* relation on $\mathcal{IC}(\Sigma)$ is defined in a similar way using the k-th term of the Johnson filtration of $\mathcal{I}(E)$ instead of its lower central series: in other words, the "twisting" homeomorphism is required to act trivially at the level of the k-th nilpotent quotient $\pi_1(E)/\Gamma_{k+1}\pi_1(E)$ of the fundamental group $\pi_1(E)$. All these equivalence relations are organized as follows:

$$\begin{array}{ccccccccccc} Y_1 & \Longleftarrow & Y_2 & \Longleftarrow & Y_3 & \Longleftarrow & \cdots & Y_k & \Longleftarrow & Y_{k+1} & \Longleftarrow & \cdots \\ \| & & \Downarrow & & \Downarrow & & & \Downarrow & & \Downarrow & & \\ J_1 & \Longleftarrow & J_2 & \Longleftarrow & J_3 & \Longleftarrow & \cdots & J_k & \Longleftarrow & J_{k+1} & \Longleftarrow & \cdots \end{array}$$

The Y_k-equivalence relations have been introduced by Goussarov and Habiro in the context of finite-type invariants [1, 4]. They have developed a surgery calculus in dimension three, which is kind of a topological analogue of the commutator calculus in groups and is called "clasper calculus" [2, 4]. The Y_k-equivalence relations can be reformulated and studied using this clasper calculus. Having this strong tool at one's disposal is a big advantage of the Y_k-equivalence relations with respect to the J_k-equivalence relations.

M2 class "Mapping class groups, braid groups and formality" by B. Enriquez and G. Massuyeau

Abstract. Mapping class groups and, in particular, braid groups are fundamental objects in Topology and Geometry, and they also constitute a very rich source of structures in Algebra. This class will propose an introduction to the study of these groups and some of their subgroups, which include the Torelli group and the pure braid group. We will present the "formal" approach which consists in replacing these discrete groups by some infinitesimal analogues (namely, their Malcev Lie algebras).

Timetable. Classes will consist of lectures, usually on Wednesday (14:00-17:00), and seminars, usually on Thursday (14:00-16:00). Here is a tentative schedule:

Week	Teacher	Theme
21-22 jan. 2015	GM	Review of surfaces and curves
28-29 jan. 2015	GM	An introduction to mapping class groups
04-05 feb. 2015	GM	An introduction to surface braid groups
11-12 feb. 2015	BE	Review of Lie theory (1/2)
18-19 feb. 2015	BE	Review of Lie theory (2/2)
25-26 feb. 2015	... BREAK ...	
04-05 mar. 2015	... BREAK ...	
11-12 mar. 2015	BE	Formality of the pure braid group
18-19 mar. 2015	BE	Formality of the elliptic pure braid group (1/2)
25-26 mar. 2015	BE	Formality of the elliptic pure braid group (2/2)
01-02 apr. 2015	GM	Formality of the Torelli group (1/2)
08-09 apr. 2015	GM	Formality of the Torelli group (2/2)
15-16 apr. 2015	... BREAK ...	
22-23 apr. 2015	... EXAM ...	

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