Categorifying the Knizhnik-Zamolodchikov connection

João Faria Martins, New (Nova) University of Lisbon

joint with:

Lucio Simone Cirio, University of Luxembourg

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On 2-Dimensional Homotopy Invariants of Complements of Knotted Surfaces

João Faria Martins

Departamento de Matemática, Instituto Superior Técnico,

Av. Rovisco Pais, 1049-001 Lisboa, Portugal

jmartins@math.ist.utl.pt

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Abstract

We prove that if M is a CW-complex and $*$ is a 0-cell of M, then the crossed module $\Pi_2(M, M^1, *) = (\pi_1(M^1, *), \pi_2(M, M^1, *), \partial, \triangleright)$ does not depend on the cellular decomposition of M up to free products with $\Pi_2(D^2, S^1, *)$, where M^1 is the 1-skeleton of M. From this it follows that if $\mathcal{G}$ is a finite crossed module and M is finite, then the number of crossed module morphisms $\Pi_2(M, M^1, *) \to \mathcal{G}$ (which is finite) can be re-scaled to a homotopy invariant $I_{\mathcal{G}}(M)$ (i. e. not dependent on the cellular decomposition of M), a construction similar to David Yetter's in [Y], or Tim Porter's in [P1, P2]. We describe an algorithm to calculate $\pi_2(M, M^{(1)}, *)$ as a crossed module over $\pi_1(M^{(1)}, *)$, in the case when M is the complement of a knotted surface in S^4 and $M^{(1)}$ is the 1-handlebody of a handle decomposition of M, which, in particular, gives a method to calculate the algebraic 2-type of M. In addition, we prove that the invariant $I_{\mathcal{G}}$ yields a non-trivial invariant of knotted surfaces.

2000 Mathematics Subject Classification: 57Q45, 57M05, 57M27.

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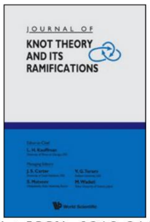

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ON THE ANALYTIC PROPERTIES OF THE z-COLOURED JONES POLYNOMIAL

JOÃO FARIA MARTINS
Departamento de Mathemática, Instituto Superior Técnico, Av Rovisco Pais, 1049-001, Lisboa, Portugal

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We analyse the possibility of defining $\mathbb{C}$-valued Knot invariants associated with infinite-dimensional unitary representations of SL(2,$\mathbb{R}$) and the Lorentz Group taking as starting point the Kontsevich integral and the notion of infinitesimal character. This yields a family of knot invariants whose target space is the set of formal power series in $\mathbb{C}$, which contained in the Melvin–Morton expansion of the coloured Jones polynomial. We verify that for some knots the series have zero radius of convergence and analyse the construction of functions of which this series are asymptotic expansions by means of Borel re-summation. Explicit calculations are done in the case of torus knots which realise an analytic extension of the values of the coloured Jones polynomial to complex spins. We present a partial answer in the general case.

Keywords: Kontsevich integral; coloured jones polynomial; Melvin–Morton expansion; non-compact Lie groups; Borel re-summation
AMSC: Primary 57M27, Secondary 17B37

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Knot theory with the Lorentz group

by

João Faria Martins (Lisboa)

Abstract. We analyse perturbative expansions of the invariants defined from unitary representations of the Quantum Lorentz Group in two different ways, namely using the Kontsevich Integral and weight systems, and the R-matrix in the Quantum Lorentz Group defined by Buffenoir and Roche. The two formulations are proved to be equivalent; and they both yield $\mathbb{C}[[h]]$-valued knot invariants related with the Melvin–Morton expansion of the Coloured Jones Polynomial.

Introduction. The main aim of this article is to show a possible path to define knot invariants from infinite-dimensional representations of the Lorentz group.

Let $\mathcal{A}$ be a Hopf algebra; its category of finite-dimensional representations is a compact monoidal category. Let q be a complex number not equal to 1 or -1. Suppose $\mathcal{A} = U_q(\mathfrak{g})$ is the Drinfeld–Jimbo algebra attached to the semisimple Lie algebra $\mathfrak{g}$. Even though $\mathcal{A}$ is not a ribbon Hopf algebra, it possesses a formal R-matrix and a formal ribbon element. These elements make sense when applied to finite-dimensional representations of $\mathcal{A}$, and thus its category of finite-dimensional representations is a ribbon category. This means we have a knot invariant attached to any finite-dimensional representation of $\mathcal{A}$. This kind of knot invariants take values in $\mathbb{C}$.

A similar situation happens in the case of the Quantum Lorentz Group $\mathcal{D}$ as defined by Woronowicz and Podleś in [PoW]. We shall use especially the further developments of its theory by Buffenoir and Roche (see [BR1] and [BR2]). Despite the fact $\mathcal{D}$ is not a Drinfeld–Jimbo algebra, its structure of a quantum double, namely $\mathcal{D} = \mathcal{D}(U_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2)))$ with $q \in (0,1)$, makes possible the definition of a formal R-matrix on it. Also, it is possible to define a heuristic ribbon element.

2000 *Mathematics Subject Classification*: 57M27, 17B37, 20G42.
Key words and phrases: Kontsevich Integral, Coloured Jones Polynomial, Quantum Lorentz Group.

Quantum Topology and The Lorentz Group

by João Nuno Gonçalves Faria Martins

Thesis submitted to the University of Nottingham for the degree of Doctor of Philosophy, July 2004

Abstract

We analyse the perturbative expansion of knot invariants related with infinite dimensional representations of $\mathfrak{sl}(2, \mathbb{R})$ and the Lorentz group taking as a starting point the Kontsevich Integral and the notion of central characters of infinite dimensional unitary representations of Lie Groups. The prime aim is to define $\mathbb{C}$-valued knot invariants. This yields a family of $\mathbb{C}[[h]]$-valued knot invariants contained in the Melvin-Morton expansion of the Coloured Jones Polynomial. It is verified that for some knots, namely torus knots, the power series obtained have a zero radius of convergence, and therefore we analyse the possibility of obtaining analytic functions of which these power series are asymptotic expansions by means of Borel re-summation. This process is complete for torus knots, and a partial answer is presented in the general case, which gives an upper bound on the growth of the coefficients of the Melvin-Morton expansion of the Coloured Jones Polynomial. In the Lorentz group case, this perturbative approach is proved to coincide with the algebraic and combinatorial approach for knot invariants defined out of the formal R-matrix and formal ribbon elements in the Quantum Lorentz Group, and its infinite dimensional unitary representations.

Infinitesimal 2-braidings and differential crossed modules

Lucio Simone Cirio[a], João Faria Martins[b]

[a]Mathematisches Institut - Universität Göttingen Bunsenstr. 3-5, 37073 Göttingen, Germany
[b]Departamento de Matemática and Centro de Matemática e Aplicações
Faculdade de Ciências e Tecnologia (Universidade Nova de Lisboa), Quinta da Torre, 2829-516 Caparica, Portugal

Abstract

We categorify the notion of an infinitesimal braiding in a linear strict symmetric monoidal category, leading to the notion of a (strict) infinitesimal 2-braiding in a linear symmetric strict monoidal 2-category. We describe the associated categorification of the 4-term relations, leading to six categorified relations. We prove that any infinitesimal 2-braiding gives rise to a flat and fake flat 2-connection in the configuration space of n particles in the complex plane, hence to a categorification of the Knizhnik-Zamolodchikov connection. We discuss infinitesimal 2-braidings in a certain monoidal 2-category naturally assigned to every differential crossed module, leading to the notion of a symmetric quasi-invariant tensor in a differential crossed module. Finally, we prove that symmetric quasi-invariant tensors exist in the differential crossed module associated to Wagemann's version of the String Lie-2-algebra. As a corollary, we obtain a more conceptual proof of the flatness of a previously constructed categorified Knizhnik-Zamolodchikov connection with values in the String Lie-2-algebra.

Keywords: Higher gauge theory, categorification, Knizhnik-Zamolodchikov equations, Zamolodchikov tetrahedron equation, infinitesimal braiding, braided monoidal 2-category, crossed module, Lie-2-algebra, 4-term relations.
2000 MSC: 16T25, 18D05 (principal); 20F36, 17B37 (secondary).

Introduction

A linear (strict) monoidal category $\mathcal{C}$ is a strict monoidal symmetric category $\mathcal{C} = (C_0, C_1, \otimes, I, B)$, with classes of objects and morphisms (C_0 and C_1, respectively), functorial strict tensor product $\otimes$ (with strict identity I) and with an involutive braiding B. The designation 'linear' comes from the fact that we suppose that, given two objects $x, y \in C_0$, the set of morphisms $\hom(x, y)$ is given a vector space structure, and the composition of morphisms is bilinear.

Given a linear strict monoidal category $\mathcal{C} = (C_0, C_1, \otimes, I, B)$, consider objects $x, y, z \in C_0$. Given a 1-morphism $f\colon x \otimes z \to x \otimes z$, we define $f^{13} = f^{13}_{x,y,z}\colon x \otimes y \otimes z \to x \otimes y \otimes z$ as being (we compose from left to right):

$$(B_{x,y} \otimes \mathrm{id}_z)(\mathrm{id}_y \otimes f)(B_{y,x} \otimes \mathrm{id}_z) = (\mathrm{id}_x \otimes B_{y,z})(f \otimes \mathrm{id}_y)(\mathrm{id}_x \otimes B_{z,y}).$$

Given $g\colon x \otimes y \to x \otimes y$ we also put $g^{12} = g^{12}_{x,y,z} = g \otimes \mathrm{id}_z$. Analogously, given $h\colon y \otimes z \to y \otimes z$ we define $\mathrm{id}_x \otimes h = h^{23}_{x,y,z} = h^{23}$.

An infinitesimal braiding in a linear strict monoidal category $\mathcal{C}$ (see [Kas, XX.4] and [Car93]) is given by a family of functorial (natural) isomorphisms $r_{x,y}\colon x \otimes y \to x \otimes y$, one for each pair of objects x and y of $\mathcal{C}$, such that:

1. For each $x, y \in C_0$ we have $B_{x,y} r_{y,x} = r_{x,y} B_{x,y}$.

Email addresses: lcirio@uni-math.gwdg.de (Lucio Simone Cirio), jn.martins@fct.unl.pt (João Faria Martins)

Link invariants from finite categorical groups and braided crossed modules

João Faria Martins and Roger Picken

Abstract

We define an invariant of tangles and framed tangles given a finite crossed module and a pair of functions, called a Reidemeister pair, satisfying natural properties. We give several examples of Reidemeister pairs derived from racks, quandles, rack and quandle cocycles, 2-crossed modules and braided crossed modules. We prove that our construction includes all rack and quandle cohomology (framed) link invariants, as well as the Eisermann invariant of knots, for which we also find a lifting by using braided crossed modules.

Introduction

In knot theory, for a knot K, the fundamental group $\pi_1(C_K)$ of the knot complement C_K, also known as the knot group, is an important invariant, which however depends only on the homotopy type of the complement of K (for which it is a complete invariant), and therefore, for example, it fails to distinguish between the square knot and the granny knot, which have homotopic, but not diffeomorphic complements. Nevertheless, a powerful knot invariant I_G can be defined from any finite group G, by counting the number of morphisms from the knot group into G. In a recent advance, Eisermann [15] constructed, from any finite group G and any $x \in G$, an invariant $E(K)$ that is closely associated to a complete invariant [36], known as the peripheral system, consisting of the knot group $\pi_1(C_K)$ and the homotopy classes of a meridian m and a longitude l. Eisermann gives examples showing that his invariant is capable of distinguishing mutant knots as well as detecting chiral (non-obversible), non-inversible and non-reversible knots (using the terminology for knot symmetries employed in [15]). Explicitly the Eisermann invariant has the form, for a knot K:

$$E(K) = \sum_{\left\{ \substack{f\colon \pi_1(C_K) \to G \\ f(m)=x} \right\}} f(l),$$

and takes values in the group algebra $\mathbb{Z}[G]$ of G.

Eisermann's invariant has in common with many other invariants that it can be calculated by summing over all the different ways of colouring knot diagrams with algebraic data. Another well-known example of such an invariant is the invariant I_G defined above, which can be calculated by counting the number of colourings of the arcs of a diagram with elements of the group G,

2010 Mathematics Subject Classification 57M25, 57M27 (primary), 18D10 (secondary)

Keywords: Knot invariant, tangle, peripheral system, quandle, rack, crossed module, categorical group, braided crossed module, 2-crossed module, non-abelian tensor product of groups

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Categorifying the $\mathfrak{sl}(2,\mathbb{C})$ Knizhnik-Zamolodchikov Connection via an Infinitesimal 2-Yang-Baxter Operator in the String Lie-2-Algebra

Lucio Simone Cirio *

Mathematics Research Unit, University of Luxembourg

6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg

lucio.cirio@uni-muenster.de

João Faria Martins

Departamento de Matemática and Centro de Matemática e Aplicações,

Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa

Quinta da Torre, 2829-516 Caparica, Portugal

jn.martins@fct.unl.pt

Abstract

We construct a flat (and fake-flat) 2-connection in the configuration space of n indistinguishable particles in the complex plane, which categorifies the $\mathfrak{sl}(2,\mathbb{C})$-Knizhnik-Zamolodchikov connection obtained from the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$. This will be done by considering the adjoint categorical representation of the string Lie 2-algebra and the notion of an infinitesimal 2-Yang-Baxter operator in a differential crossed module. Specifically, we find an infinitesimal 2-Yang-Baxter operator in the string Lie 2-algebra, proving that any (strict) categorical representation of the string Lie-2-algebra, in a chain-complex of vector spaces, yields a flat and (fake flat) 2-connection in the configuration space, categorifying the $\mathfrak{sl}(2,\mathbb{C})$-Knizhnik-Zamolodchikov connection. We will give very detailed explanation of all concepts involved, in particular discussing the relevant theory of 2-connections and their two dimensional holonomy, in the specific case of 2-groups derived from chain complexes of vector spaces.

Keywords: Higher Gauge Theory, two-dimensional holonomy, Categorification, crossed module, braid group, braided surface, configuration spaces, Knizhnik-Zamolodchikov equations, Zamolodchikov tetrahedron equation, infinitesimal braid group relations, infinitesimal relations for braid cobordisms, categorical representation.

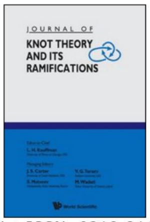
MSC2010: 16T25, 20F36 (principal); 18D05, 17B37, 53C29, 57M25, 57Q45 (secondary).

1 Introduction

Let $I = [-1, 1]$. Given a positive integer n, recall that a braid [17, 18, 37, 4], with n-strands is, by definition, a neat embedding B of the manifold $\sqcup_{i=1}^{n} I$ into $\mathbb{C} \times I$ (where $\mathbb{C}$ is the complex plane) such that the projection on the last variable (which unusually we take to be the horizontal one) is monotone. Moreover, we suppose that $B \cap (\mathbb{C} \times \{\pm 1\}) = \{1, \dots, n\} \times \{0\} \times \{\pm 1\}$. Two braids B and B' are said to be isotopic (or equivalent) if there exists a level preserving (with respect to the last variable) isotopy of $\mathbb{C} \times I$, relative to the boundary of $\mathbb{C} \times I$,

*Present address: Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Deutschland

This Issue



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CATEGORICAL GROUPS, KNOTS AND KNOTTED SURFACES

JOÃO FARIA MARTINS
Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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We define a knot invariant and a 2-knot invariant from any finite categorical group. We calculate an explicit example for the Spun Trefoil.

Keywords: Categorical group; crossed module; knots; knotted surfaces; movie moves
AMSC: 57M25, 57M27, 57Q45

Homology, Homotopy and Applications, vol. 9(1), 2007, pp.295–329

ON THE HOMOTOPY TYPE AND THE FUNDAMENTAL CROSSED COMPLEX OF THE SKELETAL FILTRATION OF A CW-COMPLEX

JOÃO FARIA MARTINS

(*communicated by Ronald Brown*)

Abstract

We prove that if M is a CW-complex, then the homotopy type of the skeletal filtration of M does not depend on the cell decomposition of M up to wedge products with n-disks D^n, when the latter are given their natural CW-decomposition with unique cells of order 0, $(n-1)$ and n, a result resembling J.H.C. Whitehead's work on simple homotopy types. From the higher homotopy van Kampen Theorem (due to R. Brown and P.J. Higgins) follows an algebraic analogue for the fundamental crossed complex $\Pi(M)$ of the skeletal filtration of M, which thus depends only on the homotopy type of M (as a space) up to free product with crossed complexes of the type $\mathcal{D}^n \doteq \Pi(D^n)$, $n \in \mathbb{N}$. This expands an old result (due to J.H.C. Whitehead) asserting that the homotopy type of $\Pi(M)$ depends only on the homotopy type of M. We use these results to define a homotopy invariant $I_{\mathcal{A}}$ of CW-complexes for each finite crossed complex $\mathcal{A}$. We interpret it in terms of the weak homotopy type of the function space $TOP\big((M,*),(|\mathcal{A}|,*)\big)$, where $|\mathcal{A}|$ is the classifying space of the crossed complex $\mathcal{A}$.

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Key words and phrases: CW-complex, skeletal filtration, crossed complex, higher homotopy van Kampen Theorem, invariants of homotopy types.

Theory and Applications of Categories, Vol. 18, No. 4, 2007, pp. 118–150.

ON YETTER'S INVARIANT AND AN EXTENSION OF THE DIJKGRAAF-WITTEN INVARIANT TO CATEGORICAL GROUPS

JOÃO FARIA MARTINS AND TIMOTHY PORTER

Abstract. We give an interpretation of Yetter's Invariant of manifolds M in terms of the homotopy type of the function space $\mathrm{TOP}(M, B(\mathcal{G}))$, where $\mathcal{G}$ is a crossed module and $B(\mathcal{G})$ is its classifying space. From this formulation, there follows that Yetter's invariant depends only on the homotopy type of M, and the weak homotopy type of the crossed module $\mathcal{G}$. We use this interpretation to define a twisting of Yetter's Invariant by cohomology classes of crossed modules, defined as cohomology classes of their classifying spaces, in the form of a state sum invariant. In particular, we obtain an extension of the Dijkgraaf-Witten Invariant of manifolds to categorical groups. The straightforward extension to crossed complexes is also considered.

1. Introduction

Let G be a finite group. In the context of Topological Gauge Field Theory, R. Dijkgraaf and E. Witten defined in [24] a 3-dimensional manifold invariant for each 3-dimensional cohomology class of G. When M is a triangulated manifold, the Dijkgraaf-Witten Invariant can be expressed in terms of a state sum model, reminiscent of Lattice Gauge Field Theory. State sum invariants of manifolds were popularised by V.G. Turaev and O.Ya. Viro, because of their construction of a non-trivial closed 3-manifold state sum invariant from quantum $6j$-symbols (see [51]), regularising the divergences of the celebrated Ponzano-Regge Model (see [41]), which is powerful enough to distinguish between manifolds which are homotopic but not homeomorphic. For categorically inclined introductions to state sum invariants of manifolds we refer the reader to [50, 5, 35].

State sum models, also known as spin foam models, appear frequently in the context of 3- and 4-dimensional Quantum Gravity, as well as Chern-Simons Theory, making them particularly interesting. See for example [3, 1] for reviews.

Let us recall the construction of the Dijkgraaf-Witten Invariant. Let G be a finite group. A 3-dimensional group cocycle is a map $\omega\colon G^3 \to U(1)$ verifying the cocycle

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Key words and phrases: Categorical Groups; Crossed Modules; Cohomology of Crossed Modules; State Sum Invariants of Manifolds; Dijkgraaf-Witten Invariant; Yetter's Invariant.

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Observables in the Turaev-Viro and Crane-Yetter models

John W. Barrett[1,a)], João Faria Martins[2] and J. Manuel García-Islas[3]
a) Electronic mail: john.barrett@nottingham.ac.uk
J. Math. Phys. 48, 093508 (2007); http://dx.doi.org/10.1063/1.2759440

Abstract

We define an invariant of graphs embedded in a 3-manifold and a partition function for 2-complexes embedded in a triangulated 4-manifold by specifying the values of variables in the Turaev-Viro and Crane-yetter state-sum models. In the case of the three-dimensional invariant, we prove a duality formula relating its Fourier transform to another invariant defined via the colored Jones polynomial. In the case of the four-dimensional partition function, we give a formula for it in terms of a regular neighborhood of the 2-complex and the signature of its complement. Some examples are computed which show that the partition function determines an invariant which can detect non locally-flat surfaces in a 4-manifold.

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Article outline:

I. INTRODUCTION II. CHAIN MAIL A. The chain-mail link B. The chain-mail invariant C. Invariance III. S^3 AND THE RELATIVISTIC INVARIANT A. Relativistic spin networks B. Graph diagrams to chain mail C. Shadow world evaluation IV. GRAPHS IN 3-MANIFOLDS A. The graph invariants on manifolds B. Proof of Theorem 1 C. Alternative representation D. Definition for arbitrary graph V. 4-MANIFOLDS A. Proof of Theorem 2 B. Examples 1. Locally flat surfaces 2. Trivial labelling

Key Topics

Manifolds

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Communications in Mathematical Physics
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Invariants of Spin Networks Embedded in Three-Manifolds

Abstract

We study the invariants of spin networks embedded in a three-dimensional manifold which are based on the path integral for **SU**(2) BF-Theory. These invariants appear naturally in Loop Quantum Gravity, and have been defined as spin-foam state sums. By using the Chain-Mail technique, we give a more general definition of these invariants, and show that the state-sum definition is a special case. This provides a rigorous proof that the state-sum invariants of spin networks are topological invariants. We derive various results about the BF-Theory spin network invariants, and we find a relation with the corresponding invariants defined from Chern-Simons Theory, i.e. the Witten-Reshetikhin-Turaev invariants. We also prove that the BF-Theory spin network invariants coincide with V. Turaev's definition of invariants of coloured graphs embedded in 3-manifolds and thick surfaces, constructed by using shadow-world evaluations. Our framework therefore provides a unified view of these invariants.

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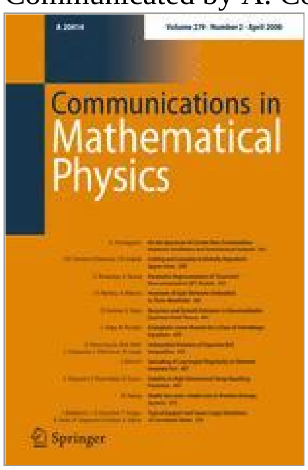

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Invariants of welded virtual knots via crossed module invariants of knotted surfaces

Louis H. Kauffman and João Faria Martins

Abstract

We define an invariant of welded virtual knots from each finite crossed module by considering crossed module invariants of ribbon knotted surfaces which are naturally associated with them. We elucidate that the invariants obtained are non-trivial by calculating explicit examples. We define welded virtual graphs and consider invariants of them defined in a similar way.

1. Introduction

Welded virtual knots were defined in [Kau99], extending the analogous construction of welded braids in [FRR97], by allowing one extra move in addition to the moves appearing in the definition of a virtual knot. This extra move preserves the (combinatorial) fundamental group of the complement, which is therefore an invariant of welded virtual knots (the knot group). Given a finite group G, one can therefore define a welded virtual knot invariant $\mathcal{H}_G$, by considering the number of morphisms from the fundamental group of the complement into G. The Wirtinger presentation of knot groups enables a quandle-type calculation of this 'counting invariant' $\mathcal{H}_G$.

Not a lot of welded virtual knot invariants are known. The aim of this paper is to introduce a new one, the 'crossed module invariant' $\mathcal{H}_\mathcal{G}$, which depends on a finite automorphic crossed module $\mathcal{G} = (E, G, \triangleright)$, in other words on a pair of groups E and G, with E abelian, and a left action of G on E by automorphisms.

The crossed module invariant $\mathcal{H}_\mathcal{G}$ reduces to the counting invariant $\mathcal{H}_G$ when $E = 0$. However, the crossed module invariant distinguishes, in some cases, between welded virtual links with the same knot group, and therefore it is strictly stronger than the counting invariant. We will assert this fact by calculating explicit examples.

Let $\mathcal{G} = (E, G, \triangleright)$ be an automorphic crossed module. Note that the counting invariant $\mathcal{H}_G$ is trivial whenever G is abelian. However, taking G to be abelian and E to be non-trivial yields a non-trivial invariant $\mathcal{H}_\mathcal{G}$, which is, as a rule, much easier to calculate than the counting invariant $\mathcal{H}_G$ where G is a generic group, and it is strong enough to tell apart some pairs of links with the same knot group. Suppose that the welded virtual link K has n components. Let $\kappa_n = \mathbb{Z}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. We will define a k_n-module $\mathrm{CM}(K)$, depending only on K, up to isomorphism and permutations of the variables $X_1, \ldots, X_n$. If G is abelian, then $\mathcal{H}_\mathcal{G}$ simply counts the number of crossed module morphisms $\mathrm{CM}(K) \to \mathcal{G}$. We prove in this paper that if K is classical then $\mathrm{CM}(K)$ coincides with the Alexander module $\mathrm{Alex}(K)$ of K. However, this is not the case if K is not classical. We will give examples of pairs of welded virtual links (K, K') with the same knot group (thus the same Alexander module) but with $\mathrm{CM}(K) \ncong \mathrm{CM}(K')$. This will happen when K and K' have the same knot group, but are distinguished by their crossed module invariants for G abelian.

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Keywords: welded virtual knots, knotted surfaces, crossed module, quandle invariants, Alexander module.

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THE FUNDAMENTAL CROSSED MODULE OF THE COMPLEMENT OF A KNOTTED SURFACE

JOÃO FARIA MARTINS

ABSTRACT. We prove that if M is a CW-complex and M^1 is its 1-skeleton, then the crossed module $\Pi_2(M, M^1)$ depends only on the homotopy type of M as a space, up to free products, in the category of crossed modules, with $\Pi_2(D^2, S^1)$. From this it follows that if $\mathcal{G}$ is a finite crossed module and M is finite, then the number of crossed module morphisms $\Pi_2(M, M^1) \to \mathcal{G}$ can be re-scaled to a homotopy invariant $I_{\mathcal{G}}(M)$, depending only on the algebraic 2-type of M. We describe an algorithm for calculating $\pi_2(M, M^{(1)})$ as a crossed module over $\pi_1(M^{(1)})$, in the case when M is the complement of a knotted surface Σ in S^4 and $M^{(1)}$ is the handlebody of a handle decomposition of M made from its 0- and 1-handles. Here, Σ is presented by a knot with bands. This in particular gives us a geometric method for calculating the algebraic 2-type of the complement of a knotted surface from a hyperbolic splitting of it. We prove in addition that the invariant $I_{\mathcal{G}}$ yields a non-trivial invariant of knotted surfaces in S^4 with good properties with regard to explicit calculations.

INTRODUCTION

Let (M, N) be a pair of based path-connected spaces. The concept of a crossed module arises from a universal description of the properties of the boundary map $\partial\colon \pi_2(M, N) \to \pi_1(N)$, together with the natural action of $\pi_1(N)$ on $\pi_2(M, N)$. These data define the crossed module $\Pi_2(M, N)$, called the "Fundamental Crossed Module of (M, N)".

Due to some strong theorems by J.H.C. Whitehead, it is possible, in principle, to calculate $\Pi_2(M, M^1)$ when M is a connected CW-complex and M^1 is its 1-skeleton. The calculability of fundamental crossed modules is, in addition, enhanced by a 2-dimensional van Kampen theorem due to R. Brown and P.J. Higgins, stating that, under mild conditions, the fundamental crossed module functor from the category of based pairs of path connected spaces to the category of crossed modules preserves colimits; see [7, 9, 5].

The crossed module $\Pi_2(M, M^1)$ determines not only $\pi_1(M)$ and $\pi_2(M)$ as a module over $\pi_1(M)$, but also it determines the k-invariant $k(M) \in H^3(\pi_1(M), \pi_2(M))$;

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Spin Foam Perturbation Theory for Three-Dimensional Quantum Gravity

Abstract

We formulate the spin foam perturbation theory for three-dimensional Euclidean Quantum Gravity with a cosmological constant. We analyse the perturbative expansion of the partition function in the dilute-gas limit and we argue that the Baez conjecture stating that the number of possible distinct topological classes of perturbative configurations is finite for the set of all triangulations of a manifold is not true. However, the conjecture is true for a special class of triangulations which are based on subdivisions of certain 3-manifold cubulations. In this case we calculate the partition function and show that the dilute-gas correction vanishes for the simplest choice of the volume operator. By slightly modifying the dilute-gas limit, we obtain a nonvanishing correction which is related to the second order perturbative correction. By assuming that the dilute-gas limit coupling constant is a function of the cosmological constant, we obtain a value for the partition function which is independent of the choice of the volume operator.

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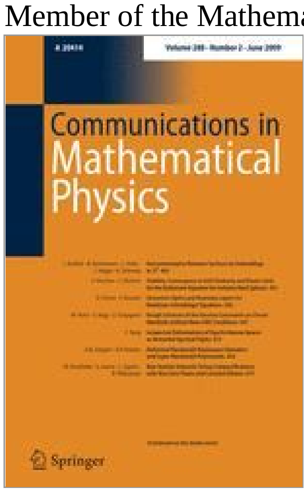
Member of the Mathematical Physics Group, University of Lisbon.



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On two-dimensional holonomy

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Abstract: We define the thin fundamental categorical group $\mathcal{P}_2(M,*)$ of a based smooth manifold $(M,*)$ as the categorical group whose objects are rank-1 homotopy classes of based loops on M and whose morphisms are rank-2 homotopy classes of homotopies between based loops on M. Here two maps are rank-n homotopic, when the rank of the differential of the homotopy between them equals n. Let $\mathcal{C}(\mathcal{G})$ be a Lie categorical group coming from a Lie crossed module $\mathcal{G}=(\partial\colon E\to G,\triangleright)$. We construct categorical holonomies, defined to be smooth morphisms $\mathcal{P}_2(M,*)\to\mathcal{C}(\mathcal{G})$, by using a notion of categorical connections, being a pair (ω,m), where ω is a connection 1-form on P, a principal G bundle over M, and m is a 2-form on P with values in the Lie algebra of E, with the pair (ω,m) satisfying suitable conditions. As a further result, we are able to define Wilson spheres in this context.

References [Enhancements On Off] (What's this?)

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The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module

João Faria Martins [a,*], Roger Picken [b]

[a] Departamento de Matemática, Faculdade de Ciências e Tecnologia (Universidade Nova de Lisboa), Quinta da Torre, 2829-516 Caparica, Portugal
[b] Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico (Universidade Técnica de Lisboa), Av. Rovisco Pais, 1049-001 Lisboa, Portugal

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A B S T R A C T

We define the thin fundamental Gray 3-groupoid $\mathcal{S}_3(M)$ of a smooth manifold M and define (by using differential geometric data) 3-dimensional holonomies, to be smooth strict Gray 3-groupoid maps $\mathcal{S}_3(M) \to \mathcal{C}(\mathcal{H})$, where $\mathcal{H}$ is a 2-crossed module of Lie groups and $\mathcal{C}(\mathcal{H})$ is the Gray 3-groupoid naturally constructed from $\mathcal{H}$. As an application, we define Wilson 3-sphere observables.

Introduction

This is the first of a series of papers on (Gray) 3-bundles and 3-dimensional holonomy (also called parallel transport), aimed at categorifying the notion of 2-bundles and non-abelian gerbes with connection, and their 2-dimensional parallel transport; see [7,48,11,28,33,37,3,50,8]. The main purpose here is to clarify the notion of 3-dimensional holonomy based on a Lie 2-crossed module, extending some of the constructions in [46–48,27], where a local 2-dimensional holonomy was defined as being a 2-functor from the fundamental 2-groupoid of a smooth manifold into a structure Lie 2-group, and the 1- and 2-gauge transformations as pseudo-natural transformations and modifications between such. Note that the category of Lie 2-groups is equivalent to the category of Lie crossed modules [19]. As analysed in [28,48], given a 2-connection on a crossed module 2-bundle, then the local 2-dimensional holonomies, together with the information coming from the transition functions (for the 2-bundle and the 2-connection) can all be patched together to give a honest global notion of a non-abelian 2-bundle 2-dimensional holonomy, independent of the chosen coordinate neighbourhoods.

To approach 3-dimensional holonomy, we will use 2-crossed modules of Lie groups [23,43,39,14], each of which naturally determines a Gray 3-groupoid with a single object (a Gray 3-group), whose definition appears, for example, in [24,36,30,31]. The first main result of this article concerns the construction of the thin fundamental (semi-strict) Gray 3-groupoid $\mathcal{S}_3(M)$

* Corresponding author.
E-mail addresses: jn.martins@fct.unl.pt (J. Faria Martins), rpicken@math.ist.utl.pt (R. Picken).

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Surface holonomy for non-abelian 2-bundles via double groupoids

João Faria Martins [a,*], Roger Picken [b]

[a] *Departamento de Matemática, Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa, Quinta da Torre, 2829-516 Caparica, Portugal*

[b] *Centro de Análise Matemática, Geometria e Sistemas Dinâmicos, Departamento de Matemática, Instituto Superior Técnico, Technical University of Lisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal*

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Abstract

In the context of non-abelian gerbes, we define a cubical version of categorical group 2-bundles with connection over a smooth manifold. We address their two-dimensional parallel transport, study its properties, and construct non-abelian Wilson surface functionals.

MSC: primary 53C29; secondary 18D05, 70S15

Keywords: Cubical set; Non-abelian gerbe; 2-Bundle; 2-Dimensional holonomy; Non-abelian integral calculus; Categorical group; Double groupoid; Higher Gauge Theory; Wilson surface

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* Corresponding author.
E-mail addresses: jn.martins@fct.unl.pt (J.F. Martins), rpicken@math.ist.utl.pt (R. Picken).

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Lie crossed modules and gauge-invariant actions for 2-BF theories

João Faria Martins *
Departamento de Matemática
Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa
Quinta da Torre, 2829-516 Caparica, Portugal
jn.martins@fct.unl.pt

Aleksandar Miković †
Departamento de Matemática
Faculdade de Engenharia e Ciências Naturais
Universidade Lusófona de Humanidades e Tecnologia,
Av do Campo Grande, 376, 1749-024 Lisboa, Portugal
amikovic@ulusofona.pt

August 3, 2010

Abstract

We generalize the BF theory action to the case of a general Lie crossed module $(\partial : H \to G, \triangleright)$, where G and H are non-abelian Lie groups. Our construction requires the existence of G-invariant non-degenerate bilinear forms on the Lie algebras of G and H and we show that there are many examples of such Lie crossed modules by using the construction of crossed modules provided by short chain complexes of vector spaces. We also generalize this construction to an arbitrary chain complex of vector spaces, of finite type. We construct two gauge-invariant actions for 2-flat and fake-flat 2-connections with auxiliary fields. The first action is of the same type as the BFCG action introduced by Girelli, Pfeiffer and Popescu for a special class of Lie crossed modules, where H is abelian. The second action is an extended BFCG action which contains an additional auxiliary field. However, these two actions are related by a field redefinition. We also construct a three-parameter deformation of the extended BFCG action, which we believe to be relevant for the construction of non-trivial invariants of knotted surfaces embedded in the four-sphere.

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THE FUNDAMENTAL 2-CROSSED COMPLEX OF A REDUCED CW-COMPLEX

JOÃO FARIA MARTINS

(*communicated by Ronald Brown*)

Abstract

We define the fundamental 2-crossed complex $\Omega^\infty(X)$ of a reduced CW-complex X from Ellis' fundamental squared complex $\rho^\infty(X)$ thereby proving that $\Omega^\infty(X)$ is totally free on the set of cells of X. This fundamental 2-crossed complex has very good properties with regard to the geometrical realisation of 2-crossed complex morphisms. After carefully discussing the homotopy theory of totally free 2-crossed complexes, we use $\Omega^\infty(X)$ to give a new proof that the homotopy category of pointed 3-types is equivalent to the homotopy category of 2-crossed modules of groups. We obtain very similar results to the ones given by Baues in the similar context of quadratic modules and quadratic chain complexes.

Introduction

A CW-complex X will be called reduced if it has a unique 0-cell, taken to be its basepoint. Let n be a positive integer. An n-type is a reduced CW-complex X, such that $\pi_i(X)$ is trivial for $i > n$. The category $\{\textbf{n-types}\}$ of n-types is defined as the category with objects the n-types and morphisms the pointed homotopy classes of pointed maps.

It is well known that the fundamental group functor yields an equivalence of categories between the category of groups and the category of 1-types, and that the fundamental crossed module of a CW-complex provides an equivalence of categories between the category of 2-types and the localisation Ho(**Xmod**), with respect to weak equivalences, of the model category **Xmod** of crossed modules; see [**16, 20**]. Note that the category of crossed modules is equivalent to the category of simplicial groups with Moore complex of length one; for a proof see [**14, 20, 23**].

The 2-crossed modules were initially defined by Conduché in [**14**], who proved that the category **2Xmod** of 2-crossed modules is equivalent to the category of simplicial

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Four-Dimensional Spin Foam Perturbation Theory

João FARIA MARTINS [†] *and Aleksandar MIKOVIĆ* [‡§]

[†] *Faculdade de Ciências e Tecnologia, Universidade Nova de Lisboa,*
Quinta da Torre, 2829-516 Caparica, Portugal
E-mail: *jn.martins@fct.unl.pt*
URL: http://ferrari.dmat.fct.unl.pt/personal/jnm/

[‡] *Departamento de Matemática, Universidade Lusófona de Humanidades e Tecnologia,*
Av do Campo Grande, 376, 1749-024 Lisboa, Portugal
E-mail: *amikovic@ulusofona.pt*

[§] *Grupo de Física Matemática da Universidade de Lisboa,*
Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal

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Abstract. We define a four-dimensional spin-foam perturbation theory for the BF-theory with a $B \wedge B$ potential term defined for a compact semi-simple Lie group G on a compact orientable 4-manifold M. This is done by using the formal spin foam perturbative series coming from the spin-foam generating functional. We then regularize the terms in the perturbative series by passing to the category of representations of the quantum group $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is the Lie algebra of G and q is a root of unity. The Chain–Mail formalism can be used to calculate the perturbative terms when the vector space of intertwiners $\Lambda \otimes \Lambda \to A$, where A is the adjoint representation of $\mathfrak{g}$, is 1-dimensional for each irrep Λ. We calculate the partition function Z in the dilute-gas limit for a special class of triangulations of restricted local complexity, which we conjecture to exist on any 4-manifold M. We prove that the first-order perturbative contribution vanishes for finite triangulations, so that we define a dilute-gas limit by using the second-order contribution. We show that Z is an analytic continuation of the Crane–Yetter partition function. Furthermore, we relate Z to the partition function for the $F \wedge F$ theory.

Key words: spin foam models; BF-theory; spin networks; dilute-gas limit; Crane–Yetter invariant; spin-foam perturbation theory

2010 Mathematics Subject Classification: 81T25; 81T45; 57R56

1 Introduction

Spin foam models are state-sum representations of the path integrals for BF theories on simplicial complexes. Spin foam models are used to define topological quantum field theories and quantum gravity theories, see [1]. However, there are also perturbed BF theories in various dimensions, whose potential terms are powers of the B field, see [10]. The corresponding spin-foam perturbation theory generating functional was formulated in [10], but further progress was hindered by the lack of the regularization procedure for the corresponding perturbative expansion and the problem of implementation of the triangulation independence.

The problem of implementation of the triangulation independence for general spin foam perturbation theory was studied in [2], and a solution was proposed, in the form of calculating the perturbation series in a special limit. This limit was called the dilute-gas limit, and it was given by $\lambda \to 0$, $N \to \infty$, such that $g = \lambda N$ is a fixed constant, where λ is the perturbation theory parameter, also called the coupling constant, N is the number of d-simplices in a simplical decomposition of a d-dimensional compact manifold M and g is the effective perturbation

Differential Geometry and its Applications

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Categorifying the Knizhnik–Zamolodchikov connection

Lucio Simone Cirio[a, 1], , João Faria Martins[b],

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Abstract

In the context of higher gauge theory, we construct a flat and fake flat 2-connection, in the configuration space of n particles in the complex plane, categorifying the Knizhnik–Zamolodchikov connection. To this end, we define the differential crossed module of horizontal 2-chord diagrams, categorifying the Lie algebra of horizontal chord diagrams in a set of n parallel copies of the interval. This therefore yields a categorification of the 4-term relation. We carefully discuss the representation theory of differential crossed modules in chain-complexes of vector spaces, which makes it possible to formulate the notion of an infinitesimal 2-R matrix in a differential crossed module.

MSC

primary, 16T25, 20F36; secondary, 18D05, 17B37, 53C29, 57Q45

Keywords

Higher gauge theory; Braided surface; Two-dimensional holonomy; Chord diagrams; Infinitesimal braiding; 4-term relation; Differential crossed module; Knizhnik–Zamolodchikov equations; Categorical representation

Corresponding author.

1 Present address: Mathematics Research Unit, University of Luxembourg, 6, rue Richard Coudenhove-Kalergi, L-1359 Luxembourg, Luxembourg.

Pointed homotopy and pointed lax homotopy of 2-crossed module maps

Björn Gohla
Centro de Matemática da Universidade do Porto, Departamento de Matemática,
Faculdade de Ciências da Universidade do Porto,
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal
b.gohla@gmx.de

João Faria Martins
Departamento de Matemática and Centro de Matemática e Aplicações,
Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa,
Quinta da Torre, 2829-516 Caparica, Portugal
jn.martins@fct.unl.pt

February 1, 2014

Abstract

We address the (pointed) homotopy theory of 2-crossed modules (of groups), which are known to faithfully represent Gray 3-groupoids [31], with a single object, and also connected homotopy 3-types [15]. The homotopy relation between 2-crossed module maps will be defined in a similar way to Crans' 1-transfors between strict Gray functors [16], however being pointed, thus this corresponds to Baues' homotopy relation between quadratic module maps, treated in [4]. Despite the fact that this homotopy relation between 2-crossed module morphisms is not, in general, an equivalence relation, we prove that if $\mathcal{A}$ and $\mathcal{A}'$ are 2-crossed modules, with the underlying group F of $\mathcal{A}$ being free (in short $\mathcal{A}$ is free up to order one), then homotopy between 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$ yields, in this case, an equivalence relation. Furthermore, if a chosen basis B is specified for F, then we can define a 2-groupoid $\mathrm{HOM}_B(\mathcal{A}, \mathcal{A}')$ of 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, homotopies connecting them, and 2-fold homotopies between homotopies, where the latter correspond to (pointed) Crans' 2-transfors between 1-transfors.

We define a partial resolution $Q^1(\mathcal{A})$, for a 2-crossed module $\mathcal{A}$, whose underlying group is free, with a chosen basis, together with a projection map $\mathrm{proj}\colon Q^1(\mathcal{A}) \to \mathcal{A}$, defining isomorphisms at the level of 2-crossed module homotopy groups. This resolution (proven to be part of a comonad in [25]) leads to a weaker notion of homotopies (lax homotopies) between 2-crossed module maps, which we fully develop and describe. In particular, given 2-crossed modules $\mathcal{A}$ and $\mathcal{A}'$, there exists a 2-groupoid $\mathcal{HOM}_{\mathrm{LAX}}(\mathcal{A}, \mathcal{A}')$ of (strict) 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, and their lax homotopies and lax 2-fold homotopies, leading to the question of whether the category of 2-crossed modules and strict maps can be enriched over the monoidal category Gray.

The associated notion of a (strict) 2-crossed module map $f\colon \mathcal{A} \to \mathcal{A}'$ to be a lax homotopy equivalence has the two-of-three property, and it is closed under retracts. This discussion leads to the issue of whether there exists a model category structure in the category of 2-crossed modules (and strict maps) where weak equivalences correspond to lax homotopy equivalences, and any free up to order one 2-crossed module is cofibrant.

Keywords Crossed module, 2-crossed module, quadratic module, homotopy 3-type, tricategory, Gray category, Peiffer lifting, simplicial group.
2000 Mathematics Subject Classification: 18D05 and 18D20 (primary), 55Q15 (secondary).

1 Introduction and simplicial group background / context

Let $\mathcal{G} = \big(G_n, d_i^n, s_i^n; i \in \{0, 1, \dots, n\}, n = 0, 1, 2, \dots\big)$ be a simplicial group; [43, 26, 35, 17]. As usual, see for example [38, 24], we say that $\mathcal{G}$ is free if each group G_n of n-simplices is a free group, with a chosen basis,

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Pointed homotopy and pointed lax homotopy of 2-crossed module maps

Björn Gohla[a], , João Faria Martins[b],

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Communicated by Ross Street

doi:10.1016/j.aim.2013.08.020

Abstract

We address the (pointed) homotopy theory of 2-crossed modules (of groups), which are known to faithfully represent Gray 3-groupoids, with a single object, and also connected homotopy 3-types. The homotopy relation between 2-crossed module maps will be defined in a similar way to Crans' 1-transfors between strict Gray functors, however being pointed, thus this corresponds to Baues' homotopy relation between quadratic module maps. Despite the fact that this homotopy relation between 2-crossed module morphisms is not, in general, an equivalence relation, we prove that if $\mathcal{A}$ and $\mathcal{A}'$ are 2-crossed modules, with the underlying group F of $\mathcal{A}$ being free (in short $\mathcal{A}$ is free up to order one), then homotopy between 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$ yields, in this case, an equivalence relation. Furthermore, if a chosen basis B is specified for F, then we can define a 2-groupoid $\mathrm{HOM}_B(\mathcal{A}, \mathcal{A}')$ of 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, homotopies connecting them, and 2-fold homotopies between homotopies, where the latter correspond to (pointed) Crans' 2-transfors between 1-transfors.

We define a partial resolution $Q^1(\mathcal{A})$, for a 2-crossed module $\mathcal{A}$, whose underlying group is free, with a canonical chosen basis, together with a projection map $\mathrm{proj}: Q^1(\mathcal{A}) \to \mathcal{A}$, defining isomorphisms at the level of 2-crossed module homotopy groups. This resolution (which is part of a comonad) leads to a weaker notion of homotopy (lax homotopy) between 2-crossed module maps, which we fully develop and describe. In particular, given 2-crossed modules $\mathcal{A}$ and $\mathcal{A}'$, there exists a 2-groupoid $\mathrm{HOM}_{\mathrm{LAX}}(\mathcal{A}, \mathcal{A}')$ of (strict) 2-crossed module maps $\mathcal{A} \to \mathcal{A}'$, and their lax homotopies and lax 2-fold homotopies, leading to the question of whether the category of 2-crossed modules and strict maps can be enriched over the monoidal category *Gray*.

The associated notion of a (strict) 2-crossed module map $f: \mathcal{A} \to \mathcal{A}'$ to be a lax homotopy equivalence has the two-of-three property, and it is closed under retracts. This discussion leads to the issue of whether there exists a model category structure in the category of 2-crossed modules (and strict maps) where weak equivalences correspond to lax homotopy equivalences, and any free up to order one 2-crossed module is cofibrant.

MSC

18D05; 18D20; 55Q15

Keywords

Crossed module; 2-crossed module; Quadratic module; Homotopy 3-type; Tricategory; Gray category; Peiffer lifting; Simplicial group

Corresponding author.

João Faria Martins

Mathematics Department

Faculty of Sciences and Technology

Universidade Nova de Lisboa.
Quinta da Torre
2829-516 Caparica
Portugal

jn.martins -at- fct dot unl dot pt

I am an assistant professor at the Mathematics Department of the Faculty of Sciences and Technology of Universidade Nova de Lisboa, in Lisbon, Portugal.

I am also a member of the research unit CMA (Centro de Matemática e Aplicações da Universidade Nova de Lisboa), and a collaborator at the research unit Grupo de Física Matemática da Universidade de Lisboa.

Previously I was an assistant researcher at the Centro de Matemática da Universidade do Porto, in Oporto, Portugal. I was offered that position under the programme Ciência 2007 of Fundação para a Ciência e Tecnologia.

Before that I was a postdoctoral fellow at the Mathematics Department of Instituto Superior Técnico, in Lisbon, Portugal, affiliated with the Center for Mathematical Analysis, Geometry, and Dynamical Systems, and financially supported by Fundação para a Ciência e Tecnologia.

I did my PhD at the School of Mathematical Sciences of Nottingham University. My supervisor was John W. Barrett.

Here are my pages at Mathematical Reviews, scholar.google; inspire (hep-th); ORCID; ResearcherID.

Official page at the Mathematics Department of the Faculty of Sciences and Technology of Universidade Nova de Lisboa.

Research Interests:

-Algebraic and Geometric Topology in dimensions 3 and 4.
-Higher categories.
-Quantum groups and low dimensional topology.
-Algebraic models for homotopy types.
-Differential geometry (higher Lie and gauge theory).
-Mathematical aspects of Chern-Simons Theory and BF-theory.

Publications:

-Pointed homotopy and pointed lax homotopy of 2-crossed module maps, with Björn Gohla, Advances in Mathematics Volume 248, 25 November 2013, Pages 986–1049. Preliminary version. Very preliminary version.

-Spin Foam State Sums and Chern-Simons Theory, with Aleksandar Miković. Chern-Simons gauge theory: 20 years after, 277–284, AMS/IP Stud. Adv. Math., 50, Amer. Math. Soc., Providence, RI, 2011. Preliminary version.

-Categorifying the Knizhnik–Zamolodchikov connection, with Lucio Simone Cirio, Differential Geometry and its Applications, Volume 30, Issue 3, June 2012, Pages 238–261. Preliminary version.

-Four-Dimensional Spin Foam Perturbation Theory, with Aleksandar Miković. Symmetry, Integrability and Geometry: Methods and Applications (SIGMA). 7 (2011), 094, 22 pages.

-The fundamental 2-crossed complex of a reduced CW-complex. *Homology, Homotopy and Applications, Vol. 13 (2011), No. 2, pp.129-157.*

-Lie crossed modules and gauge-invariant actions for 2-BF theories, with Aleksandar Miković. *Advances in Theoretical and Mathematical Physics, Volume 15, Number 4 (August 2011) p.1059-1084. Preliminary version*

-Surface Holonomy for Non-Abelian 2-Bundles via Double Groupoids, with Roger Picken. *Advances in Mathematics Volume 226, Issue 4, 1 March 2011, Pages 3309-3366. Preliminary version. Very preliminary version.*

-The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module, with Roger Picken. *Differential Geometry and its Applications Volume 29, Issue 2, March 2011, Pages 179-206. Preliminary version. Very preliminary version.*

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-The Fundamental Crossed Module of the Complement of a Knotted Surface. *Transactions of the American Mathematical Society.* 361 (2009), 4593-4630.

-Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, with Louis H. Kauffman. *Compositio Mathematica*. Volume 144, Issue 04, July 2008, pp 1046-1080.

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-Observables in the Turaev-Viro and Crane-Yetter models, with John W. Barrett and J. Manuel Garcia Islas. *Journal of Mathematical Physics*, **48**, 093508 (2007) (*18 pages*).

-On Yetter's Invariant and an Extension of the Dijkgraaf-Witten Invariant to Categorical Groups, with Timothy Porter. *Theory and Application of Categories*, Vol. 18, 2007, No. 4, pp 118-150.

-On the Homotopy Type and the Fundamental Crossed Complex of the Skeletal Filtration of a CW-Complex. *Homology Homotopy and Applications*, Vol. 9 (2007), No. 1, pp.295-329.

-Categorical Groups, Knots and Knotted Surfaces, J. Knot Theory Ramifications 16 (2007), no 9, 1181-1217.

Preprints (comments are very welcome):

-Categorifying the $sl(2,C)$ Knizhnik-Zamolodchikov Connection via an Infinitesimal 2-Yang-Baxter Operator in the String Lie-2-Algebra, with Lucio Simone Cirio.

-Link invariants from finite categorical groups and braided crossed modules, with Roger Picken.

-Infinitesimal 2-braidings and differential crossed modules with Lucio Simone Cirio.

PhD Thesis:

-Quantum Topology and the Lorentz Group.

Thesis related publications

-Knot Theory with the Lorentz Group, FUNDAMENTA MATHEMATICAE 188 (2005), 59-93. (Special volume based on papers submitted by the participants of the Second International Conference “Knots in Poland 2003”).

-On the Analytic Properties of the z-coloured Jones Polynomial, J. Knot Theory Ramifications 14 (2005), no. 4, 435--466.

Other work (comments are very welcome):

-On 2-Dimensional Homotopy Invariants of Complements of Knotted Surfaces. This is an archaic version of "The Fundamental Crossed Module of the Complement of a Knotted Surface", using movies with bands and spanners rather than hyperbolic splittings of knotted surfaces in the construction. May still be of interest.

Conferences / workshops (past and future):

- XVIIIth Oporto Meeting on Geometry, Topology and Physics (July 8-12, 2009) at the Oporto University Main Theme: Symplectic and Poisson Geometry

-Higher Gauge Theory, TQFT and Quantum Gravity, Lisbon, 10-13 February, 2011 (Workshop), 7-13 February, 2011 (School)

- Recent Advances in Topological Quantum Field Theory (TQFT-2012) A one-week workshop on Topological Quantum Field Theory and Quantum Gravity, to be held at the University of Lisbon on September 10–14, 2012

Talk slides (comments are very welcome):

Categorifying the Knizhnik-Zamolodchikov Connections 3Quantum: Algebra Geometry Information, Tallinn 10-13 July 2012 , session: Geometry and topology in quantum gravity