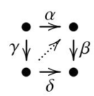
Pensieve header: An implementation of the partial quadratic signature formalism for tangles; with Jessica Liu.

230109 **Def.** Given a v.s. V, a Partial Quadratic (PQ) Q on V is a symmetric bilinear form Q on a subspace $\mathcal{D}(Q) \subset V$. For $U \subset \mathcal{D}(Q)$, denote $\operatorname{ann}_Q(U) \coloneqq \{v \in \mathcal{D}(Q) : Q(U, v) = 0\}$ and $\operatorname{rad} Q \coloneqq \operatorname{ann}_Q(\mathcal{D}(Q))$.
Def. $Q_1 + Q_2$ is with $\mathcal{D}(Q_1 + Q_2) = \mathcal{D}(Q_1) \cap \mathcal{D}(Q_2)$.
Def. Given a linear $\psi\colon V \to W$ and a PQ Q on W, the pullback is $(\psi^*Q)(v_1, v_2) = Q(\psi v_1, \psi v_2)$ with $\mathcal{D}(\psi^*Q) = \phi^{-1}(\mathcal{D}(Q))$.
Def. Given $\phi\colon V \to W$ and a PQ Q on V the pushforward $\phi_* Q$ is with $\mathcal{D}(\phi_*Q) = \phi(\operatorname{ann}_Q(\operatorname{rad} Q|_{\ker\phi}))$ and $(\phi_*Q)(w_1, w_2) = Q(v_1, v_2)$, where v_i are s.t. $\phi(v_i) = w_i$ and $Q(v_i, \operatorname{rad} Q|_{\ker\phi}) = 0$.

Thm(?). ψ^* and ϕ_* are well-defined and functorial, and if $\alpha /\!\!/ \beta = \gamma /\!\!/ \delta$, then $\gamma^* /\!\!/ \alpha_* = \delta_* /\!\!/ \beta^*$. ψ^* is additive but ϕ_* isn't.

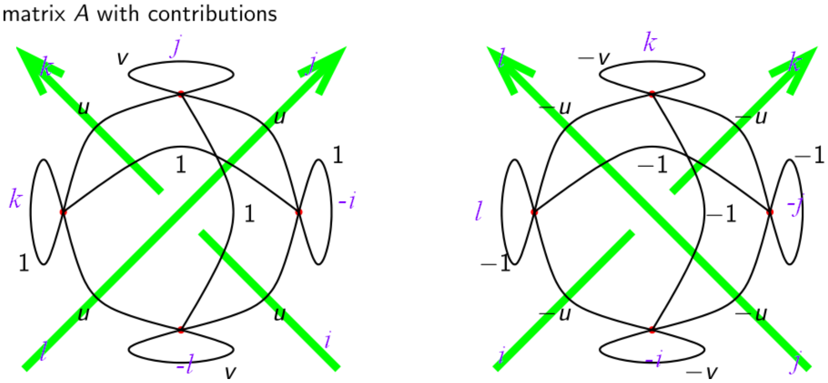


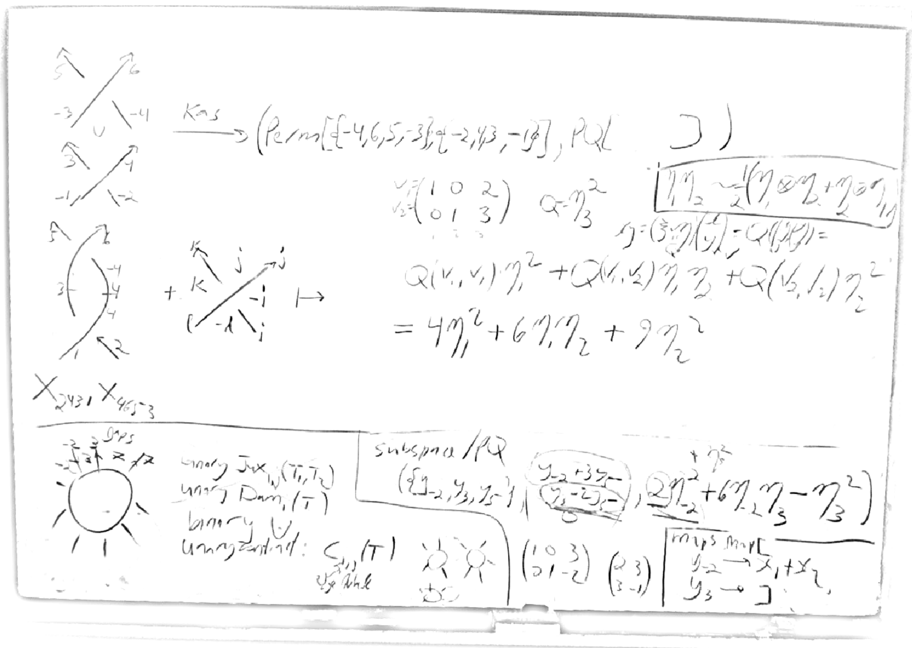
Thm(?). Over $\mathbb{R}$, given $\phi\colon V \to W$ and PQs Q on V and C on W,

$$\operatorname{sign}_V(Q + \phi^*C) = \operatorname{sign}_{\ker\phi}(\iota^*Q) + \operatorname{sign}_W(C + \phi_*Q).$$

221228 **Missing.** A fully defined theory of pushing forward Gaussians (better with determinants and signatures).

For a knot K and a complex unit ω set $u = \Re(\omega^{1/2})$, $v = \Re(\omega)$, make an $F \times F$ matrix A with contributions





In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\LiuJ"];
<< KnotTheory`
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

In[∘]:=
```
Kas[X[i_, j_, k_, l_]] := If[PositiveQ@X[i, j, k, l],
  Kas[Perm[{-i, j, k, -l}], PQ[Subspace[{y-i, yj, yk, y-l}, {y-i, yj, yk, y-l}],
    η-i^2 + 2 u η-i ηj + v ηj^2 + 2 η-i ηk + 2 u ηj ηk + ηk^2 + 2 u η-i η-l + 2 ηj η-l + 2 u ηk η-l + v η-l^2]],
  Kas[Perm[{-i, -j, k, l}], PQ[Subspace[{y-j, yk, yl, y-i}, {y-j, yk, yl, y-i}],
    -v η-i^2 - 2 u η-i η-j - η-j^2 - 2 η-i ηk - 2 u η-j ηk - v ηk^2 - 2 u η-i ηl - 2 η-j ηl - 2 u ηk ηl - ηl^2]]
 ]
```

In[∘]:=
```
CF[Subspace[{}, {0 ...}]] := Subspace[{}, {}];
CF[Subspace[vs_, {}]] := Subspace[Sort[vs], {}];
CF[Subspace[vs_, gens_]] := Module[{cvs = Sort[vs]},
  Subspace[cvs,
   DeleteCases[RowReduce[Table[Coefficient[g, v], {g, gens}, {v, cvs}]].cvs, 0]
  ]]
```

In[∘]:=
```
CF[Subspace[{y, z, x, w}, {x + y, x - y + z, x + 2 y + w}]]
```

Out[∘]=

$$\text{Subspace}\left[\{w, x, y, z\}, \left\{w + \frac{z}{2}, x + \frac{z}{2}, y - \frac{z}{2}\right\}\right]$$

In[∘]:=
```
Eval[Q_, v_, w_] := Expand[Q v w / 2] //. {η_i_ y_i_ :> 1, η_i_^2 y_i_^2 :> 2} /. (η | y)_ → 0;
Eval[ϕ_, v_] := Expand[ϕ v] /. η_i_ y_j_ :> If[i == j, 1, 0];
```

In[∘]:=
```
Eval[u η1^2 + v η1 η2, y1 + y2, y1 + y2]
```

Out[∘]=

$u + v$

In[∘]:=
```
Pivot[v_Plus] := v[[1]]; Pivot[v_] := v;
y_i_^* := η_i; η_i_^* := y_i; (vs_List)^* := Table[v^*, {v, vs}];
```

In[∘]:=
```
CF[PQ[sub_Subspace, Q_]] := Module[{csub, cvs, cgens},
  {cvs, cgens} = List @@ (csub = CF[sub]);
  PQ[csub, Sum[Eval[Q, v, w] Pivot[v]^* Pivot[w]^*, {v, cgens}, {w, cgens}]]
 ]
```

In[∘]:=
```
CF[PQ[Subspace[{y1, y2, y3}, {y1 + 2 y3, y2 + 3 y3}], η3^2]]
```

Out[∘]=

$PQ[Subspace[\{y_1, y_2, y_3\}, \{y_1 + 2 y_3, y_2 + 3 y_3\}], 4 \eta_1^2 + 12 \eta_1 \eta_2 + 9 \eta_2^2]$

In[∘]:=
```
Eval[η3^2, y1 + 2 y3, y2 + 3 y3]
```

Out[∘]=

6

In[∘]:=
```
Eval[4 η1^2 + 12 η1 η2 + 9 η2^2, y1 + 2 y3, y2 + 3 y3]
```

Out[∘]=

6

In[∘]:=
```
Eval[4 η1^2 + 12 η1 η2 + 9 η2^2, y1, y2]
```

Out[∘]=

6

In[∘]:=
```
Eval[12 η1 η2, y1, y2]
```

Out[∘]=

6

In[∘]:=
```
Perp[Subspace[vs_, gens_]] := Module[{pp},
  pp = Complement[vs, Pivot /@ gens]^*;
  CF@Subspace[vs^*,
    Table[p - Sum[Coefficient[g, p^*] Pivot[g]^*, {g, gens}], {p, pp}]
   ]
 ]
```

In[∘]:=
```
Perp@Subspace[{y1, y2, y3}, {y1 - y2}]
```

Out[∘]=

$Subspace[\{\eta_1, \eta_2, \eta_3\}, \{\eta_1 + \eta_2, \eta_3\}]$

In[∘]:=
```
Perp@Perp@Subspace[{y1, y2, y3}, {y1 - y2}]
```
Out[∘]=

$\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 - y_2\}]$

In[∘]:=
```
Id[vs_] := LT[vs, vs, Table[v → v, {v, vs}]]
```

In[∘]:=
```
Id[{y1, y2}]
```
Out[∘]=

$\text{LT}[\{y_1, y_2\}, \{y_1, y_2\}, \{y_1 \to y_1, y_2 \to y_2\}]$

In[∘]:=
```
LT[dom_, ran_, rs_]*[Subspace[ran_, gens_]] := Perp@CF@Subspace[dom*, Table[
    Sum[Eval[p, v /. rs] v*, {v, dom}],
    {p, Perp[Subspace[ran, gens]][[2]]}
  ]]
```

In[∘]:=
```
LT[{y-1, y-2, y-3}, {y1, y2, y3}, {y-1 → y1 + 2 y3, y-2 → 2 y2 - y3, y-3 → y3}]*[
 Subspace[{y1, y2, y3}, {y1 - y2} ]]
```
Out[∘]=

$$\text{Subspace}\left[\{y_{-3}, y_{-2}, y_{-1}\}, \left\{y_{-3} + \frac{y_{-2}}{5} - \frac{2\,y_{-1}}{5}\right\}\right]$$

In[∘]:=
```
LT[dom_, ran_, rs_]*[PQ[sub_, Q_]] := CF@PQ[
   LT[dom, ran, rs]*[sub],
   Sum[Eval[Q, v1 /. rs, v2 /. rs] v1* v2*, {v1, dom}, {v2, dom}]
  ]
```

In[∘]:=
```
Id[{y1, y2, y3}]*[PQ[Subspace[{y1, y2, y3}, {y1 + 2 y3, y2 + 3 y3}], 4 η1^2 + 12 η1 η2 + 9 η2^2]]
```
Out[∘]=

$$\text{PQ}\left[\text{Subspace}[\{y_1, y_2, y_3\}, \{y_1 + 2\,y_3, y_2 + 3\,y_3\}], 4\,\eta_1^2 + 12\,\eta_1\,\eta_2 + 9\,\eta_2^2\right]$$

In[∘]:=
```
LT[{y-1, y-2, y-3}, {y1, y2, y3}, {y-1 → y1 + 2 y3, y-2 → 2 y2 - y3, y-3 → y3}]*[
 PQ[Subspace[{y1, y2, y3}, {y1 + 2 y3, y2 + 3 y3}], 4 η1^2 + 12 η1 η2 + 9 η2^2]]
```
Out[∘]=

$$\text{PQ}\left[\text{Subspace}\left[\{y_{-3}, y_{-2}, y_{-1}\}, \left\{y_{-3} + \frac{y_{-2}}{7}, y_{-1}\right\}\right], \frac{36\,\eta_{-3}^2}{49} + \frac{24}{7}\,\eta_{-3}\,\eta_{-1} + 4\,\eta_{-1}^2\right]$$

In[∘]:=

$$\text{Eval}\left[\frac{36\,\eta_{-3}^2}{49} + \frac{24}{7}\,\eta_{-3}\,\eta_{-1} + 4\,\eta_{-1}^2, y_{-3} + \frac{y_{-2}}{7}, y_{-3} + \frac{y_{-2}}{7}\right]$$

Out[∘]=

$$\frac{36}{49}$$

In[•]:=
```
Eval[4 η₁^2 + 12 η₁ η₂ + 9 η₂^2,
  y₋₃ + y₋₂/7 /. {y₋₁ → y₁ + 2 y₃, y₋₂ → 2 y₂ - y₃, y₋₃ → y₃},
  y₋₃ + y₋₂/7 /. {y₋₁ → y₁ + 2 y₃, y₋₂ → 2 y₂ - y₃, y₋₃ → y₃}]
```

Out[•]=

$$\frac{36}{49}$$

```
Ann_PQ[𝒟_Subspace,Q_][U : Subspace[vs_, gens_]] /; 𝒟⟦1⟧ === vs := 𝒟 ⋂ Perp[
   CF@Subspace[vs*, Table[Eval[Q, g], {g, gens}]]
  ]
```