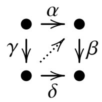
Def. Given a v.s. V, a Partial Quadratic (PQ) Q on V is a symmetric bilinear form Q on a subspace $\mathcal{D}(Q) \subset V$. For $U \subset \mathcal{D}(Q)$, denote $\operatorname{ann}_Q(U) := \{v \in \mathcal{D}(Q) : Q(U, v) = 0\}$.

Def. $Q_1 + Q_2$ is with $\mathcal{D}(Q_1 + Q_2) = \mathcal{D}(Q_1) \cap \mathcal{D}(Q_2)$.

Def. Given a linear $\psi\colon V \to W$ and a PQ Q on W, the pullback is $(\psi^*Q)(v_1, v_2) = Q(\psi v_1, \psi v_2)$ with $\mathcal{D}(\psi^*Q) = \phi^{-1}(\mathcal{D}(Q))$.

Def. Given $\phi\colon V \to W$ and a PQ Q on V the pushforward ϕ_*Q is with $\mathcal{D}(\phi_*Q) = \phi(\operatorname{ann}_Q(\mathcal{D}(Q) \cap \ker\phi))$ and $(\phi_*Q)(w_1, w_2) = Q(v_1, v_2)$, where v_i are s.t. $\phi(v_i) = w_i$ and $Q(v_i, \operatorname{rad} Q|_{\ker\phi}) = 0$.

Thm(?). ψ^* and ϕ_* are well-defined and functorial, and if $\alpha /\!\!/ \beta = \gamma /\!\!/ \delta$, then $\gamma^* /\!\!/ \alpha_* = \delta_* /\!\!/ \beta^*$. ψ^* is additive but ϕ_* isn't.



Thm(?). Over $\mathbb{R}$, given $\phi\colon V \to W$ and PQs Q on V and C on W,

$$\operatorname{sign}_V(Q + \phi^*C) = \operatorname{sign}_{\ker\phi}(\iota^*Q) + \operatorname{sign}_W(C + \phi_*Q).$$

```
Kas[X[i_, j_, k_, l_]] :=
 If[PositiveQ@X[i, j, k, l],
  Kas[Perm[{-i, j, k, -l}],
   PQ[Subspace[{y_{-i}, y_j, y_k, y_{-l}},
     {y_{-i}, y_j, y_k, y_{-l}}],
     1/2 (η_{-i}^2 + 2 u η_{-i} η_j + v η_j^2 + 2 η_{-i} η_k + 2 u η_j η_k +
       η_k^2 + 2 u η_{-i} η_{-l} + 2 η_j η_{-l} + 2 u η_k η_{-l} + v η_{-l}^2)]],
  Kas[Perm[{-i, -j, k, l}],
   PQ[Subspace[{y_{-j}, y_k, y_l, y_{-i}},
     {y_{-j}, y_k, y_l, y_{-i}}],
     1/2 (-v η_{-i}^2 - 2 u η_{-i} η_{-j} - η_{-j}^2 - 2 η_{-i} η_k -
       2 u η_{-j} η_k - v η_k^2 - 2 u η_{-i} η_l - 2 η_{-j} η_l -
       2 u η_k η_l - η_l^2)]]
 ]
```

```
CF[Subspace[{}, {0 ...}]] := Subspace[{}, {}];
CF[Subspace[vs_, {}]] := Subspace[Sort[vs], {}];
CF[Subspace[vs_, gens_]] :=
 Module[{cvs = Sort[vs]},
  Subspace[cvs,
   DeleteCases[
    RowReduce[Table[Coefficient[g, v],
      {g, gens}, {v, cvs}]].cvs, 0]
  ]]
```

```
Eval[Q_, v_, w_] :=
  Expand[Q v w] //. {η_i_ y_i_ :> 1, η_i_^2 y_i_^2 :> 2} /.
   (η | y)_ → 0;
Eval[ϕ_, v_] :=
  Expand[ϕ v] /. {η_i_ y_i_ :> 1, η_i_^2 y_i_ :> 2 η_i} /.
   y_ → 0;
```

```
Pivot[v_Plus] := v⟦1⟧; Pivot[v_] := v;
y_i_^* := η_i; η_i_^* := y_i;
(vs_List)^* := Table[v^*, {v, vs}];
```

```
CF[PQ[sub_Subspace, Q_]] :=
 Module[{csub, cvs, cgens},
  {cvs, cgens} = List @@ (csub = CF[sub]);
  PQ[csub, Sum[Eval[Q, v, w] Pivot[v]^* Pivot[w]^* / 2,
    {v, cgens}, {w, cgens}]]
 ]
```

```
Perp[Subsp_] := Module[{pp, cvs, cgens},
  {cvs, cgens} = List @@ CF@Subsp;
  pp = Complement[cvs, Pivot /@ cgens]^*;
  CF@Subspace[cvs^*,
   Table[p - Sum[Coefficient[g, p^*] Pivot[g]^*,
      {g, cgens}], {p, pp}]
   ]
 ]
```

```
Id[vs_] := LT[vs, vs, Table[v → v, {v, vs}]]
```

```
LT[dom_, ran_, rs_]^*[Subspace[ran_, gens_]] :=
 Perp@CF@Subspace[dom^*, Table[
    Sum[Eval[p, v /. rs] v^*, {v, dom}],
    {p, Perp[Subspace[ran, gens]]⟦2⟧}
   ]]
```

```
LT[dom_, ran_, rs_]_*[Subspace[dom_, gens_]] :=
 CF@Subspace[ran, gens /. rs]
```

```
LT[dom_, ran_, rs_]^*[PQ[sub_, Q_]] := CF@PQ[
   LT[dom, ran, rs]^*[sub],
   Sum[Eval[Q, v1 /. rs, v2 /. rs] v1^* v2^* / 2,
    {v1, dom}, {v2, dom}]
  ]
```

```
Subspace /: Subspace[vs_, gen1s_] +
   Subspace[vs_, gen2s_] :=
  CF@Subspace[vs, gen1s ⋃ gen2s]
Subspace /: sub1_Subspace ⋂ sub2_Subspace :=
  Perp[Perp[sub1] + Perp[sub2]]
```

```
Subspace /: v_ ∈ Subspace[vs_, gens_] :=
  (Subspace[vs, gens] ⋂ Subspace[vs, {v}])⟦2⟧ =!=
   {}
```

```
Ann_{PQ[𝒟_Subspace, Q_]}[Subspace[vs_, gens_]] :=
  𝒟 ⋂
   Perp@Subspace[vs^*, Table[Eval[Q, g], {g, gens}]]
```

```
Ker[LT[{}, _, _]] := Subspace[{}, {}];
Ker[LT[dom_, {}, _]] := Subspace[dom, dom];
Ker[LT[dom_, ran_, rs_]] := Module[{ns},
   ns = NullSpace[Table[Coefficient[d /. rs, r],
      {r, ran}, {d, dom}]];
   If[Length@ns > 0, CF@Subspace[dom, ns.dom],
    Subspace[dom, {}]]
  ]
```

```
Section[LT[dom_, ran, rs_]]
Section[LT[Subspace[___], ran, rs_]]
```