COMPUTING SIGNATURES

Let U, V be finite dimensional vector spaces and let $A : U \to U^*$, $B : V \to U^*$, $C : V \to V^*$ be such that A and C are self-adjoint. Define the self-adjoint transformation $M(A,B,C)$ on $U \oplus V$ by $M(u+v) := A(u) + B^*(u) + B(v) + C(v)$. That is, let $M(A,B,C)$ be given by the matrix

$0 \to \ker A \to U \to U/\ker A \to 0$

$M(A,B,C) :=$

	U	V
U^*	A	B
V^*	B^*	C

Now pick isomorphisms

$$U \xrightarrow{f} (U/\ker A) \oplus (\ker A \cap \ker B^*) \oplus (\ker A/\ker B^*)$$
$$V \xrightarrow{g} (V/B^{-1}(\mathrm{im}A)) \oplus B^{-1}(\mathrm{im}A)$$

that make the corresponding short exact sequences split. That is, we require:

- $U \xrightarrow{f} U/\ker A$ is the natural projection
- $(\ker A \cap \ker B^*) \xrightarrow{f^{-1}} U$ is the natural inclusion
- $\ker A \xrightarrow{f} \ker A/\ker B^*$ is the natural projection
- $V \xrightarrow{g} V/B^{-1}(\mathrm{im}A)$ is the natural projection
- $B^{-1}(\mathrm{im}A) \xrightarrow{g^{-1}} V$ is the natural inclusion

By taking the dual map of the isomorphisms above, we also get isomorphisms for U^* and V^*:

$$U^* \cong \mathrm{im}A \oplus (U^*/(\mathrm{im}A + \mathrm{im}B)) \oplus \mathrm{im}B/\mathrm{im}A$$
$$V^* \cong B^*(\ker A) \oplus (V^*/B^*(\ker A))$$

where the spaces of the same colour are naturally dual to each other. Then A, B, C induce maps $A_1, B_1, B_2, C_1, C_2, C_3$ on the summands in the direct sums as in the matrix below:

	$(U/\ker A)$	$(\ker A \cap \ker B^*)$	$(\ker A/\ker B^*)$	$(V/B^{-1}(\mathrm{im}A))$	$B^{-1}(\mathrm{im}A)$
$\mathrm{im}A$	A_1				B_1
$(U^*/(\mathrm{im}A + \mathrm{im}B))$					
$\mathrm{im}B/\mathrm{im}A$				B_2	
$B^*(\ker A)$			B_2^*	C_1	C_2
$(V^*/B^*(\ker A))$	B_1^*			C_2^*	C_3

For example, B induces the map $B_2 : (V/B^{-1}(\mathrm{im}A)) \to \mathrm{im}B/\mathrm{im}A$ by first restricting to $(V/B^{-1}(\mathrm{im}A))$ and then projecting to $\mathrm{im}B/\mathrm{im}A$. Let the empty spots in the matrix above be 0. Then the matrix above represents a map M_1, which is just the map M after applying the isomorphisms on U, U^*, V, and V^*. Thus M and M_1 have the same signature.

Note that A_1 and B_2 are invertible. Then let Q be the invertible map:

$$Q : (u_1 + u_2 + u_3 + v_1 + v_2) \mapsto (u_1 - A_1^{-1}B_1v_2 + u_2 + u_3 - (B_2^*)^{-1}C_2v_2 + v_1 + v_2)$$

We can verify that Q^*M_1Q is represented by the following matrix:

	$(U/\ker A)$	$(\ker A \cap \ker B^*)$	$(\ker A/\ker B^*)$	$(V/B^{-1}(\mathrm{im}A))$	$B^{-1}(\mathrm{im}A)$
$\mathrm{im}A$	A_1				
$(U^*/(\mathrm{im}A + \mathrm{im}B))$					
$\mathrm{im}B/\mathrm{im}A$				B_2	
$B^*(\ker A)$			B_2^*	C_1	
$(V^*/B^*(\ker A))$					$C_3 - B_1^*A_1^{-1}B_1$

Thus the signature of $M(A,B,C)$ can be computed by

$$\begin{aligned}\sigma(M(A,B,C)) &= \sigma(A_1) + \sigma\left(\begin{array}{c|c} & B_2 \\ \hline B_2^* & C_1\end{array}\right) + \sigma(C_3 - B_1^*A_1^{-1}B_1) \\ &= \sigma(A) + \sigma(C_3 - B_1^*A_1^{-1}B_1)\end{aligned}$$

Since $\sigma(A) = \sigma(A_1)$ and anything of the form $\left(\begin{array}{c|c} & B_2 \\ \hline B_2^* & C_1\end{array}\right)$ has signature 0. [A BETTER DESCRIPTION OF $C_3 - B_1^*A_1^{-1}B_1$]?

Theorem 0.1.

$$\sigma \begin{pmatrix} A & B & \\ B^* & C+D & E \\ & E^* & F \end{pmatrix} = \sigma(A) + \sigma(F) + \sigma(\pi(C+D-B_1^* A_1^{-1} B - E_1 F_1^{-1} E^*)|_{(E^*)^{-1}(imF)\cap B^{-1}(imA)})$$

where:

- π *is projection onto* $V^*/(B^*(\ker A) + E(\ker F))$*;*
- A_1^{-1} *and* F_1^{-1} *are the inverses of the induced isomorphisms* $A_1 : U/\ker A \xrightarrow{\cong} imA$ *and* $F_1 : W/\ker F \xrightarrow{\cong} imF$*; and*
- B_1^* *and* E_1 *are the induced maps on the quotients* $B_1^* : U/\ker A \to V^*/B(\ker A)$ *and* $E_1 : W/\ker F \to V^*/E(\ker F)$