

Pensieve Header: KV1 in FAA language.

Preliminaries

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FreeLie.m
<< AwCalculus.m
<< FAA.m
```

FreeLie` implements / extends

{*, +, **, \$SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, ℏ, ↦, ↪}.

FreeLie` is in the public domain. Dror Bar-Natan is committed

to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,

hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Δ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed

to support it within reason until July 15, 2022. This is version 150909.

```
In[*]:= Rs[a_, b_] := Es[⟨a → LS[0], b → LS[LW@a]⟩, CWS[0]];
α = LS[{x, y}, αs]; β = LS[{x, y}, βs]; γ = CWS[{x, y}, γs];
V = Es[⟨x → α, y → β⟩, γ];
κ = CWS[{x}, κs]; Cap = Es[⟨x → LS[0]⟩, κ];
R4Eqn = V ** (Rs[x, z] // dΔ[x, x, y]) ≡ Rs[y, z] ** Rs[x, z] ** V;
UnitarityEqn = (V ** (V // dA[x] // dA[y]) ≡ Es[⟨x → LS[0], y → LS[0]⟩, CWS[0]]);
CapEqn = ((V ** (Cap // dΔ[x, x, y]) // dc[x] // dc[y]) ≡
(Cap * (Cap // dσ[x, y]) // dc[x] // dc[y]));
βs[x] = 1 / 2; βs[y] = 0;
SeriesSolve[{α, β, γ, κ}, (ℏ-1 R4Eqn) && UnitarityEqn && CapEqn];
{V, κ}
```

SeriesSolve: In degree 1 arbitrarily setting {κs[x] → 0}.

SeriesSolve: In degree 3 arbitrarily setting {αs[x, y] → 0}.

Out[*]=

$$\left\{ \text{Es} \left[\left\langle x \rightarrow \text{LS} \left[\theta, -\frac{\overline{xy}}{24}, \theta, \dots \right], y \rightarrow \text{LS} \left[\frac{\overline{x}}{2}, -\frac{\overline{xy}}{12}, \theta, \dots \right] \right\rangle, \text{CWS} \left[\theta, -\frac{\overline{xy}}{48}, \theta, \dots \right] \right], \right. \\ \left. \text{CWS} \left[\theta, -\frac{\overline{xx}}{96}, \theta, \dots \right] \right\}$$

```
In[*]:= logF = V[[1]] // dσ[{x, y} → {y, x}]
logF@{4}
```

```
Out[*]=
{ x → LS[ $\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \theta, \dots$ ], y → LS[ $\theta, \frac{\overline{xy}}{24}, \theta, \dots$ ] }
```

```
Out[*]=
{ x → LS[ $\frac{\overline{y}}{2}, \frac{\overline{xy}}{12}, \theta, -\frac{1}{720} \overline{x x xy} + \frac{1}{720} \overline{x xy y} - \frac{\overline{xy y y}}{5760}, \dots$ ],
y → LS[ $\theta, \frac{\overline{xy}}{24}, \theta, -\frac{\overline{x x xy}}{1440} + \frac{\overline{7 x xy y}}{5760} - \frac{\overline{7 xy y y}}{5760}, \dots$ ] }
```

```
In[*]:= AWExp_d[_] := Module[{t = AW[]},
AW[] + Sum[t = Expand[t ** L / k] /. a_AW /; Length[a] > d := 0, {k, d}]
]
```

Testing

```
In[*]:= d = 4
```

```
Out[*]=
4
```

```
In[*]:= lhs = Sum[AW_1@@Table[x, k] / k!, {k, 0, d}] // FA[x → x + y]
```

```
Out[*]=
AW_1[] + AW_1[x] + AW_1[y] +  $\frac{1}{2}$  AW_1[x, x] +  $\frac{1}{2}$  AW_1[x, y] +  $\frac{1}{2}$  AW_1[y, x] +
 $\frac{1}{2}$  AW_1[y, y] +  $\frac{1}{6}$  AW_1[x, x, x] +  $\frac{1}{6}$  AW_1[x, x, y] +  $\frac{1}{6}$  AW_1[x, y, x] +  $\frac{1}{6}$  AW_1[x, y, y] +
 $\frac{1}{6}$  AW_1[y, x, x] +  $\frac{1}{6}$  AW_1[y, x, y] +  $\frac{1}{6}$  AW_1[y, y, x] +  $\frac{1}{6}$  AW_1[y, y, y] +
 $\frac{1}{24}$  AW_1[x, x, x, x] +  $\frac{1}{24}$  AW_1[x, x, x, y] +  $\frac{1}{24}$  AW_1[x, x, y, x] +  $\frac{1}{24}$  AW_1[x, x, y, y] +
 $\frac{1}{24}$  AW_1[x, y, x, x] +  $\frac{1}{24}$  AW_1[x, y, x, y] +  $\frac{1}{24}$  AW_1[x, y, y, x] +  $\frac{1}{24}$  AW_1[x, y, y, y] +
 $\frac{1}{24}$  AW_1[y, x, x, x] +  $\frac{1}{24}$  AW_1[y, x, x, y] +  $\frac{1}{24}$  AW_1[y, x, y, x] +  $\frac{1}{24}$  AW_1[y, x, y, y] +
 $\frac{1}{24}$  AW_1[y, y, x, x] +  $\frac{1}{24}$  AW_1[y, y, x, y] +  $\frac{1}{24}$  AW_1[y, y, y, x] +  $\frac{1}{24}$  AW_1[y, y, y, y]
```

In[*]:= rhs1 = AWE_d[Plus@@L[Ad[x /. logF[[1]]][LW@x]]@{d}] /. AW → AW₁

Out[*]=

$$\begin{aligned}
 & AW_1[] + AW_1[x] + \frac{1}{2} AW_1[x, x] - \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \frac{1}{6} AW_1[x, x, x] - \\
 & \frac{1}{3} AW_1[x, x, y] + \frac{1}{6} AW_1[x, y, x] + \frac{1}{8} AW_1[x, y, y] + \frac{1}{6} AW_1[y, x, x] - \\
 & \frac{1}{4} AW_1[y, x, y] + \frac{1}{8} AW_1[y, y, x] + \frac{1}{24} AW_1[x, x, x, x] - \frac{1}{8} AW_1[x, x, x, y] + \\
 & \frac{1}{24} AW_1[x, x, y, x] + \frac{1}{12} AW_1[x, x, y, y] + \frac{1}{24} AW_1[x, y, x, x] - \frac{1}{24} AW_1[x, y, x, y] - \\
 & \frac{1}{48} AW_1[x, y, y, y] + \frac{1}{24} AW_1[y, x, x, x] - \frac{1}{8} AW_1[y, x, x, y] + \frac{1}{24} AW_1[y, x, y, x] + \\
 & \frac{1}{16} AW_1[y, x, y, y] + \frac{1}{24} AW_1[y, y, x, x] - \frac{1}{16} AW_1[y, y, x, y] + \frac{1}{48} AW_1[y, y, y, x]
 \end{aligned}$$

In[*]:= rhs2 = AWE_d[Plus@@L[Ad[y /. logF[[2]]][LW@y]]@{d}] /. AW → AW₂

Out[*]=

$$\begin{aligned}
 & AW_2[] + AW_2[y] + \frac{1}{2} AW_2[y, y] + \frac{1}{24} AW_2[x, y, y] - \frac{1}{12} AW_2[y, x, y] + \\
 & \frac{1}{24} AW_2[y, y, x] + \frac{1}{6} AW_2[y, y, y] + \frac{1}{48} AW_2[x, y, y, y] - \frac{1}{48} AW_2[y, x, y, y] - \\
 & \frac{1}{48} AW_2[y, y, x, y] + \frac{1}{48} AW_2[y, y, y, x] + \frac{1}{24} AW_2[y, y, y, y]
 \end{aligned}$$

In[*]:= rhs = (Expand[rhs1 * rhs2] // m_{1,2→1}) /. AW₁[w____] /; Length@{w} > d ⇒ 0

Out[*]=

$$\begin{aligned}
 & AW_1[] + AW_1[x] + AW_1[y] + \frac{1}{2} AW_1[x, x] + \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \\
 & \frac{1}{2} AW_1[y, y] + \frac{1}{6} AW_1[x, x, x] + \frac{1}{6} AW_1[x, x, y] + \frac{1}{6} AW_1[x, y, x] + \frac{1}{6} AW_1[x, y, y] + \\
 & \frac{1}{6} AW_1[y, x, x] + \frac{1}{6} AW_1[y, x, y] + \frac{1}{6} AW_1[y, y, x] + \frac{1}{6} AW_1[y, y, y] + \\
 & \frac{1}{24} AW_1[x, x, x, x] + \frac{1}{24} AW_1[x, x, x, y] + \frac{1}{24} AW_1[x, x, y, x] + \frac{1}{24} AW_1[x, x, y, y] + \\
 & \frac{1}{24} AW_1[x, y, x, x] + \frac{1}{24} AW_1[x, y, x, y] + \frac{1}{24} AW_1[x, y, y, x] + \frac{1}{24} AW_1[x, y, y, y] + \\
 & \frac{1}{24} AW_1[y, x, x, x] + \frac{1}{24} AW_1[y, x, x, y] + \frac{1}{24} AW_1[y, x, y, x] + \frac{1}{24} AW_1[y, x, y, y] + \\
 & \frac{1}{24} AW_1[y, y, x, x] + \frac{1}{24} AW_1[y, y, x, y] + \frac{1}{24} AW_1[y, y, y, x] + \frac{1}{24} AW_1[y, y, y, y]
 \end{aligned}$$

In[*]:= lhs == rhs

Out[*]=

True