

# Free Lie Algebras Routines

Pensieve header: A free-Lie calculator, pensieve://Projects/WKO4/ branch (most current), continues pensieve://Project-s/WKO4. Now understands the Drinfel'd-Kohno algebra.

Non-current versions of this package are available at <http://drorbn.net/AcademicPensieve/Projects/WKO4/Archive/>.

## To Do

- ToDo* \* Add "?" help lines and/or tooltip popups.
- ToDo* \* Revise LW and CW I/O; have declarable “generators”.
- ToDo* \* Consider “trees and wheels printing”.

## Prolog

```

BeginPackage["FreeLie`"];
Print["FreeLie` implements / extends ",
  Sort@{"*", "+", "**", "$SeriesShowDegree", "<>", "f", "≡", ad, Ad, adSeries, AllCyclicWords,
  AllLyndonWords, AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS,
  CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries,
  EulerE, Exp, FreeLieFormatting, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW,
  LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve,
  Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, τ, Γ, ℓ, Δ, σ, ħ, ↦, ↵},
  "."];
Print["FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 240218."];
$SeriesShowDegree = 3;
Begin["`Private`"];

```

## “New”

```

SetAttributes[New, HoldAll];
(*New[type_[srn_], def_] := Module[ (* "srn" for "self-reference name" *)
  {un}, (* "un" for "unique name" *)
  un=Unique[ToString[type]<>"$"];
  ReleaseHold[Hold[def] /. srn → un];
  type[un]
];*)
New[type_[srn_], def_] := ( (* "srn" for "self-reference name" *)
  (*Print["new object of type ", type//ToString, " and srn ", srn];*)
  ReleaseHold[Hold[def; type[srn]] /. srn → Unique[ToString[type]<>"$"]
  );

```

## Words and Lyndon Words

A Lyndon word is a word lexicographically smaller than all of its proper right factors.

```
AllWords[n_, ab_List] := AW @@@ Tuples[ab, n];
LW /: LW[] < LW[___] = True;
LW /: _LW < LW[] = False;
LW /: LW[x_, xs___] < LW[x_, ys___] := LW[xs] < LW[ys];
LW /: LW[x_, ___] < LW[y_, ___] /; (x != y) := OrderedQ[{x, y}];
LW /: x_LW > y_LW := y < x;
LW /: x_LW ≤ y_LW := !(y < x);
LW /: x_LW ≥ y_LW := !(x < y);
LyndonQ[w_AW] := And @@ (
  (LW@@w < LW@@#) & /@ Table[Drop[w, i], {i, 1, Length[w] - 1}]);
AllLyndonWords[n_Integer, ab_List] := AllLyndonWords[n, ab] =
  LW @@@ Select[AllWords[n, ab /. LW[w_] := w], LyndonQ];
AllLyndonWords[{n_}, ab_List] := Join@@Table[AllLyndonWords[k, ab], {k, n}];
Deg[w_LW] := Length[w];
LyndonFactorization[w_LW /; Deg[w] == 1] := w;
LyndonFactorization[w_LW /; Deg[w] > 1] := Module[{rf},
  rf = First[Sort[Table[Drop[w, i], {i, 1, Length[w] - 1}], Less]];
  LW /@ {Drop[w, -Length[rf]], rf}];
LW[w_LW] := w;
BracketForm[LW[c_]] := ToString[c];
BracketForm[w_LW] := BracketForm[w] = StringJoin[Flatten[{
  "[", BracketForm /@ LyndonFactorization[w], "]"
}]];
BracketForm[expr_] := expr /. w_LW := BracketForm[w];
topbracketform[LW[c_]] := c;
topbracketform[w_LW] := topbracketform[w] = Overscript[
  Row[Riffle[topbracketform /@ LyndonFactorization[w], ""], -];
TopBracketForm[LW[c_]] := Overscript[c, -];
TopBracketForm[w_LW] := topbracketform[w];
TopBracketForm[w_CW] := Overscript[StringJoin@@(ToString /@ w), -];
TopBracketForm[expr_] := expr /. w_LW | w_CW | w_DK := TopBracketForm[w];
```

## The Bracket for Lie Elements

```
b[0, _] = 0; b[_, 0] = 0;
b[c_* (x_AW | x_LW), y_] := Expand[c b[x, y]];
b[x_, c_* (y_AW | y_LW)] := Expand[c b[x, y]];
b[x_Plus, y_] := b[#, y] & /@ x;
b[x_, y_Plus] := b[x, #] & /@ y;
b[w_LW, z_LW] := LWAdjoint[w][z];
ad[x_][y_] := b[x, y];
```

```

LWAdjoint[w_] := LWAdjoint[w] = Module[{u},
  u = Unique[LWAct];
  u[z_] := u[z] = Which[
    w === z, 0,
    z < w, Expand[-b[z, w]],
    Deg[w] == 1, Join[w, z],
    True, Module[{x, y},
      {x, y} = LyndonFactorization[w];
      If[y ≥ z,
        Join[w, z],
        b[x, LWAdjoint[y][z]] + b[LWAdjoint[x][z], y]
      ]
    ]
  ];
  u
];

```

## BooleanSequence

```

BS[v : (True | False)] := BS[v] = New[BooleanSequence[ser], ser[_] = v];
BooleanSequence[bs_Symbol][{dd_Integer}] := TopBracketForm[Append[
  BS@@(Table[bs[d], {d, 0, dd}] // {n_ True, True, rest___} => {(n+1) True, rest}),
  "..."]
];
BooleanSequence[bs_Symbol][e___] := bs[e];
BooleanSequence /: And[bss__BooleanSequence] := New[BooleanSequence[bs],
  bs[d_Integer] := bs[d] = And @@ ((#[d]) & /@ {bss})
];
BooleanSequence /:  $\hbar^k \cdot$  BooleanSequence[bs_Symbol] := New[BooleanSequence[nbs],
  nbs[d_Integer] := bs[d - k]
];
BooleanSequence /: TrueQ[bs_BooleanSequence] := New[BooleanSequence[nbs],
  nbs[d_Integer] := nbs[d] = TrueQ[bs[d]]
];

```

## LieSeries

```

LieSeries[ser_Symbol][{dd_Integer}] := Append[TopBracketForm[LS@@Table[ser[d], {d, dd}]], "..."];
LieSeries[ser_Symbol][e___] := ser[e];
ShowLieSeries[d_Integer][s_LieSeries] := s[{d}];
LS[s_] := MakeLieSeries[s];
MakeLieSeries[s_LieSeries] := s;
MakeLieSeries[expr_] := MakeLieSeries[expr] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Expand[expr /. w_LW /; Deg[w] ≠ d → 0]
];
s1_LieSeries ≡ s2_LieSeries := New[BooleanSequence[bs],
  bs[0] = True;
  bs[d_Integer] := bs[d - 1] && (s1[d] == s2[d]);
];
Crop[s_LieSeries, d_Integer] := Crop[s, d] = New[LieSeries[ser],
  ser[dd_Integer] := If[dd ≤ d, s[dd], 0]
];
RandomLieSeries[ab_List, opts__Rule] := Module[
  {rand = Randomizer /. {opts} /. Randomizer →  $\left(\frac{\text{RandomInteger}[\{-2 \text{Deg}[\#]!, 2 \text{Deg}[\#]!\}]}{\text{Deg}[\#]!} \&\right)$ },
  New[LieSeries[ser],
    ser[d_Integer] := ser[d] = Plus @@ ((rand[#] * #) & /@ AllLyndonWords[d, ab])
  ]
];

```

```

AddLieSeries[ss__LieSeries] := AddLieSeries[ss] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Plus @@ ((#[d]) & /@ {ss})
];
LieSeries /: Plus[ss__LieSeries] := AddLieSeries[ss];
ScaleLieSeries[c_, s_LieSeries] := ScaleLieSeries[c, s] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Expand[c * s[d]]
];
LieSeries /: c_ * s_LieSeries := ScaleLieSeries[c, s];
LieSeries /: D[ls_LieSeries, vars__] := New[LieSeries[ser],
  ser[d_Integer] := ser[d] = D[ls[d], vars]
];
IntegrateLieSeries[ls_LieSeries, {s_, s0_, s1_}] :=
  IntegrateLieSeries[ls, {s, s0, s1}] = New[LieSeries[ser],
    ser[d_Integer] := ser[d] = Expand $\left[\int_{s0}^{s1} ls[d] ds\right]$ 
  ];
LieSeries /: Integrate[ls_LieSeries, {s_, s0_, s1_}] := IntegrateLieSeries[ls, {s, s0, s1}];

```

```

b[s1_LieSeries, s2_LieSeries] := b[s1, s2] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Sum[b[s1[k], s2[d-k]]
    ];
b[s_LieSeries, y_] := b[s, MakeLieSeries[y]];
b[x_, s_LieSeries] := b[MakeLieSeries[x], s];

```

## EulerE, DegreeScale

```

LieSeries /: EulerE[s_LieSeries] := New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Expand[d * s[d]]
];
DegreeScale[h_][s_LieSeries | s_CWSeries] := New[Head[s][ser],
  ser[d_Integer] := ser[d] = Expand[h^d s[d]]
];

```

## adSeries, and Ad

Convention:  $\text{ad}(x)(z)=[x,z]$ . Satisfies  $\text{ad}([x,y])=[\text{ad}(x),\text{ad}(y)]$ .

```

adSeries[1, x_LieSeries][psi_LieSeries] := adSeries[1, x][psi] = psi;
adSeries[ad^n., x_LieSeries][psi_LieSeries] := adSeries[ad^n, x][psi] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = b[x, adSeries[ad^{n-1}, x][psi]][d]
];
adSeries[f_, x_LieSeries][psi_LieSeries] := adSeries[f, x][psi] = New[LieSeries[ser],
  ser[d_Integer] := ser[d] = Module[{c},
    Expand[Sum[
      c = SeriesCoefficient[f, {ad, 0, k}];
      If[c === 0, 0, c * adSeries[ad^k, x][psi][d]],
      {k, 0, d-1}
    ]]]
];
adSeries[f_, x_][psi_] := adSeries[f, MakeLieSeries[x]][MakeLieSeries[psi]];
Ad[x_] := adSeries[E^ad, x];

```

## LieDerivation, DerivationPower, DerivationSeries

```

LieDerivation[der_][es___] := der[es];
LieDerivation[rules__Rule] := LieDerivation[{rules}];
LieDerivation[rules_List] := LieDerivation[rules] = New[LieDerivation[der],
  der[Support] = First /@ rules;
  (der[w_LW] /; Deg[w] == 1) := (der[w] = MakeLieSeries[w /. Append[rules, _LW -> 0]]);
  der[w_LW] := der[w] = Module[{x, y},
    {x, y} = LyndonFactorization[w];
    AddLieSeries[b[der[x], y], b[x, der[y]]]
  ];
  der[s_LieSeries] := der[s] = New[LieSeries[ser],
    ser[d_] := ser[d] = Sum[der[s[k]][d], {k, 1, d}];
  der[as_ASeries] := der[as] = New[ASeries[ser],
    ser[d_] := ser[d] = Sum[
      Expand[as[k] /. w_AW -> Sum[
        NonCommutativeMultiply[
          Take[w, j - 1],
          L[der[LW[w[[j]]][d - k + 1]],
          Drop[w, j]
        ],
        {j, k}
      ],
      {k, 1, d}
    ],
  ];
  der[cws_CWSeries] := der[cws] = New[CWSeries[ser],
    ser[d_] := ser[d] = Sum[
      Expand[cws[k] /. w_CW -> Sum[
        tr[NonCommutativeMultiply[
          AW@@Take[w, j - 1],
          L[der[LW[w[[j]]][d - k + 1]],
          AW@@Drop[w, j]
        ],
        {j, k}
      ],
      {k, 1, d}
    ],
  ];
  der[expr_][d_] := Expand[expr /. {w_LW -> der[w][d], s_LieSeries -> der[s][d]};
];

```

```

LieDerivation /: Plus[ders_LieDerivation] := LieDerivation[Table[
  u → Total[#[u] & /@ {ders}],
  {u, Union@@ (#[Support] & /@ {ders})}
]];
LieDerivation /: c_*der_LieDerivation := LieDerivation[Table[
  u → (c der[u]),
  {u, der@Support}
]];

```

```

b[der1_LieDerivation, der2_LieDerivation] := LieDerivation[Table[
  u → der1[der2[u]] - der2[der1[u]],
  {u, (der1@Support) ∪ (der2@Support)}
]];

```

```

DerivationPower[0, der_LieDerivation][ψ_LieSeries | ψ_CWSeries] :=
  DerivationPower[0, der][ψ] = New[Head[ψ][ser],
  ser[d_Integer] := ser[d] = ψ[d]
  ];
DerivationPower[n_Integer, der_LieDerivation][ψ_LieSeries | ψ_CWSeries] :=
  DerivationPower[n, x][ψ] = New[Head[ψ][ser],
  ser[d_Integer] := ser[d] = der[DerivationPower[n - 1, der][ψ]][d]
  ];
DerivationSeries[___][0] = 0;
DerivationSeries[f_, Ld_LieDerivation][ψ_LieSeries | ψ_CWSeries] :=
  DerivationSeries[f, Ld][ψ] = New[Head[ψ][ser],
  ser[d_Integer] := ser[d] = Module[{c},
  Expand[Sum[
    c = SeriesCoefficient[f, {der, 0, k}];
    If[c == 0, 0, c * DerivationPower[k, Ld][ψ][d]],
    {k, 0, d}
  ]]
  ]
  ];
DerivationExp[Ld_LieDerivation] := DerivationSeries[E^der, Ld];
LieDerivation /: Exp[Ld_LieDerivation] := DerivationExp[Ld];
LieDerivation /: eLd_LieDerivation := DerivationExp[Ld];

```

## LieMorphism

```

LieMorphism[mor_][es___] := mor[es];
LieMorphism[rules__Rule] := LieMorphism[{rules}];
LieMorphism[rules_List] := LieMorphism[rules] = New[LieMorphism[mor],
  mor[Support] = First /@ rules;
  (mor[w_LW] /; Deg[w] == 1) := (mor[w] = MakeLieSeries[w /. rules]);
  mor[w_LW] := (mor[w] = b @@ (mor /@ LyndonFactorization[w]));
  mor[AW[]] = MakeASeries[AW[]];
  mor[AW[w_]] := mor[AW[w]] = L[MakeLieSeries[LW[w] /. rules]];
  mor[w_AW] := mor[w] = Module[{w1, w2},
    w1 = Take[w, Floor[Length[w] / 2]];
    w2 = Drop[w, Floor[Length[w] / 2]];
    mor[w1] ** mor[w2]
  ];
  mor[w_CW] := tr[mor[AW@@w]];
  mor[s_LieSeries] := mor[s] = New[LieSeries[ser],
    ser[d_] := ser[d] = Sum[mor[s[k]][d], {k, 1, d}];
  mor[as_ASeries] := mor[as] = New[ASeries[ser],
    ser[d_] := ser[d] = Sum[mor[as[k]][d], {k, 1, d}];
  mor[cws_CWSeries] := mor[cws] = New[CWSeries[ser],
    ser[d_] := ser[d] = Sum[mor[cws[k]][d], {k, 1, d}];
  mor[expr_][d_] := Expand[expr /. (w_LW | w_AW | w_CW) -> mor[w][d]];
  (* Added 150217, commented out later that day *)
  (*mor[expr_] := Expand[expr /. (w_LW|w_AW|w_CW) -> mor[w]]*)
];

```

```

LieMorphism /: Inverse[mor_LieMorphism] := InvertLieMorphism[mor];
InvertLieMorphism[mor_LieMorphism] := InvertLieMorphism[mor] =
  LieMorphism[Table[
    LW@u -> New[LieSeries[uimg],
      uimg[1] = LW@u;
      uimg[d_Integer] /; d > 1 := uimg[d] = -Sum[mor[uimg[k]][d], {k, 1, d-1}
    ],
    {u, mor[Support]}
  ]]

```



## StableApply

```
StableApply[mor_LieMorphism, (type : (LieSeries | ASeries | CWSeries)) [s_]] := (
  StableApply[mor, type[s]] = New[type[ser],
    ser[d_] := ser[d] = Nest[mor, type[s], d][d]
  ]
);
```

## BCH

```
BCHBase = New[LieSeries[bch],
  bch[1] = LW@"x" + LW@"y";
  bch[d_Integer] := bch[d] = Expand[Plus[
    adSeries[e-ad, MakeLieSeries[LW@"y"]] [MakeLieSeries[LW@"x"]][d],
    -adSeries[ $\frac{1 - e^{-ad}}{ad} - 1$ , LieSeries[bch]] [EulerE[LieSeries[bch]]][d]
  ] / d];
BCH[x_, y_] := LieMorphism[{LW["x"] → x, LW["y"] → y}][BCHBase];
```

## AW, ASeries, $\iota$ , $\tau$ , $\text{tr}_y$

```
Unprotect[NonCommutativeMultiply];
x_ ** 0 = 0; 0 ** y_ = 0;
(c_ * x_AW) ** y_ := Expand[c (x ** y)];
x_ ** (c_ * y_AW) := Expand[c (x ** y)];
x_Plus ** y_ := (# ** y) & /@ x;
x_ ** y_Plus := (x ** #) & /@ y;
Deg[w_AW] := Length[w];
AW[AW[w_]] := AW[w];
AW[w1___] ** AW[w2___] := AW[w1, w2];
b[w_AW, z_AW] := w ** z - z ** w;
```

```

ASeries[ser_Symbol][{dd_Integer}] := AS@@Table[ser[d], {d, 0, dd}];
ASeries[as_Symbol][es___] := as[es];
AS[s_] := MakeASeries[s];
MakeASeries[as_CWSeries] := as;
MakeASeries[expr_] := MakeASeries[expr] = New[ASeries[ser],
  ser[d_Integer] := ser[d] = Expand[expr /. w_AW /; Deg[w] ≠ d → 0]
];
(s1_ASeries ** s2_ASeries) := (s1 ** s2) = New[ASeries[ser],
  ser[d_Integer] := ser[d] =  $\sum_{k=0}^d s1[k] ** s2[d-k]$ ;
s1_ASeries ≡ s2_ASeries := New[BooleanSequence[bs],
  bs[0] = True;
  bs[d_Integer] := bs[d-1] && (s1[d] == s2[d]);
];

```

```

AddASeries[ss___ASeries] := AddASeries[ss] = New[ASeries[ser],
  ser[d_Integer] := ser[d] = Plus@@({# [d]} & /@ {ss})
];
ASeries /: Plus[ss___ASeries] := AddASeries[ss];
ScaleASeries[c_, s_ASeries] := ScaleASeries[c, s] = New[ASeries[ser],
  ser[d_Integer] := ser[d] = Expand[c * s[d]]
];
ASeries /: c_ * s_ASeries := ScaleASeries[c, s];

```

```

L[w_LW] /; Deg[w] == 1 := AW@@w;
L[w_LW] := L[w] = b@@(L /@ LyndonFactorization[w]);
L[expr_] := Expand[expr /. w_LW => L[w]];
L[ls_LieSeries] := L[ls] = New[ASeries[as],
  as[0] = 0;
  as[d_] /; d > 0 := as[d] = L[ls[d]]
];

```

```

τ[y_LW, w_LW] /; Deg[y] == 1 := τ[y, w] = Which[
  y === w, AW[],
  Deg[w] === 1, 0,
  True, Module[{w1, w2},
    {w1, w2} = LyndonFactorization[w];
    L[w1] ** τ[y, w2] - L[w2] ** τ[y, w1]
  ]
];
τ[y_, ls_LieSeries] := τ[y, ls] = New[ASeries[as],
  as[d_] := as[d] = τ[LW[y], ls[d+1]]
];
τ[y_, expr_] := Expand[expr /. w_LW => τ[LW[y], w]];

```

```

tr_y_[expr_] /; Deg[LW@y] == 1 := tr[τ[LW@y, expr]];

```

## CW, cw, CWSeries, tr, div

```

RotateToMinimal[L_] := Module[
  {best1 = L, rotated1 = RotateLeft[L]},
  While[rotated1 != L,
    best1 = First[Sort[{best1, rotated1}]];
    rotated1 = RotateLeft[rotated1]
  ];
  best1
];

```

```

Deg[w_CW] := Length[w];
AllCyclicWords[d_Integer, ab_List] := AllCyclicWords[d, ab] = Union[tr[AllWords[d, ab]]];
CWSeries[cws_Symbol][es___] := cws[es];
CWSeries[ser_Symbol][{dd_Integer}] :=
  Append[TopBracketForm[CWS @@ Table[ser[d], {d, dd}]], "..."];
CWS[cws_] := MakeCWSeries[cws];
MakeCWSeries[cws_CWSeries] := cws;
MakeCWSeries[expr_] := MakeCWSeries[expr] = New[CWSeries[ser],
  ser[d_Integer] := ser[d] = Expand[
    expr /. w_CW => If[Deg[w] == d, RotateToMinimal@w, 0]
  ]
];
RandomCWSeries[ab_List, opts___Rule] := New[CWSeries[ser],
  Module[
    {
      rand = Randomizer /. {opts} /. Randomizer ->  $\left( \frac{\text{RandomInteger}[\{-2 \text{Deg}[\#]!, 2 \text{Deg}[\#]!\}]}{\text{Deg}[\#]!} \& \right)$ ,
      ser[d_Integer] := ser[d] = Plus @@ ((rand[#] * #) & /@ AllCyclicWords[d, ab])
    }
  ]
];
s1_CWSeries == s2_CWSeries := New[BooleanSequence[bs],
  bs[0] = True;
  bs[d_Integer] := bs[d - 1] && (s1[d] == s2[d]);
];
AddCWSeries[ss___CWSeries] := AddCWSeries[ss] = New[CWSeries[ser],
  ser[d_Integer] := ser[d] = Plus @@ ((#[d]) & /@ {ss})
];
CWSeries /: Plus[ss_CWSeries] := AddCWSeries[ss];
ScaleCWSeries[c_, s_CWSeries] := ScaleCWSeries[c, s] = New[CWSeries[ser],
  ser[d_Integer] := ser[d] = Expand[c * s[d]]
];
CWSeries /: c_ * s_CWSeries := ScaleCWSeries[c, s];
IntegrateCWSeries[cws_CWSeries, {s_, s0_, s1_}] :=
  IntegrateCWSeries[cws, {s, s0, s1}] = New[CWSeries[ser],
    ser[d_Integer] := ser[d] = Expand  $\left[ \int_{s0}^{s1} cws[d] ds \right]$ ;
];
CWSeries /: Integrate[cws_CWSeries, {s_, s0_, s1_}] := IntegrateCWSeries[cws, {s, s0, s1}];

```

```

cw[gs___] := CW @@ Replace[RotateToMinimal@{gs}, LW[g_] => g, {1}];

```

```

tr[w_AW] := tr[w] = cw@@w;
tr[expr_] := expr /. aw_AW => tr[aw];
tr[as_ASeries] := tr[as] = New[CWSeries[cws], cws[d_] := cws[d] = tr[as[d]]];

```

```

div[LW[y_], w_LW] := div[LW@y, w] = tr[(AW@y) ** τ[LW@y, w]];
div[y_, ls_LieSeries] := div[y, ls] = New[CWSeries[cws],
  cws[d_] := cws[d] = div[LW[y], ls[d]]
];
div[y_, expr_] := Expand[expr /. w_LW => div[LW[y], w]];
div[y_, p_Plus] := Total[div[LW[y], List@@p]];
div_y[expr_] := div[y, expr];

```

## The Meta-Cocycle JA

```

JA[-1, ___] = MakeCWSeries[0];
JA[n_, y_LW, μ_LieSeries, ss_] := JA[n, y, μ, ss] = Module[
  {s, sμ, μs},
  sμ = ScaleLieSeries[s, μ];
  μs = StableApply[LieMorphism[{y → Ad[ScaleLieSeries[1, sμ]][LW[z]]}], μ];
  μs = μs // LieMorphism[{LW[z] → y}];
  IntegrateCWSeries[
    AddCWSeries[
      JA[n-1, y, μ, s] // LieDerivation[{y → b[μs, y]}],
      div[y, μs]
    ],
    {s, 0, ss}
  ]
];
JA[y_LW, μ_LieSeries] := JA[y, μ] = Module[{s}, New[CWSeries[cws],
  cws[d_Integer] := cws[d] = JA[d-1, y, μ, s][d] /. s → 1
]];

```

<...>

```

SetAttributes[AngleBracket, Orderless];
<{λ___}> := <λ>;
(*<x_ → 0, rest___> := <x→LS[0], rest>;*)
Support[<λ___>] := First /@ {λ};
λ \ key_ := DeleteCases[λ, key → _];
λ \ keys_List := Fold[#1 \ #2 &, λ, keys];
<λ___>_s := s /. {λ};
(λ_AngleBracket)[d_] := <Table[u → λ_u[d], {u, Support[λ]}]>;
AngleBracket /: EulerE[λ_AngleBracket] := MapAt[EulerE, λ, {All, 2}];
Crop[λ_AngleBracket, d_] := MapAt[Crop[#, d] &, λ, {All, 2}];
AngleBracket /: λ1_AngleBracket ≡ λ2_AngleBracket /; Support[λ1] == Support[λ2] :=
  And @@ Table[λ1_s ≡ λ2_s, {s, Support[λ1]}];
AngleBracket /: Plus[λs__AngleBracket] := <Table[
  u → Total[#u & /@ {λs}],
  {u, Union@@(Support[#] & /@ {λs})}
]>;
AngleBracket /: c_ * λ_AngleBracket := <Table[
  u → Expand[c λ_u],
  {u, Support[λ]}
]>;
AngleBracket /: λ_AngleBracket // der_LieDerivation := MapAt[der, λ, {All, 2}];
AngleBracket /: λ_AngleBracket // DerivationSeries[f_, ld_LieDerivation] :=
  MapAt[DerivationSeries[f, ld], λ, {All, 2}];
AngleBracket /: λ_AngleBracket // mor_LieMorphism := MapAt[mor, λ, {All, 2}];
AngleBracket /: Integrate[λ_AngleBracket, {s_, s0_, s1_}] :=
  <Table[u → Integrate[λ_u, {s, s0, s1}], {u, Support[λ]}>;
λ_AngleBracket // DegreeScale[h_] := MapAt[DegreeScale[h], λ, {All, 2}];
b[λ1_AngleBracket, λ2_AngleBracket] := <Table[s → b[λ1_s, λ2_s], {s, Support[λ1] ∪ Support[λ2]}>;
AngleBracket /: D[λ_AngleBracket, vars__] := <Table[s → D[λ_s, vars], {s, Support[λ]}>;

```

## Tangential Derivations $D_\lambda$ , the tangential bracket $tb$

```

AngleBracket /: D[λ_AngleBracket] := LieDerivation[List@@λ /. (s_ → λ_s) ⇒ (LW[s] → b[LW[s], λ_s])];
AngleBracket /: D_λ_AngleBracket := D[λ];

```

```

tb[λ1_AngleBracket, λ2_AngleBracket] := <Table[
  s → b[λ1_s, λ2_s] + D_λ1[λ2_s] - D_λ2[λ1_s],
  {s, Support[λ1] ∪ Support[λ2]}
]>;

```

CC, RC,  $\text{ad}_u$ 

```

AngleBracket /: CC[⟨λ___⟩] := LieMorphism[{λ} /. (s_ → λs_) ⇒ (LW[s] → Ad[λs][LW[s]])];
CC[u_, γ_] := CC[⟨u → γ⟩];
CC_u[γ_] := CC[u, γ];
RC[λ_AngleBracket] := Inverse[CC[-λ]];
RC[u_, γ_] := RC[⟨u → γ⟩];
RC_u[γ_] := RC[u, γ];
ad[u_, γ_LieSeries] := LieDerivation[u → b[γ, u]];
ad_u[γ_] := ad[u, γ];

```

## J, div, j

JDef

```

J_u[γ_] := J_u[γ] = Module[{s}, ∫_0^1 (γ // RC_u[s γ] // div_u // CC_u[-s γ]) ds];

```

```

J[u_, γ_] := J_u[γ];

```

divDef

```

div[λ_AngleBracket] := Sum[div_a[λ_a], {a, Support[λ]}];
j[λ_AngleBracket] := div[λ] // DerivationSeries[ $\frac{e^{\text{der}} - 1}{\text{der}}$ , D_λ];

```

## Evaluating Lie Series in ⟨...⟩

```

LieMorphism[rules__Rule, keys_AngleBracket, br_] := LieMorphism[{rules}, keys, br];
LieMorphism[rules_List, keys_AngleBracket, br_] := New[LieMorphism[mor],
  mor[Support] = First /@ rules;
  mor[0] := mor[0] = AngleBracket @@ Table[s → LS[0], {s, List @@ keys}];
  (mor[w_LW] /; Deg[w] == 1) := (mor[w] = w /. rules);
  mor[w_LW] := (mor[w] = br @@ (mor /@ LyndonFactorization[w]));
  mor[expr_][d_] := Expand[expr /. w_LW ⇒ mor[w][d]];
  mor[ls_LieSeries] := mor[ls] = Module[{ser},
    {
      Table[
        ReleaseHold[Hold[
          ser[] = Hold[AngleBracketFromLieMorphism[mor, ls, ss]];
          ser[d_Integer] := ser[d] =  $\left(\sum_{k=1}^d \text{mor}[ls[k]][d]\right)_{ss}$ ;
          ss → LieSeries[ser]
        ] /. {ss → s, ser → Unique[LieSeriesInAngleBracket]}],
        {s, List @@ keys}
      ]
    ]
  ];

```

```

BCH[x_, y_, keys_AngleBracket, br_] := LieMorphism[{LW["x"] → x, LW["y"] → y}, keys, br][BCHBase];
BCH_br[λ1_AngleBracket, λ2_AngleBracket] := BCH[λ1, λ2, <Support[λ1] ∪ Support[λ2]>, br];
adSeries[f_, λ1_AngleBracket, br_][λ2_AngleBracket] :=
  LieMorphism[{LW["x"] → λ1, LW["y"] → λ2}, <Support[λ1] ∪ Support[λ2]>, br][
    adSeries[f, LW["x"]][LW["y"]]];
adSeries[f_, λ1_AngleBracket][λ2_AngleBracket] := adSeries[f, λ1, br][λ2];

```

## Γ and Λ

```

Γ_{0,t}[λ_AngleBracket] := t λ;
Γ_{n,t}[λ_AngleBracket] := Γ_{n,t}[λ] = ∫_0^t (λ // e^{-τ D_λ} // adSeries[ $\frac{ad}{e^{ad}-1}$ , Γ_{n-1,τ}[λ]]) dτ;
Γ_t[λ_AngleBracket] := Γ_t[λ] = <Table[
  s → New[LieSeries[ser],
    With[{s = s}, ser[d_Integer] := ser[d] = (Γ_{d-1,t}[λ]_s)[d]];
  ],
  {s, Support[λ]}
]>;
Γ[λ_AngleBracket] := Γ_1[λ];
(*Γ_{-1,t}[λ_AngleBracket] := <Table[s→MakeLieSeries[0], {s, Support[λ]}]>;*)
(*Γ_{n,t}[λ_AngleBracket] := Γ_{n,t}[λ] = Module[{τ},
  ∫_0^t (λ // e^{-τ D_λ} // adSeries[ $\frac{ad}{e^{ad}-1}$ , Γ_{n-1,τ}[λ]]) dτ
];*)
(*Γ_{n,t}[λ_AngleBracket] := Γ_{n,t}[λ] = Module[{τ, Γ0},
  Γ0 = Γ_{n-1,τ}[λ];
  <Table[
    s → ∫_0^t (λ_s // e^{-τ D_λ} // adSeries[ $\frac{ad}{e^{ad}-1}$ , Γ0_s]) dτ,
    {s, Support[λ]}
  ]>
];*)

```

```

Λ_{0,t}[λ_AngleBracket] := t λ;
Λ_{n,t}[λ_AngleBracket] := Λ_{n,t}[λ] = Module[{τ, Λ0},
  Λ0 = Λ_{n-1,τ}[λ];
  ∫_0^t (λ // e^{D_λ τ} // adSeries[ $\frac{ad}{e^{ad}-1}$ , Λ0, tb]) dτ
];
Λ_t[λ_AngleBracket] := Λ_t[λ] = <Table[
  s → New[LieSeries[ser],
    With[{s = s}, ser[d_Integer] := ser[d] = (Λ_{d-1,t}[λ]_s)[d]];
  ],
  {s, Support[λ]}
]>;
Λ[λ_AngleBracket] := Λ_1[λ];

```

## SeriesSolve

```

LS[ab_List, coefs_] := New[LieSeries[ser],
  ser[setter] = Null;
  ser[d_Integer, UndeterminedCoefficients] := Cases[coefs @@@ AllLyndonWords[d, ab], _coefs];
  ser[d_Integer] := If[ser[setter] != Null,
    ser[setter][d]; ser[d],
    Plus @@ ((# * coefs @@ #) & /@ AllLyndonWords[d, ab])
  ];
];
AS[ab_List, coefs_] := New[ASeries[ser],
  ser[setter] = Null;
  ser[d_Integer, UndeterminedCoefficients] := Cases[coefs @@@ AllWords[d, ab], _coefs];
  ser[d_Integer] := If[ser[setter] != Null,
    ser[setter][d]; ser[d],
    Plus @@ ((# * coefs @@ #) & /@ AllWords[d, ab])
  ];
];
CWS[ab_List, coefs_] := New[CWSeries[ser],
  ser[setter] = Null;
  ser[d_Integer, UndeterminedCoefficients] := Cases[coefs @@@ AllCyclicWords[d, ab], _coefs];
  ser[d_Integer] := If[ser[setter] != Null,
    ser[setter][d]; ser[d],
    Plus @@ ((# * coefs @@ #) & /@ AllCyclicWords[d, ab])
  ];
];

```



```

Options[SeriesSolve] = {
  Arbitrator → 0 (* should be 0 or a pure function that takes a
    list of unsettled variables and returns a list of their arbitrated values *)
};
SeriesSolve::ArbitrarilySetting = "In degree `1` arbitrarily setting `2`.";
SeriesSolve::NoSolution = "No solution in degree `1`.";
SeriesSolve[unknown_, eqns_, opts___] /; Head[unknown] != List :=
  SeriesSolve[{unknown}, eqns, opts];
SeriesSolve[unknowns_List, eqns_, opts___] := Module[{
  arbitrator = Arbitrator /. {opts} /. Options[SeriesSolve],
  msgs, MessageStream, solver, lineqs, gens, vars, sol, fvars, avals, d
},
  If[arbitrator === 0, arbitrator = Replace[#, _ → 0, 1] &];
  msgs = {};
  MessageStream /: Read[MessageStream] := msgs;
  solver[0] = Null;
  solver[n_] := (
    solver[n - 1];
    (#[setter] = Null) & /@ (First /@ unknowns);
    lineqs = eqns[n];
    lineqs = If[Head[lineqs] === And, List @@ lineqs, List@lineqs];
    gens = Union[Cases[lineqs, _LW | _CW | _AW, ∞]];
    lineqs = Flatten[Replace[lineqs,
      Lhs_ == rhs_ ⇒ ((Coefficient[Lhs, #] == Coefficient[rhs, #]) & /@ gens),
      {1}]];
    vars = Union@@((#[n, UndeterminedCoefficients] & /@ unknowns);
    sol = Quiet[Solve[lineqs, vars], {Solve::svars}];
    If[sol === {},
      AppendTo[msgs, {"NoSolution", n}]; Message[SeriesSolve::NoSolution, n]; Abort[],
      (* Else *) sol = sol[[1]];
    fvars = Complement[vars, First /@ sol];
    avals = arbitrator[fvars];
    AppendTo[msgs, {"ArbitrarilySetting", n, Thread[(Hold /@ fvars) → avals]}];
    If[fvars != {}, Message[SeriesSolve::ArbitrarilySetting, n, Thread[fvars → avals]]];
    MapThread[(#1 = #2) &, {fvars, avals}];
    sol /. (Rule → Set);
    (#[n] = #[n]) & /@ (First /@ unknowns);
    (#[setter] = solver) & /@ (First /@ unknowns);
    solver[n] = Null
  ]
];
MessageStream
];

```

## DK (Drinfel'd-Kohno Elements)

```

DK[_ , 0] = 0;
DK /: DK[k_ , x1_] + DK[k_ , x2_] := DK[k , x1 + x2];
DK /: c_ * DK[k_ , x_] := DK[k , Expand[c x]];
DK /: b[DK[k1_ , c_ * x1_LW], DK[k2_ , x2_]] := Expand[c b[DK[k1 , x1], DK[k2 , x2]]];
DK /: b[DK[k1_ , x1_LW], DK[k2_ , c_ * x2_LW]] := Expand[c b[DK[k1 , x1], DK[k2 , x2]]];
DK /: b[DK[k1_ , x1_Plus], DK[k2_ , x2_]] := b[DK[k1 , #], DK[k2 , x2]] & /@ x1;
DK /: b[DK[k1_ , x1_], DK[k2_ , x2_Plus]] := b[DK[k1 , x1], DK[k2 , #]] & /@ x2;
DK /: b[DK[k_ , x1_], DK[k_ , x2_]] := DK[k , b[x1 , x2]];
DK /: b[DK[k1_ , x1_], DK[k2_ , x2_]] /; k1 > k2 := b[DK[k2 , Expand[-x2]], DK[k1 , x1]];
DK /: b[DK[k1_ , LW@i1_], DK[k2_ , LW@i2_]] /; k1 < k2 := b[DK[k1 , LW@i1], DK[k2 , LW@i2]] = Which[
  i1 == i2, DK[k2 , -b[LW@k1 , LW@i2]],
  k1 == i2, DK[k2 , -b[LW@i1 , LW@i2]],
  True, 0
];
DK /: b[DK[k1_ , w1_LW], DK[k2_ , w2_LW]] /; Deg[w1] > 1 := b[DK[k1 , w1], DK[k2 , w2]] = Module[{x , y},
  {x , y} = LyndonFactorization[w1];
  b[b[DK[k1 , x], DK[k2 , w2]], DK[k1 , y]] + b[DK[k1 , x], b[DK[k1 , y], DK[k2 , w2]]]
];
DK /: b[DK[k1_ , w1_LW], DK[k2_ , w2_LW]] /; Deg[w2] > 1 := b[DK[k1 , w1], DK[k2 , w2]] = Module[{x , y},
  {x , y} = LyndonFactorization[w2];
  b[b[DK[k1 , w1], DK[k2 , x]], DK[k2 , y]] + b[DK[k2 , x], b[DK[k1 , w1], DK[k2 , y]]]
];
t[i_ , j_] := DK[Max[i , j], LW@Min[i , j]];
TopBracketForm[DK[k_ , x_]] := x // LieMorphism[Table[LW@i → LW@t10 i+k, {i , k - 1}]];

```

 $\sigma$ 

```

σ /: exprs-σ := expr // s;
σ[lft___ , i_Integer , rgt___] := σ[lft , IntegerDigits[i] , rgt];
_σ[0] = 0;
x_Plus // s_σ := s[#] & /@ x;
DK[k_ , x_Plus] // s_σ := s[DK[k , #]] & /@ x;
DK[k_ , c_ * w_LW] // s_σ := Expand[c * s[DK[k , w]]];
DK[k_ , LW@i_] // s_σ := Sum[t[α , β], {α , s[[i]]}, {β , s[[k]]}];
DK[k_ , w_LW] // s_σ := b@@(s[DK[k , #]] & /@ LyndonFactorization[w]);

```

## DKSeries

```

DKSeries[ser_Symbol][{dd_Integer}] := TopBracketForm[Append[
  DKS @@ Table[
    ser[d] /. DK[k_, x_] => (x // LieMorphism[Table[LW@i -> LW@t10i+k, {i, k-1}]]@d,
    {d, dd}
  ],
  "..."];
DKSeries[ser_Symbol][e___] := ser[e];
b[dk1_DKSeries, dk2_DKSeries] := b[dk1, dk2] = New[DKSeries[ser],
  ser[d_Integer] := ser[d] = Sum[b[dk1[j], dk2[d-j]]
    {j, 1, d-1}
];
DKS[dk_Series] := dks;
DKS[expr_] := DKS[expr] = New[DKSeries[ser],
  ser[d_Integer] := ser[d] = Expand[expr /. w_LW /; Deg[w] ≠ d -> 0]
];
DKS[k_Integer, coefs_] := New[DKSeries[ser],
  ser[setter] = Null;
  ser[d_Integer, UndeterminedCoefficients] := Cases[
    Join @@ Table[coefs[j, Sequence @@ #] & /@ AllLyndonWords[d, Range[j-1]], {j, 2, k}],
    _coefs];
  ser[d_Integer] := If[ser[setter] != Null,
    ser[setter][d]; ser[d],
    Sum[
      Plus @@ ((DK[j, #] * coefs[j, Sequence @@ #]) & /@ AllLyndonWords[d, Range[j-1]]),
      {j, 2, k}
    ]
];
AddDKSeries[ss___DKSeries] := AddDKSeries[ss] = New[DKSeries[ser],
  ser[d_Integer] := ser[d] = Plus @@ ((#[d]) & /@ {ss})
];
DKSeries /: Plus[ss___DKSeries] := AddDKSeries[ss];
ScaleDKSeries[c_, s_DKSeries] := ScaleDKSeries[c, s] = New[DKSeries[ser],
  ser[d_Integer] := ser[d] = Expand[c * s[d]]
];
DKSeries /: c_ * s_DKSeries := ScaleDKSeries[c, s];
s1_DKSeries ≡ s2_DKSeries := New[BooleanSequence[bs],
  bs[0] = True;
  bs[d_Integer] := bs[d-1] &&
    (And @@ Cases[{Expand[s1[d] - s2[d]]}, DK[_ , x_] => (x == 0), ∞]);
];
dks_DKSeries // s_σ := dks // s = New[DKSeries[ser],
  ser[d_Integer] := ser[d] = dks[d] // s
];
(dks_DKSeries)_k_Integer := New[LieSeries[ser],
  ser[d_Integer] := ser[d] = dks[d] /. {DK[k, x_] => x, DK[___] -> 0}
];

```

## Morphisms $LS \rightarrow DKS$

```

Morphism[mor_][es___] := mor[es];
Morphism[LS, DKS, rules__Rule] := Morphism[LS, DKS, {rules}];
Morphism[LS, DKS, rules_List] := New[Morphism[mor],
  mor[Support] = First /@ rules;
  (mor[w_LW] /; Deg[w] == 1) := (mor[w] = w /. rules);
  mor[w_LW] := (mor[w] = b @@ (mor /@ LyndonFactorization[w]));
  mor[expr_][d_] := Expand[expr /. w_LW => mor[w][d]];
  mor[ls_LieSeries] := mor[ls] = New[DKSeries[ser],
    ser[d_] := ser[d] = Sum[mor[ls[k]][d], {k, 1, d}];
  ];
BCH[x_DKSeries, y_DKSeries] := BCH[LW@"x", LW@"y"] // Morphism[LS, DKS, LW@"x" -> x, LW@"y" -> y];
DKSeries /: x_DKSeries ** y_DKSeries := BCH[x, y];

```

## The $\alpha$ Map

```

alphaMap[S_List][dks_DKSeries] := Sum[
  LieMorphism[
    Table[
      LW@S[[i]] -> AngleBracket @@ Table[
        S[[j]] -> LS[Switch[j,
          i, LW@S[[k],
            k, LW@S[[i],
              _, 0
            ]]],
        {j, Length[S]}],
      {i, k - 1}
    ],
    AngleBracket @@ S, tb
  ][dks_S[[k]],
  {k, 2, Length[S]}
];
alphaMap[S___][dks_DKSeries] := alphaMap[{S}][dks];

```

## Formatting

```

FreeLieFormatting[True] := (
  Format[w_LW] := TopBracketForm[w];
  Format[bs_BooleanSequence, StandardForm] := bs[{$SeriesShowDegree}];
  Format[s_LieSeries, StandardForm] := TopBracketForm[s[{$SeriesShowDegree}]];
  Format[s_ASeries, StandardForm] := s[{$SeriesShowDegree}];
  Format[w_CW] := TopBracketForm[w];
  Format[s_CWSeries, StandardForm] := TopBracketForm[s[{$SeriesShowDegree}]];
  Format[dks_DKSeries, StandardForm] := dks[{$SeriesShowDegree}];
);
FreeLieFormatting[False] := (
  Format[w_LW] =.;
  Format[bs_BooleanSequence, StandardForm] =.;
  Format[s_LieSeries, StandardForm] =.;
  Format[s_ASeries, StandardForm] =.;
  Format[w_CW] =.;
  Format[s_CWSeries, StandardForm] =.;
  Format[dks_DKSeries, StandardForm] =.;
);
FreeLieFormatting[True]

```

## Epilog

```
End[]; EndPackage[];
```