

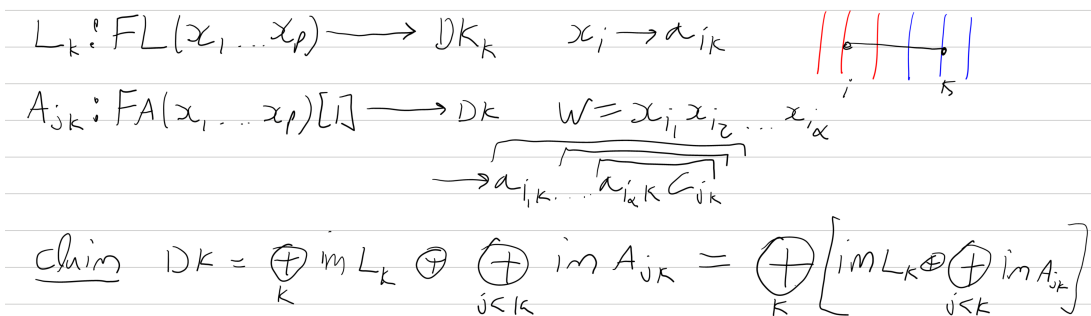
```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FreeLie.m
```

FreeLie` implements / extends  
 {\*, +, \*\*, \$SeriesShowDegree, <>, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords, AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop, cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE, Exp, FreeLieFormatting, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization, Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support, t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ℓ, Δ, σ, τ, ħ, ↦, ↵}.

FreeLie` is in the public domain. Dror Bar-Natan is committed to support it within reason until July 15, 2022. This is version 240218.

```
In[*]:= Antipode[w_AW] := Antipode[w] = (-1)^Length[w] Reverse[w];
Antipode[expr_] := Expand[expr /. w_AW -> Antipode[w]]
```

Goal: Implement the “Emergent Drinfel’d-Kohno Algebra” EDK following



EDK[k\_Integer, x\_FL]: an element in the image of  $L_k$ .

EDK[j\_Integer, k\_Integer, w\_AW]: an element in the image of  $A_{jk}$ .

EDKBasis[ps, s, d] produces the basis of the EDK algebra with poles ps, strands 1,...,s, and at degree d.

```
In[*]:= EDKBasis[ps_List, s_Integer, d_Integer] := Flatten@{
  Table[EDK[j, k, #] & /@ AllWords[d - 1, ps], {j, 1, s - 1}, {k, j + 1, s}],
  Table[EDK[k, #] & /@ AllLyndonWords[d, ps], {k, s}]
}
```

```
In[*]:= EDKBasis[{x, y}, 3, 3]
```

```
Out[*]= {EDK[1, 2, AW[x, x]], EDK[1, 2, AW[x, y]], EDK[1, 2, AW[y, x]], EDK[1, 2, AW[y, y]],
  EDK[1, 3, AW[x, x]], EDK[1, 3, AW[x, y]], EDK[1, 3, AW[y, x]], EDK[1, 3, AW[y, y]],
  EDK[2, 3, AW[x, x]], EDK[2, 3, AW[x, y]], EDK[2, 3, AW[y, x]], EDK[2, 3, AW[y, y]],
  EDK[1, x̄xy], EDK[1, x̄yȳ], EDK[2, x̄xy], EDK[2, x̄yȳ], EDK[3, x̄xy], EDK[3, x̄yȳ]}
```

```
In[*]:= EDK[j_, k_, w_AW] /; j > k := EDK[k, j, Antipode[w]];
```

```
In[*]:= EDK[_, 0] = 0;
EDK /: EDK[jk_, x1_] + EDK[jk_, x2_] := EDK[jk, x1 + x2];
EDK /: c_ * EDK[jk_, x_] := EDK[jk, Expand[c x]];
EDK /: b[EDK[jk1_, c_ * (x1_LW | x1_AW)], EDK[jk2_, x2_]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1_, x1_], EDK[jk2_, c_ * (x2_LW | x2_AW)]] :=
  Expand[c b[EDK[jk1, x1], EDK[jk2, x2]]];
EDK /: b[EDK[jk1_, x1_], EDK[jk2_, x2_Plus]] :=
  b[EDK[jk1, x1], EDK[jk2, #]] & /@ x2;
EDK /: b[EDK[jk1_, x1_Plus], EDK[jk2_, x2_]] := b[EDK[jk1, #], EDK[jk2, x2]] & /@ x1;
```

$$\left[ A_{j_1 k_1}(w_1), A_{j_2 k_2}(w_2) \right] = 0 \text{ obvious}$$

```
In[*]:= EDK /: b[EDK[_, _, _], EDK[_, _, _]] := 0
```

Claim  $[L_{k_1}(w_1), A_{j_2, k_2}(w_2)]$

$k_1 \notin \{j_2, k_2\}$

$[L(w_1), A(w_2)] = 0$

$k_1 = j_2$        $k_1 = k_2$

$[L_{k_1}(w_1), A_{j_2, k_2}(w_2)] = A_{j_2, k_2}(w_2 \bar{w}_1)$

$[L_{k_1}(w_1), A_{j_2, k_2}(w_2)] = A_{j_2, k_2}(w_1 w_2)$

$= -A_{j_2, k_2}(w_2 w_1)$

```

In[ ] := EDK /: b[EDK[k1_, w1_LW], EDK[j2_, k2_, w2_AW]] := Which[
  k1 == j2, -EDK[j2, k2, w2 ** L[w1]],
  k1 == k2, EDK[j2, k2, L[w1] ** w2],
  True, 0
];
EDK /: b[EDK[j2_, k2_, w2_AW], EDK[k1_, w1_LW]] :=
  Expand[-b[EDK[k1, w1], EDK[j2, k2, w2]]]
  
```

```

In[ ] := b[EDK[1, LW[x, y]], EDK[1, 2, AW[z, z, x, y]]]
  
```

```

Out[ ] := EDK[1, 2, -AW[z, z, x, y, x, y] + AW[z, z, x, y, y, x]]
  
```

In[\*]:= Cases[{AW[1], LW[2], Yu[3]}, x\_LW | x\_AW]

Out[\*]=

{AW[1], 2}

Claim  $[L_{k_1}(w_1), L_{k_2}(w_2)]$

$(k_1 < k_2)$

$k_1 = k_2 (=k)$

$[L_k(w_1), L_k(w_2)] = L_k(w_1, w_2)$

$\sum_{i_1, i_2} A_{k_1, k_2}(x_{i_2})$

$k_1 < k_2$

(i)  $[L_{k_1}(x_{i_1}), L_{k_2}(x_{i_2})] = [a_{i_1 k_1}, a_{i_2 k_2}] = \sum_{i_1, i_2} A_{i_1, i_2} [a_{i_2 k_2}, c_{k_1, k_2}]$

(ii)  $[L_{k_1}(w_1, w'_1), L_{k_2}(w_2)]$  should be  $[L_{k_1}(w_1), [L_{k_1}(w'_1), L_{k_2}(w_2)]] - [L_{k_1}(w'_1), [L_{k_1}(w_1), L_{k_2}(w_2)]]$

(iii)  $[L_{k_1}(w_1), L_{k_2}(w_2, w'_2)]$  should be  $-[L_{k_2}(w'_2), [L_{k_1}(w_1), L_{k_2}(w_2)]] + [L_{k_2}(w_2), [L_{k_1}(w_1), L_{k_2}(w'_2)]]$

Prop  $[L_{k_1}(w_1), L_{k_2}(w_2)] = A_{k_1, k_2} \left( \sum_i (a_i w_2) x_i \overline{(a_i w_1)} \right)$

Satisfies the above three conditions

In[\*]:= AllLyndonWords[4, {x, y, z}] /. w\_LW => Union[List@@w]

Out[\*]=

{x, y}, {x, z}, {x, y}, {x, y, z}, {x, y, z}, {x, z}, {x, y, z}, {x, y}, {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, y, z}, {x, z}, {y, z}, {y, z}, {y, z}

```
EDK /: b[EDK[k1_, w1_LW], EDK[k2_, w2_LW]] := Which[
  k1 == k2, EDK[k1, b[w1, w2]],
  k1 < k2, EDK[k1, k2, Sum[tau[LW@p, w2] ** AW[p] ** Antipode[tau[LW@p, w1]],
    {p, Union[List@@w1] ∩ Union[List@@w2]}]],
  k2 < k1, EDK[k2, k1, Sum[-tau[LW@p, w1] ** AW[p] ** Antipode[tau[LW@p, w2]],
    {p, Union[List@@w1] ∩ Union[List@@w2]}]]
]
```

Testing anti-symmetry and Jacobi for EDK:

```

In[*]:= bas = EDKBasis[{x, y, z}, 3, 4]
Union@Table[{α, β} = αβ; b[α, β] + b[β, α], {αβ, Subsets[bas, {2}]}]

Out[*]=
{EDK[1, 2, AW[x, x, x]], EDK[1, 2, AW[x, x, y]], EDK[1, 2, AW[x, x, z]],
EDK[1, 2, AW[x, y, x]], EDK[1, 2, AW[x, y, y]], EDK[1, 2, AW[x, y, z]],
EDK[1, 2, AW[x, z, x]], EDK[1, 2, AW[x, z, y]], EDK[1, 2, AW[x, z, z]],
EDK[1, 2, AW[y, x, x]], EDK[1, 2, AW[y, x, y]], EDK[1, 2, AW[y, x, z]],
EDK[1, 2, AW[y, y, x]], EDK[1, 2, AW[y, y, y]], EDK[1, 2, AW[y, y, z]],
EDK[1, 2, AW[y, z, x]], EDK[1, 2, AW[y, z, y]], EDK[1, 2, AW[y, z, z]],
EDK[1, 2, AW[z, x, x]], EDK[1, 2, AW[z, x, y]], EDK[1, 2, AW[z, x, z]],
EDK[1, 2, AW[z, y, x]], EDK[1, 2, AW[z, y, y]], EDK[1, 2, AW[z, y, z]],
EDK[1, 2, AW[z, z, x]], EDK[1, 2, AW[z, z, y]], EDK[1, 2, AW[z, z, z]],
EDK[1, 3, AW[x, x, x]], EDK[1, 3, AW[x, x, y]], EDK[1, 3, AW[x, x, z]],
EDK[1, 3, AW[x, y, x]], EDK[1, 3, AW[x, y, y]], EDK[1, 3, AW[x, y, z]],
EDK[1, 3, AW[x, z, x]], EDK[1, 3, AW[x, z, y]], EDK[1, 3, AW[x, z, z]],
EDK[1, 3, AW[y, x, x]], EDK[1, 3, AW[y, x, y]], EDK[1, 3, AW[y, x, z]],
EDK[1, 3, AW[y, y, x]], EDK[1, 3, AW[y, y, y]], EDK[1, 3, AW[y, y, z]],
EDK[1, 3, AW[y, z, x]], EDK[1, 3, AW[y, z, y]], EDK[1, 3, AW[y, z, z]],
EDK[1, 3, AW[z, x, x]], EDK[1, 3, AW[z, x, y]], EDK[1, 3, AW[z, x, z]],
EDK[1, 3, AW[z, y, x]], EDK[1, 3, AW[z, y, y]], EDK[1, 3, AW[z, y, z]],
EDK[1, 3, AW[z, z, x]], EDK[1, 3, AW[z, z, y]], EDK[1, 3, AW[z, z, z]],
EDK[2, 3, AW[x, x, x]], EDK[2, 3, AW[x, x, y]], EDK[2, 3, AW[x, x, z]],
EDK[2, 3, AW[x, y, x]], EDK[2, 3, AW[x, y, y]], EDK[2, 3, AW[x, y, z]],
EDK[2, 3, AW[x, z, x]], EDK[2, 3, AW[x, z, y]], EDK[2, 3, AW[x, z, z]],
EDK[2, 3, AW[y, x, x]], EDK[2, 3, AW[y, x, y]], EDK[2, 3, AW[y, x, z]],
EDK[2, 3, AW[y, y, x]], EDK[2, 3, AW[y, y, y]], EDK[2, 3, AW[y, y, z]],
EDK[2, 3, AW[y, z, x]], EDK[2, 3, AW[y, z, y]], EDK[2, 3, AW[y, z, z]],
EDK[2, 3, AW[z, x, x]], EDK[2, 3, AW[z, x, y]], EDK[2, 3, AW[z, x, z]],
EDK[2, 3, AW[z, y, x]], EDK[2, 3, AW[z, y, y]], EDK[2, 3, AW[z, y, z]],
EDK[2, 3, AW[z, z, x]], EDK[2, 3, AW[z, z, y]], EDK[2, 3, AW[z, z, z]],
EDK[1, x x x̄y], EDK[1, x x x̄z], EDK[1, x x̄y y], EDK[1, x x x̄y z],
EDK[1, x x̄z y], EDK[1, x x̄z z], EDK[1, x̄y x̄z], EDK[1, x̄y y y], EDK[1, x y ȳz],
EDK[1, x ȳz y], EDK[1, x ȳz z], EDK[1, x̄z y y], EDK[1, x̄z ȳz], EDK[1, x̄z z y],
EDK[1, x̄z z z], EDK[1, y y ȳz], EDK[1, y ȳz z], EDK[1, ȳz z z], EDK[2, x x x̄y],
EDK[2, x x x̄z], EDK[2, x x̄y y], EDK[2, x x x̄y z], EDK[2, x x̄z y], EDK[2, x x̄z z],
EDK[2, x̄y x̄z], EDK[2, x̄y y y], EDK[2, x y ȳz], EDK[2, x ȳz y], EDK[2, x ȳz z],
EDK[2, x̄z y y], EDK[2, x̄z ȳz], EDK[2, x̄z z y], EDK[2, x̄z z z], EDK[2, y y ȳz],
EDK[2, y ȳz z], EDK[2, ȳz z z], EDK[3, x x x̄y], EDK[3, x x x̄z], EDK[3, x x̄y y],

```



EDK[1,  $\overline{xz z z}$ ], EDK[1,  $\overline{y y y z}$ ], EDK[1,  $\overline{y y z z}$ ], EDK[1,  $\overline{y z z z}$ ], EDK[2,  $\overline{x x x y}$ ],  
 EDK[2,  $\overline{x x x z}$ ], EDK[2,  $\overline{x x y y}$ ], EDK[2,  $\overline{x x y z}$ ], EDK[2,  $\overline{x x z y}$ ], EDK[2,  $\overline{x x z z}$ ],  
 EDK[2,  $\overline{x y x z}$ ], EDK[2,  $\overline{x y y y}$ ], EDK[2,  $\overline{x y y z}$ ], EDK[2,  $\overline{x y z y}$ ], EDK[2,  $\overline{x y z z}$ ],  
 EDK[2,  $\overline{x z y y}$ ], EDK[2,  $\overline{x z y z}$ ], EDK[2,  $\overline{x z z y}$ ], EDK[2,  $\overline{x z z z}$ ], EDK[2,  $\overline{y y y z}$ ],  
 EDK[2,  $\overline{y y z z}$ ], EDK[2,  $\overline{y z z z}$ ], EDK[3,  $\overline{x x x y}$ ], EDK[3,  $\overline{x x x z}$ ], EDK[3,  $\overline{x x y y}$ ],  
 EDK[3,  $\overline{x x y z}$ ], EDK[3,  $\overline{x x z y}$ ], EDK[3,  $\overline{x x z z}$ ], EDK[3,  $\overline{x y x z}$ ], EDK[3,  $\overline{x y y y}$ ],  
 EDK[3,  $\overline{x y y z}$ ], EDK[3,  $\overline{x y z y}$ ], EDK[3,  $\overline{x y z z}$ ], EDK[3,  $\overline{x z y y}$ ], EDK[3,  $\overline{x z y z}$ ],  
 EDK[3,  $\overline{x z z y}$ ], EDK[3,  $\overline{x z z z}$ ], EDK[3,  $\overline{y y y z}$ ], EDK[3,  $\overline{y y z z}$ ], EDK[3,  $\overline{y z z z}$ ]

Out[ ] =  
 {0}

$\sigma$

$\Phi^{\sigma[1,23,4]} \rightarrow \Phi^{\sigma[\{1\},\{2,3\},\{4\}]} \rightarrow \sigma[\{1\}, \{2, 3\}, \{4\}] [\Phi]$

$\Phi^{\sigma[2,3,1]}$

$\Phi^{\sigma[1,x23,4]}$

$\Delta[i \rightarrow jk]$

EDK[j1, sum of Lie words] + EDK[j2, sum of Lie words] +  
 EDK[j, k, sum of Associative words] +  
 p2s

```
In[ ] :=  $\sigma$  /:  $\mathcal{E}^{\mathcal{S}-\sigma} := s[\mathcal{E}]$ ;
 $\sigma$ [lft___, i_Integer, rgt___] :=  $\sigma$ [lft, IntegerDigits[i], rgt];
 $\sigma$ [___][0] = 0;
x_Plus //  $s_{-\sigma} := s[\#] \& /@ x$ ;
EDK[jk_, x_Plus] //  $s_{-\sigma} := s$ [EDK[jk, #]] & /@ x;
EDK[jk_, c_ * (w_LW | w_AW)] //  $s_{-\sigma} := \text{Expand}[c * s$ [EDK[jk, w]]];
EDK[k_, LW[x_]] //  $s_{-\sigma} := \text{Sum}$ [EDK[ $\alpha$ , LW[x]], { $\alpha$ , s[[k]]}];
EDK[k_, w_LW] //  $s_{-\sigma} := b@@(s$ [EDK[k, #]] & /@ LyndonFactorization[w]);
EDK[j_, k_, AW[]] //  $s_{-\sigma} := \text{Sum}$ [EDK[ $\alpha$ ,  $\beta$ , AW[]], { $\alpha$ , s[[j]]}, { $\beta$ , s[[k]]}];
EDK[j_, k_, AW[x_, w___]] //  $s_{-\sigma} := b$ [s@EDK[k, LW@x], s@EDK[j, k, AW[w]]]
```

```
In[ ] := bas = EDKBasis[{x, y}, 1, 1];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12]}$ , { $\alpha$ , bas}]
```

Out[ ] =  
 EDK[1,  $\overline{x}$ ]  $\rightarrow$  EDK[1,  $\overline{x}$ ] + EDK[2,  $\overline{x}$ ]  
 EDK[1,  $\overline{y}$ ]  $\rightarrow$  EDK[1,  $\overline{y}$ ] + EDK[2,  $\overline{y}$ ]

```
In[*]:= bas = EDKBasis[{x, y}, 2, 1];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[]]  $\rightarrow$  EDK[1, 3, AW[]] + EDK[2, 3, AW[]]
EDK[1,  $\overline{x}$ ]  $\rightarrow$  EDK[1,  $\overline{x}$ ] + EDK[2,  $\overline{x}$ ]
EDK[1,  $\overline{y}$ ]  $\rightarrow$  EDK[1,  $\overline{y}$ ] + EDK[2,  $\overline{y}$ ]
EDK[2,  $\overline{x}$ ]  $\rightarrow$  EDK[3,  $\overline{x}$ ]
EDK[2,  $\overline{y}$ ]  $\rightarrow$  EDK[3,  $\overline{y}$ ]
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 2];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[x]]  $\rightarrow$  EDK[1, 3, AW[x]] + EDK[2, 3, AW[x]]
EDK[1, 2, AW[y]]  $\rightarrow$  EDK[1, 3, AW[y]] + EDK[2, 3, AW[y]]
EDK[1,  $\overline{xy}$ ]  $\rightarrow$  EDK[1,  $\overline{xy}$ ] + EDK[2,  $\overline{xy}$ ]
EDK[2,  $\overline{xy}$ ]  $\rightarrow$  EDK[3,  $\overline{xy}$ ]
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 3];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[12,3]}$ , { $\alpha$ , bas}]
```

```
Out[*]=
```

```
EDK[1, 2, AW[x, x]]  $\rightarrow$  EDK[1, 3, AW[x, x]] + EDK[2, 3, AW[x, x]]
EDK[1, 2, AW[x, y]]  $\rightarrow$  EDK[1, 3, AW[x, y]] + EDK[2, 3, AW[x, y]]
EDK[1, 2, AW[y, x]]  $\rightarrow$  EDK[1, 3, AW[y, x]] + EDK[2, 3, AW[y, x]]
EDK[1, 2, AW[y, y]]  $\rightarrow$  EDK[1, 3, AW[y, y]] + EDK[2, 3, AW[y, y]]
EDK[1,  $x \overline{xy}$ ]  $\rightarrow$  EDK[1,  $x \overline{xy}$ ] + EDK[2,  $x \overline{xy}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[1,  $\overline{xy}y$ ]  $\rightarrow$  EDK[1,  $\overline{xy}y$ ] + EDK[2,  $\overline{xy}y$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[2,  $x \overline{xy}$ ]  $\rightarrow$  EDK[3,  $x \overline{xy}$ ]
EDK[2,  $\overline{xy}y$ ]  $\rightarrow$  EDK[3,  $\overline{xy}y$ ]
```



```
In[ ]:= bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → ασ[12,3], {α, bas}]
```

Out[ ]:=

$$\begin{aligned}
 & \text{EDK}[1, 2, \text{AW}[x, x, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, x, x]] + \text{EDK}[2, 3, \text{AW}[x, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, x, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, x, y]] + \text{EDK}[2, 3, \text{AW}[x, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, y, x]] + \text{EDK}[2, 3, \text{AW}[x, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[x, y, y]] + \text{EDK}[2, 3, \text{AW}[x, y, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, x, x]] + \text{EDK}[2, 3, \text{AW}[y, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, x, y]] + \text{EDK}[2, 3, \text{AW}[y, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, x]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, y, x]] + \text{EDK}[2, 3, \text{AW}[y, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, y]] \rightarrow \text{EDK}[1, 3, \text{AW}[y, y, y]] + \text{EDK}[2, 3, \text{AW}[y, y, y]] \\
 & \text{EDK}\left[1, x \overline{x \overline{xy}}\right] \rightarrow \text{EDK}\left[1, x \overline{x \overline{xy}}\right] + \text{EDK}\left[2, x \overline{x \overline{xy}}\right] + \text{EDK}[1, 2, -2 \text{AW}[x, x, y] + 2 \text{AW}[y, x, x]] \\
 & \text{EDK}\left[1, x \overline{xyy}\right] \rightarrow \text{EDK}\left[1, x \overline{xyy}\right] + \text{EDK}\left[2, x \overline{xyy}\right] + \\
 & \quad \text{EDK}[1, 2, -\text{AW}[x, x, y] - \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, y, x]] \\
 & \text{EDK}\left[1, \overline{xyy}y\right] \rightarrow \text{EDK}\left[1, \overline{xyy}y\right] + \text{EDK}\left[2, \overline{xyy}y\right] + \text{EDK}[1, 2, -2 \text{AW}[x, y, y] + 2 \text{AW}[y, y, x]] \\
 & \text{EDK}\left[2, x \overline{x \overline{xy}}\right] \rightarrow \text{EDK}\left[3, x \overline{x \overline{xy}}\right] \\
 & \text{EDK}\left[2, x \overline{xyy}\right] \rightarrow \text{EDK}\left[3, x \overline{xyy}\right] \\
 & \text{EDK}\left[2, \overline{xyy}y\right] \rightarrow \text{EDK}\left[3, \overline{xyy}y\right]
 \end{aligned}$$

```
In[ ]:= bas = EDKBasis[{x, y}, 2, 4];
Column@Table[α → ασ[1,23], {α, bas}]
```

Out[ ]:=

$$\begin{aligned}
 & \text{EDK}[1, 2, \text{AW}[x, x, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, x, x]] + \text{EDK}[1, 3, \text{AW}[x, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, x, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, x, y]] + \text{EDK}[1, 3, \text{AW}[x, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, y, x]] + \text{EDK}[1, 3, \text{AW}[x, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[x, y, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[x, y, y]] + \text{EDK}[1, 3, \text{AW}[x, y, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, x, x]] + \text{EDK}[1, 3, \text{AW}[y, x, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, x, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, x, y]] + \text{EDK}[1, 3, \text{AW}[y, x, y]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, x]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, y, x]] + \text{EDK}[1, 3, \text{AW}[y, y, x]] \\
 & \text{EDK}[1, 2, \text{AW}[y, y, y]] \rightarrow \text{EDK}[1, 2, \text{AW}[y, y, y]] + \text{EDK}[1, 3, \text{AW}[y, y, y]] \\
 & \text{EDK}\left[1, x \overline{x \overline{xy}}\right] \rightarrow \text{EDK}\left[1, x \overline{x \overline{xy}}\right] \\
 & \text{EDK}\left[1, x \overline{xyy}\right] \rightarrow \text{EDK}\left[1, x \overline{xyy}\right] \\
 & \text{EDK}\left[1, \overline{xyy}y\right] \rightarrow \text{EDK}\left[1, \overline{xyy}y\right] \\
 & \text{EDK}\left[2, x \overline{x \overline{xy}}\right] \rightarrow \text{EDK}\left[2, x \overline{x \overline{xy}}\right] + \text{EDK}\left[3, x \overline{x \overline{xy}}\right] + \text{EDK}[2, 3, -2 \text{AW}[x, x, y] + 2 \text{AW}[y, x, x]] \\
 & \text{EDK}\left[2, x \overline{xyy}\right] \rightarrow \text{EDK}\left[2, x \overline{xyy}\right] + \text{EDK}\left[3, x \overline{xyy}\right] + \\
 & \quad \text{EDK}[2, 3, -\text{AW}[x, x, y] - \text{AW}[x, y, y] + \text{AW}[y, x, x] + \text{AW}[y, y, x]] \\
 & \text{EDK}\left[2, \overline{xyy}y\right] \rightarrow \text{EDK}\left[2, \overline{xyy}y\right] + \text{EDK}\left[3, \overline{xyy}y\right] + \text{EDK}[2, 3, -2 \text{AW}[x, y, y] + 2 \text{AW}[y, y, x]]
 \end{aligned}$$

```

In[*]:= bas = EDKBasis[{x, y}, 2, 2];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[3,12]}$ , { $\alpha$ , bas}]

Out[*]=
EDK[1, 2, AW[x]]  $\rightarrow$  EDK[1, 3, -AW[x]] + EDK[2, 3, -AW[x]]
EDK[1, 2, AW[y]]  $\rightarrow$  EDK[1, 3, -AW[y]] + EDK[2, 3, -AW[y]]
EDK[1,  $\overline{xy}$ ]  $\rightarrow$  EDK[3,  $\overline{xy}$ ]
EDK[2,  $\overline{xy}$ ]  $\rightarrow$  EDK[1,  $\overline{xy}$ ] + EDK[2,  $\overline{xy}$ ]

In[*]:= bas = EDKBasis[{x, y}, 2, 3];
Column@Table[ $\alpha \rightarrow \alpha^{\sigma[3,12]}$ , { $\alpha$ , bas}]

Out[*]=
EDK[1, 2, AW[x, x]]  $\rightarrow$  EDK[1, 3, AW[x, x]] + EDK[2, 3, AW[x, x]]
EDK[1, 2, AW[x, y]]  $\rightarrow$  EDK[1, 3, AW[y, x]] + EDK[2, 3, AW[y, x]]
EDK[1, 2, AW[y, x]]  $\rightarrow$  EDK[1, 3, AW[x, y]] + EDK[2, 3, AW[x, y]]
EDK[1, 2, AW[y, y]]  $\rightarrow$  EDK[1, 3, AW[y, y]] + EDK[2, 3, AW[y, y]]
EDK[1,  $\overline{x \overline{xy}}$ ]  $\rightarrow$  EDK[3,  $\overline{x \overline{xy}}$ ]
EDK[1,  $\overline{xyy}$ ]  $\rightarrow$  EDK[3,  $\overline{xyy}$ ]
EDK[2,  $\overline{x \overline{xy}}$ ]  $\rightarrow$  EDK[1,  $\overline{x \overline{xy}}$ ] + EDK[2,  $\overline{x \overline{xy}}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]
EDK[2,  $\overline{xyy}$ ]  $\rightarrow$  EDK[1,  $\overline{xyy}$ ] + EDK[2,  $\overline{xyy}$ ] + EDK[1, 2, -AW[x, y] - AW[y, x]]

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[12, 3]$ ;
Column@
DeleteCases[_  $\rightarrow$  0][Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

Out[*]=

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[1, 23]$ ;
Column@
DeleteCases[_  $\rightarrow$  0][Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

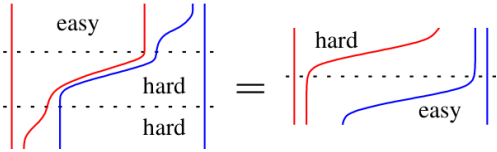
Out[*]=

In[*]:= bas1 = EDKBasis[{x, y}, 2, 4];
bas2 = EDKBasis[{x, y}, 2, 4];
s =  $\sigma[23, 1]$ ;
Column@
DeleteCases[_  $\rightarrow$  0][Flatten@Table[ $\{\alpha, \beta\} \rightarrow \mathbf{b}[\alpha, \beta]^s - \mathbf{b}[\alpha^s, \beta^s]$ , { $\alpha$ , bas1}, { $\beta$ , bas2}]]

Out[*]=

```

The Full Pole-Strand “Operad”.



Goal: Implement the pentagon,

$$\Phi ** \Phi^{\sigma[y \rightarrow y+1, 2]} ** \Phi^{\sigma[x \rightarrow y, y \rightarrow 1, 2]} == \Phi^{\sigma[x \rightarrow x+y, y \rightarrow 1, 2]} ** \Phi^{\sigma[12]}$$

A more modest goal: Implement  $\text{ps}\Delta_s[y]$ , which shifts all the strands (1..s) one unit to the right and creates a new strand 1 by doubling pole  $y$ .

The pentagon will now be:

$$\Phi ** \text{ps}\Delta[y][\Phi] ** \dots$$

```
In[*]:= psDelta[_][0] = 0;
x_Plus // psDelta[y_] := psDelta[y] /@ x;
EDK[k_, x_] // psDelta[y_] := EDK[k+1, x] + EDK[1, k+1, tau[LW@y, x]];
EDK[j_, k_, x_] // psDelta[_] := EDK[j+1, k+1, x];
```

```
In[*]:= bas = EDKBasis[{x, y}, 2, 3];
Table[alpha -> psDelta[y][alpha], {alpha, bas}] // Column
```

```
Out[*]=
EDK[1, 2, AW[x, x]] -> EDK[2, 3, AW[x, x]]
EDK[1, 2, AW[x, y]] -> EDK[2, 3, AW[x, y]]
EDK[1, 2, AW[y, x]] -> EDK[2, 3, AW[y, x]]
EDK[1, 2, AW[y, y]] -> EDK[2, 3, AW[y, y]]
EDK[1, x Overline[x y]] -> EDK[2, x Overline[x y]] + EDK[1, 2, AW[x, x]]
EDK[1, x Overline[y y]] -> EDK[2, x Overline[y y]] + EDK[1, 2, AW[x, y] - 2 AW[y, x]]
EDK[2, x Overline[x y]] -> EDK[3, x Overline[x y]] + EDK[1, 3, AW[x, x]]
EDK[2, x Overline[y y]] -> EDK[3, x Overline[y y]] + EDK[1, 3, AW[x, y] - 2 AW[y, x]]
```

```
In[*]:= bas1 = EDKBasis[{x, y}, 2, 3];
bas2 = EDKBasis[{x, y}, 2, 4];
Column@DeleteCases[_ -> 0][Flatten@
  Table[{alpha, beta} -> psDelta[y][b[alpha, beta]] - b[psDelta[y][alpha], psDelta[y][beta]], {alpha, bas1}, {beta, bas2}]]
```

```
Out[*]=
```

```

In[*]:= LM[___][0] = 0;
x_Plus // Lm_LM := Lm /@ x;
Lm_LM[c_w_LW] := Expand[c Lm[w]];
Lm_LM[c_w_AW] := Expand[c Lm[w]];
Lm_LM[EDK[jk___, x_]] := EDK[jk, Expand[Lm[x]]];
LM[rules___][LW[w]] := Distribute[LW[w] /. {rules}] /. LW[0] -> 0;
LM[rules___][w_LW] :=
  (LM[rules][w] = b @@ (LM[rules] /@ LyndonFactorization[w]));
LM[rules___][AW[]] = AW[];
LM[rules___][AW[w]] := Distribute[AW[w] /. {rules}] /. AW[0] -> 0;
LM[rules___][w_AW] := LM[rules][w] = Module[{w1, w2},
  w1 = Take[w, Floor[Length[w] / 2]];
  w2 = Drop[w, Floor[Length[w] / 2]];
  LM[rules][w1] ** LM[rules][w2]
];

```

```
In[*]:= LM[x -> 0][LW[x, y]]
```

```
Out[*]=
0
```

```
In[*]:= LM[x -> x + y, y -> z][LW[x]]
```

```
Out[*]=
 $\overline{x + y}$ 
```

```
In[*]:= LM[x -> x + y, y -> z][LW[y]]
```

```
Out[*]=
 $\overline{z}$ 
```

```
In[*]:= LM[x -> x + y, y -> z][LW[x, y]]
```

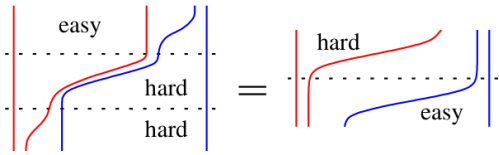
```
Out[*]=
 $\overline{xz} + \overline{yz}$ 
```

```
In[*]:= LM[x -> x + y, y -> z][AW[x, x, y, x]]
```

```
Out[*]=
AW[x, x, z, x] + AW[x, x, z, y] + AW[x, y, z, x] + AW[x, y, z, y] +
  AW[y, x, z, x] + AW[y, x, z, y] + AW[y, y, z, x] + AW[y, y, z, y]
```

```
In[*]:= AW[x, y, x] // LM[x -> x + y, y -> z] // LM[z -> z1 + z2]
```

```
Out[*]=
AW[x, z1, x] + AW[x, z1, y] + AW[x, z2, x] + AW[x, z2, y] +
  AW[y, z1, x] + AW[y, z1, y] + AW[y, z2, x] + AW[y, z2, y]
```



$$\Phi ** \Phi^{\sigma[y \rightarrow y+1, 2]} ** \Phi^{\sigma[x \rightarrow y, y \rightarrow 1, 2]} == \Phi^{\sigma[x \rightarrow x+y, y \rightarrow 1, 2]} ** \Phi^{\sigma[12]}$$

```
In[ ]:= bas = EDKBasis[{x, y}, 1, 5];
```

```
Table[α → (α // psΔ[y] // LM[x → y, y → 0]), {α, bas}]
```

Out[ ]:=

$$\{ \text{EDK} \left[ 1, x \overline{x \overline{x \overline{xy}}} \right] \rightarrow \text{EDK} [1, 2, \text{AW}[y, y, y, y]], \text{EDK} \left[ 1, x \overline{x \overline{xy} y} \right] \rightarrow \emptyset, \\ \text{EDK} \left[ 1, x \overline{xy \overline{xy}} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, x \overline{xy y y} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, \overline{xy \overline{xy} y} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, \overline{xy y y y} \right] \rightarrow \emptyset \}$$

```
In[ ]:= bas = EDKBasis[{x, y}, 1, 5];
```

```
Table[α → (α // psΔ[y] // LM[x → x+y, y → 0]), {α, bas}]
```

Out[ ]:=

$$\{ \text{EDK} \left[ 1, x \overline{x \overline{x \overline{xy}}} \right] \rightarrow \\ \text{EDK} [1, 2, \text{AW}[x, x, x, x] + \text{AW}[x, x, x, y] + \text{AW}[x, x, y, x] + \text{AW}[x, x, y, y] + \text{AW}[x, y, x, x] + \\ \text{AW}[x, y, x, y] + \text{AW}[x, y, y, x] + \text{AW}[x, y, y, y] + \text{AW}[y, x, x, x] + \text{AW}[y, x, x, y] + \text{AW}[y, x, \\ y, x] + \text{AW}[y, x, y, y] + \text{AW}[y, y, x, x] + \text{AW}[y, y, x, y] + \text{AW}[y, y, y, x] + \text{AW}[y, y, y, y]], \\ \text{EDK} \left[ 1, x \overline{x \overline{xy} y} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, x \overline{xy \overline{xy}} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, x \overline{xy y y} \right] \rightarrow \emptyset, \\ \text{EDK} \left[ 1, \overline{xy \overline{xy} y} \right] \rightarrow \emptyset, \text{EDK} \left[ 1, \overline{xy y y y} \right] \rightarrow \emptyset \}$$

```
In[ ]:= bas = EDKBasis[{x, y}, 1, 3]
```

Out[ ]:=

$$\{ \text{EDK} \left[ 1, x \overline{xy} \right], \text{EDK} \left[ 1, \overline{xy} y \right] \}$$

In[\*]:= Table[ $\alpha \rightarrow$  Column@{ $\alpha$ ,  $\alpha$  // ps $\Delta$ [y],  $\alpha$  // ps $\Delta$ [y] // LM[x  $\rightarrow$  y, y  $\rightarrow$   $\theta$ ],  
 $\alpha$  // ps $\Delta$ [y] // LM[x  $\rightarrow$  x + y, y  $\rightarrow$   $\theta$ ],  $\alpha^{\sigma[12]}$ }, { $\alpha$ , bas}] // Column

Out[\*]=

$$\begin{aligned}
 & \text{EDK} \left[ 1, \overline{x \ xy} \right] \\
 \text{EDK} \left[ 1, \overline{x \ xy} \right] & \rightarrow \text{EDK} \left[ 2, \overline{x \ xy} \right] + \text{EDK} [1, 2, \text{AW}[x, x]] \\
 & \text{EDK} [1, 2, \text{AW}[y, y]] \\
 & \text{EDK} [1, 2, \text{AW}[x, x] + \text{AW}[x, y] + \text{AW}[y, x] + \text{AW}[y, y]] \\
 & \text{EDK} \left[ 1, \overline{x \ xy} \right] + \text{EDK} \left[ 2, \overline{x \ xy} \right] + \text{EDK} [1, 2, -\text{AW}[x, y] - \text{AW}[y, x]] \\
 & \text{EDK} \left[ 1, \overline{xy \ y} \right] \\
 \text{EDK} \left[ 1, \overline{xy \ y} \right] & \rightarrow \text{EDK} \left[ 2, \overline{xy \ y} \right] + \text{EDK} [1, 2, \text{AW}[x, y] - 2 \text{AW}[y, x]] \\
 & \theta \\
 & \theta \\
 & \text{EDK} \left[ 1, \overline{xy \ y} \right] + \text{EDK} \left[ 2, \overline{xy \ y} \right] + \text{EDK} [1, 2, -\text{AW}[x, y] - \text{AW}[y, x]]
 \end{aligned}$$