Pensieve Header: Testing EmergentChordDiagrams.nb

In[∘]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< EmergentChordDiagrams.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, ⟨⟩, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ℏ, ⌐, ⌢}.
```

```
FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.
```

```
AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.
```

```
AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

Bases

In[∘]:= $\mathrm{Basis}_2[\mathbb{O}_{AR,\{x,y\},\{1,2\}}]$

Out[∘]=

$$\{\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[]\,AW_2[x,x]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[]\,AW_2[x,y]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[]\,AW_2[y,x]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[]\,AW_2[y,y]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[x]\,AW_2[x]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[x]\,AW_2[y]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[y]\,AW_2[x]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[y]\,AW_2[y]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[x,x]\,AW_2[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[x,y]\,AW_2[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[y,x]\,AW_2[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_0[AW_1[y,y]\,AW_2[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[]\,AW_2[]\,AW_{\bar{1}}[x]\,AW_{\tilde{1}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[]\,AW_2[]\,AW_{\bar{1}}[y]\,AW_{\tilde{1}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[]\,AW_2[x]\,AW_{\bar{1}}[]\,AW_{\tilde{1}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[]\,AW_2[y]\,AW_{\bar{1}}[]\,AW_{\tilde{1}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[x]\,AW_2[]\,AW_{\bar{1}}[]\,AW_{\tilde{1}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1]}[AW_1[y]\,AW_2[]\,AW_{\bar{1}}[]\,AW_{\tilde{1}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[]\,AW_2[]\,AW_{\bar{2}}[x]\,AW_{\tilde{2}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[]\,AW_2[]\,AW_{\bar{2}}[y]\,AW_{\tilde{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[]\,AW_2[x]\,AW_{\bar{2}}[]\,AW_{\tilde{2}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[]\,AW_2[y]\,AW_{\bar{2}}[]\,AW_{\tilde{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[x]\,AW_2[]\,AW_{\bar{2}}[]\,AW_{\tilde{2}}[]]],\ \mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[2]}[AW_1[y]\,AW_2[]\,AW_{\bar{2}}[]\,AW_{\tilde{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[]\,AW_2[]\,AW_{\bar{1}}[x]\,AW_{\bar{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[]\,AW_2[]\,AW_{\bar{1}}[y]\,AW_{\bar{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[]\,AW_2[x]\,AW_{\bar{1}}[]\,AW_{\bar{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[]\,AW_2[y]\,AW_{\bar{1}}[]\,AW_{\bar{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[x]\,AW_2[]\,AW_{\bar{1}}[]\,AW_{\bar{2}}[]]],$$
$$\mathbb{O}_{AR,\{x,y\},\{1,2\}}[\mathcal{A}_{c[1,2]}[AW_1[y]\,AW_2[]\,AW_{\bar{1}}[]\,AW_{\bar{2}}[]]]\}$$

AR: Reduction in A

```
In[∘]:= D1 = 𝕆_{AR,{x,y,z},{1,2}}[
           𝒜_0[AW_1[x, y, x] AW_2[x, x, y]] +
            𝒜_{c[1,2]}[AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[z] AW_{\bar{2}}[x, y]]
          ] // CF
Out[∘]=
        𝕆_{AR,{x,y,z},{1,2}}[𝒜_0[AW_1[x, y, x] AW_2[x, x, y]] +
          𝒜_{c[1,2]}[AW_1[x, y, x, y] AW_2[y, x] AW_{\bar{1}}[z] AW_{\bar{2}}[] + AW_1[x, y, y] AW_2[y, x, x] AW_{\bar{1}}[z] AW_{\bar{2}}[] +
            AW_1[x, y, x] AW_2[y, x, y] AW_{\bar{1}}[z] AW_{\bar{2}}[] + AW_1[x, y] AW_2[y, x, x, y] AW_{\bar{1}}[z] AW_{\bar{2}}[] -
            AW_1[x, y, y] AW_2[y, x] AW_{\bar{1}}[x, z] AW_{\bar{2}}[] - AW_1[x, y] AW_2[y, x, y] AW_{\bar{1}}[x, z] AW_{\bar{2}}[] -
            AW_1[x, y, x] AW_2[y, x] AW_{\bar{1}}[y, z] AW_{\bar{2}}[] - AW_1[x, y] AW_2[y, x, x] AW_{\bar{1}}[y, z] AW_{\bar{2}}[] +
            AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[y, x, z] AW_{\bar{2}}[]]]
```

HR: Reduction in the H Quotient

```
In[∘]:= D2 = 𝕆_{HR,{x,y,z},{1,2}}[
           𝒜_0[AW_1[x, y, x] AW_2[x, x, y]] +
            𝒜_{c[1,2]}[AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[z] AW_{\bar{2}}[x, y]]
          ] // CF
Out[∘]=
        𝕆_{HR,{x,y,z},{1,2}}[𝒜_0[AW_1[x, y, x] AW_2[x, x, y]] + 𝒜_{c[1,2]}[AW_1[x, y, x, y] AW_2[y, x, z] AW_{\bar{1}}[] AW_{\bar{2}}[]]]
```

Reordering strands

```
In[∘]:= 𝕆_{AR,{x,y},{1,2}}[𝒜_0[AW_1[x, y, y] AW_2[x]]] // 𝕆_{2,1}
Out[∘]=
        𝕆_{AR,{x,y},{2,1}}[
         𝒜_0[AW_1[x, y, y] AW_2[x]] + 𝒜_{c[2,1]}[-AW_1[x, y, y] AW_2[] AW_{\bar{1}}[] AW_{\bar{2}}[] + AW_1[y, y] AW_2[x] AW_{\bar{1}}[] AW_{\bar{2}}[] -
            2 AW_1[x, y] AW_2[y] AW_{\bar{1}}[] AW_{\bar{2}}[] + 2 AW_1[y] AW_2[x, y] AW_{\bar{1}}[] AW_{\bar{2}}[] -
            AW_1[x] AW_2[y, y] AW_{\bar{1}}[] AW_{\bar{2}}[] + AW_1[] AW_2[x, y, y] AW_{\bar{1}}[] AW_{\bar{2}}[] +
            2 AW_1[x, y] AW_2[] AW_{\bar{1}}[] AW_{\bar{2}}[y] - 2 AW_1[y] AW_2[x] AW_{\bar{1}}[] AW_{\bar{2}}[y] + 2 AW_1[x] AW_2[y] AW_{\bar{1}}[] AW_{\bar{2}}[y] -
            2 AW_1[] AW_2[x, y] AW_{\bar{1}}[] AW_{\bar{2}}[y] - AW_1[x] AW_2[] AW_{\bar{1}}[] AW_{\bar{2}}[y, y] + AW_1[] AW_2[x] AW_{\bar{1}}[] AW_{\bar{2}}[y, y]]]
```

```
In[∘]:= Total@Table[B == (B // 𝕆_{2,1,3} // 𝕆_{1,2,3}), {B, Basis_3[𝕆_{AR,{x,y},{1,2,3}}]}]
Out[∘]=
        320 True
```

```
In[∘]:= Total@Table[B == (B // 𝕆_{2,1,3} // 𝕆_{1,2,3}), {B, Basis_4[𝕆_{AR,{x,y},{1,2,3}}]}]
Out[∘]=
        1200 True
```

```
In[∘]:= Total@Table[B == (B // 𝕆_{2,1,3} // 𝕆_{1,2,3}), {B, Basis_5[𝕆_{AR,{x,y},{1,2,3}}]}]
Out[∘]=
        4032 True
```

Associativity of Strand Multiplication

```
In[∘]:= Total@Table[(B // sm_{1,2→1} // sm_{1,3→1}) == (B // sm_{2,3→2} // sm_{1,2→1}), {B, Basis_3[𝕆_{AR,{x,y},{1,2,3}}]}]
Out[∘]=
        320 True
```

```
In[∘]:= Total@Table[(B // sm_{3,2→2} // sm_{2,1→1}) == (B // sm_{2,1→1} // sm_{3,1→1}), {B, Basis_3[𝕆_{AR,{x,y},{1,2,3}}]}]
Out[∘]=
        320 True
```

Co-Associativity of Strand Doubling

```
In[∘]:= SortBy[
         DeleteCases[
          Table[B → ((B // sΔ_{3→3,4} // sΔ_{4→4,5}) - (B // sΔ_{3→4,5} // sΔ_{4→3,4})),
           {B, Basis_3[𝕆_{AR,{x,y},{1,2,3}}]}],
          _ → 0
         ],
         LeafCount
        ]
Out[∘]=
        { }
```

The Hopf Axiom: Compatibility of Strand Doubling with Strand Multiplication

```
In[∘]:= SortBy[
         DeleteCases[
          Table[B → ((B // sΔ_{3→5,6} // sΔ_{4→7,8} // sm_{5,7→3} // sm_{6,8→4}) - (B // sm_{3,4→3} // sΔ_{3→3,4})),
           {B, Basis_3[𝕆_{AR,{x,y},{1,2,3,4}}]}],
          _ → 0
         ],
         LeafCount
        ]
Out[∘]=
        { }
```